Jet and Rocket Propulsion
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Lecture - 16

Welcome back. So, in the last lecture, we have been discussing orbital mechanics. So, let us continue from that discussion on orbital mechanics.

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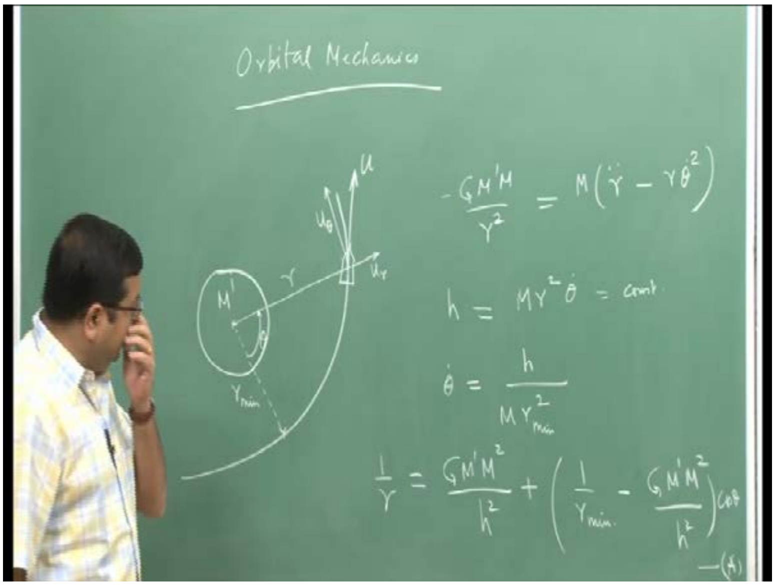

Let me draw the problem statement that we have a heavenly body, quite massive and with mass M dash, and an artificial body – a satellite or something is moving around it in a predetermined path. Let us say this is the satellite. The satellite let us say is moving with a velocity u like this. The minimum distance of this satellite path from the centre of this heavenly body is r min. The distance of the satellite at any instance of time from the heavenly body center is given by r; and the angle that it is making at a particular instance with r min is given by theta. The satellite is moving in a plane and we are considering polar coordinate system. So, the plane… That is why we are not considering the z axis. So, essentially the coordinate system is primarily focused on r m theta. The velocity is in this directions; it will have two components: one is u r; other is u theta.

Now, this is the problem statement. The vehicle is moving because of that. First if I look at the forces acting on this vehicle at a particular instance depending on this distance, the gravitational force acting on this is this, which is the gravitational force acting on this vehicle. And this gravitational force is balanced by the motion of this vehicle, which is given as M r double dot minus r theta dot square. We had worked out this and we have also shown that the angular momentum of this vehicle is constant, because there is no force in the theta direction. So, if there is no force, there is no acceleration. Therefore, there is no rate of change of momentum. So, angular momentum remains constant. So, the angular momentum we have shown is equal to M r square theta dot; which is equal to a constant. And from here we get an expression for theta dot is equal to h upon M r min square.

Now, we put this back into this equation and simplify and rearrange this equation to get an expression for r. So, r here is our intended path, which will be in terms r n theta. So, we get an expression for r, that is, the path at given instance. So, essentially, what is r? r is the distance of the satellite from the center of this heavenly body at any instance. So, then, after we have simplified this, we have shown that 1 by r is equal to G M dash M square divided by h square plus 1 upon r min minus G M dash M square upon h square cos theta. And, we had call this equation A. So, this is the equation for the orbit, which this satellite is following. And, we see that, this orbit equation is given in terms of the mass of the vehicles and the angular momentum and the minimum distance.

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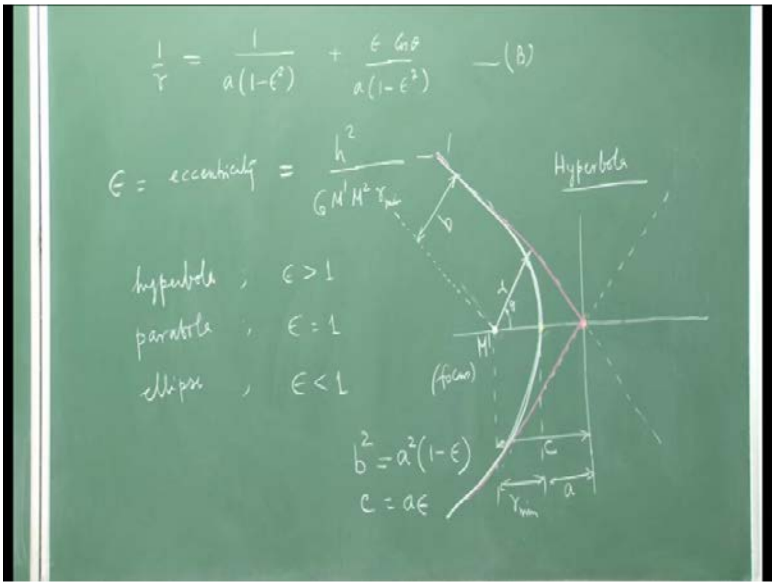

Now, last time we had just started discussing the conic sections. We have given the general expression for a conic section, which is given as 1 by r equal to 1 upon a 1 minus epsilon square plus epsilon cos theta upon a 1 minus epsilon square. And, we had called this equation B. Now, if I look at equation A and B, remember this we derived from force balance and this is the general equation of conic section. We see that, these two expressions are very similar. Therefore, we can equate the terms that are appearing here directly; and then, we can now get these values. These are the designed parameters, which will go into our rocket design.

Now, we can get these values. So, looking back at this, first of all, epsilon as I have mentioned last time, is called eccentricity is equal to h square upon G M dash M square r min minus 1. How do we get this? We get this by essentially equating the terms here. But, a is also present there. So, I have to get an expression for a also. For that, let us go back to the basic conic sections. So, the conic sections that we have are hyperbola, parabola and ellipse. These are the conic sections that we have. For hyperbola, eccentricity is greater than 1; for a parabola, eccentricity is equal to 1; and for an ellipse, eccentricity is less than 1. These are the basic definitions of conic sections.

Now, let me look at a specific case of a hyperbola. Let me draw the path. This is the axis. This is what typically a hyperbola is. And, this point here is where the big ((Refer Time: 08:20)) is sitting, which is mass M dash. This point is called focus – focus of the hyperbola. So, the hyperbola actually is focusing on this or originating from this. At any instant of time, let us say the vehicle is following this hyperbolic path. So, at any instant of time, the distance of this from the focus is r; and the angle that it is making with respect to this horizontal is called theta. Now, if I project it back to that definition, then this distance is the minimum path as we can see here – minimum distance between focus and the path. So, that is the distance between these two, is the r min according to that picture. Therefore, this is the path that we are following. I will define some more properties.

The distance of the focus from the origin is c and the distance between the origin and this minimum... So, the distance between this point, which is the origin and this point is a. So, this distance is a. Now, there is one more thing that I like to point out that, from this focus, if I drop a parallel to this line… This is a 45 degree line. If I drop a parallel to this… Actually not 45 degree line; it is a given path; it is the tangent to the hyperbola.

So, from this, actually if I drop a tangent to this path… So, drop a line parallel to the tangent to the hyperbola passing through the origin. Then, by the way, this – what is this origin? The origin is the point of intersection between the tangent to this and this. So, this is I am calling it origin. This is the point of intersection between this tangent and this tangent. As we can see that, this is a tangent to the path; and this is another tangent to the path. So, the point of intersection between these two is the origin. And, then this is the path.

Now, if I draw a parallel passing through the focus to this tangent, then the distance between this parallel and the tangent is called b. So, this is the basic geometry of a hyperbola. So, once again, here what I have is; I have this hyperbolic path; I draw tangents to this hyperbolic path. These two intersect at this point, which is the origin. This point is the focus, where the bigger ((Refer Time: 11:54)) is sitting. How do we get the focus? Focus is the distance between this point, which is crossing the horizontal; and from this point, the minimum distance r min is the focus. So, this is the focus. Then, at any point here on the path or trajectory, the distance between the focus and that point is r; and the angle it makes from the horizontal is theta. The distance between this point of intersection and the minimum point is a. And, the distance between this point of intersection and the focus is c.

Now, with this now, we can define the eccentricity epsilon. For the hyperbola, you can look at any geometry book; it gives b square is equal to a square 1 minus epsilon; and c is equal to a epsilon. So, if a hyperbolic trajectory is given; looking at this, from this, trajectory, we can get b; we can get c; then, we have two equations here with two unknowns a and epsilon. We can solve for these two: a and epsilon. We can take that and put back into this equation. And now, we have 1 by r in terms of the known quantities. Once we have that, then we can go back to this. And now, we have equations for the… Here the unknowns are h square and r min. So, I can solve for h square and r min here. And, we have now the full solution. So, this is how we use the geometry of a conic section to determine the given mission requirement. This was for a hyperbola. I will come back to the hyperbola again. I have said at the beginning that, the most commonly used orbit is an elliptical orbit. So, next, let us look at the geometry of an ellipse. So, we consider an elliptical orbit. Let us look at the geometry of an ellipse. So, once again, the focus is to get these parameters.

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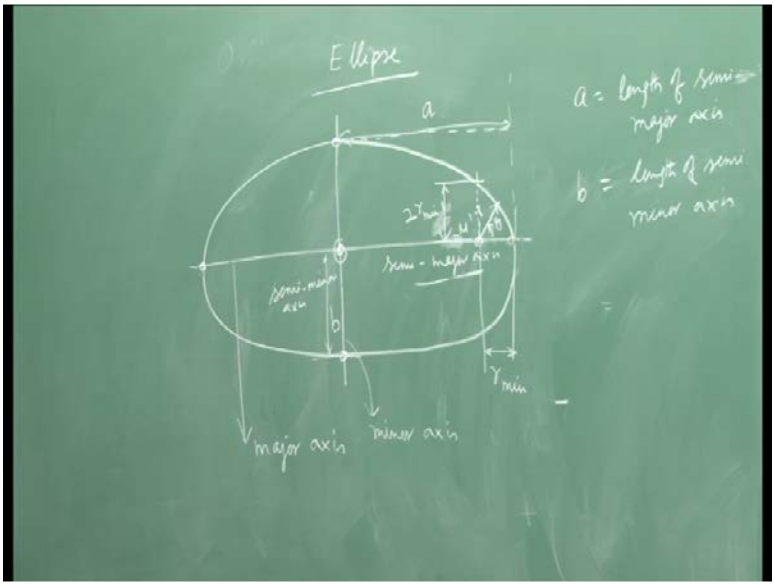

So, geometry that we are going to discuss is an ellipse. Ellipse we have two axes: one is called a major axis; other is called a minor axis. So, this is typically the ellipse that we have. Now, this axis is the major axis and this axis is the minor axis. As we can see that, this axis – major axis is longer than the minor axis; minor axis is shorter. Then, half of the major axis… So, we have… Now, if I look at this as the center; then, this and this points are the minimum points; and this and this points are the maximum points. So, when the satellite is at this point, it is at the minimum distance from the center; but it is at this point, it is at the maximum distance from the center.

So, now therefore the major axis is twice the distance between the maximum distance. So, we call this half; we take half of it – this half; this is called semi major axis. As you know that, semi means half. So, this is semi major axis. Then, this distance of the semi major axis is a. So, once again, what is a? a is the length of semi major axis; and this is the minor axis. So, half of it from here to here is the semi minor axis; and that length is called b. Therefore, a is the length of semi major axis – semi major axis; b is the length of semi minor axis.

Now, here all these definitions are with respect to the center of this ellipse; the point of intersection between the major and minor axis. But, the heavenly body may not be sitting there; it is sitting somewhere else; it is somewhere else. Let us consider that, the heavenly body somewhere here. This is the M bar – M dash. Then, from this, what we see is that… Actually, it will be closer here; somewhere here – M dash. Then, the minimum distance of this body of the artificial satellite from the heavenly body is this much. So, this is the r min. And then, at any instance of time, this is the r; and this is theta. So, once again, we have brought back our r and theta, etcetera. So, now, what we see is that, the satellite is moving here; this is the minimum distance; and the maximum distance is this – other end of the major axis. So, minimum distance is when the satellite is at this end of the major axis; the location between the satellite and the heavenly body is r min. So, now, we have defined this parameter a, b, c, d, etcetera.

Now, from the geometry of ellipse, one more thing can be very simply obtained. If I drop a perpendicular to this, it will cut at this point. Then, from the geometry of the ellipse, it can be shown that, this distance is twice this distance. So, once again, the location of this heavenly body is not arbitrary. If this is the minimum distance; then, if I drop a perpendicular from here cutting the path, the point, where it is the perpendicular intersects the path should be at twice the distance from this. So, that is how the location of the heavenly body is determined. Or, in other words, the artificial satellite then should put in such a way that, if this is the heavenly body, the minimum distance is r min, so that when it comes here, this distance is twice r min. Then, it is an elliptic path. Now, this is how we have defined. But, for any conic section, what we need to do is we have to get an expression for a and epsilon with respect to the conic section definition that we had already discussed. So, next, let us find the value of eccentricity and a for an ellipse.

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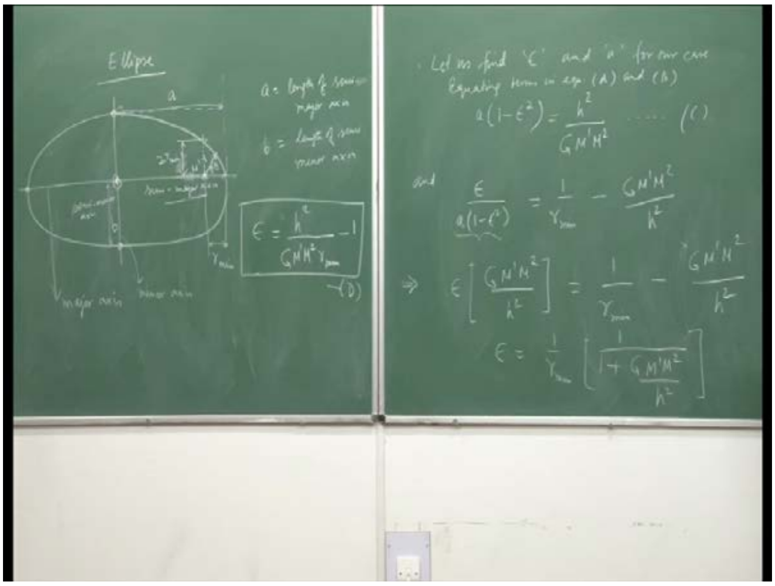

Let us find epsilon and a for our case. What is our case? That we are talking about a vehicle moving in a conic section. We want to find an expression for epsilon and a for that. So, once again, go back to the equation A and B. What was the equation A? Equation A was from the force balance. It is the orbital mechanics. Equation B was from the definition of conic section. So, if I equate the two… There are two terms in both of them. If I equate the first term of equation A with the first term of equation B and second term of equation A with the second term of equation B, I get two relationships. So, these relationships are… This we will be getting by equating terms in equation A and B. We get a 1 minus epsilon square equal to h square divided by G M dash M square. Let me call this equation C. And, the second term there was epsilon upon a 1 minus epsilon square. This was equal to 1 upon r min minus G M dash M square by h square. So, we are equating the two terms.

Now, let us simplify this little more. This implies epsilon times G M dash M square upon h square equal to 1 upon r min minus G M dash M square upon h square. So, what we have done here is; as we can notice here, a 1 upon epsilon square is this term. So, I have just taken this and put it into this equation; taken this term and put it into this equation. I get this expression. Then, we simplify… Now, here in this equation, the epsilon is the only unknown for the time being. So, epsilon then can be written as 1 upon r min 1 upon 1 plus G M bar M square upon h square. So, we just simplified it got an expression for epsilon in terms of r min and other stuff.

Now, we can further simplify it and get an expression correct. Now, we can further simplify it and get the expression that we had already shown that, epsilon is equal to h square upon G M dash M square r min minus 1. This is what I had given you last time as the homework. So, this is the derivation of that. Let me call this equation D. Now, we have shown, we have derived an expression for epsilon from here. Now, if I take this expression for epsilon and put it into equation C, then we eliminate epsilon and get an expression for a.

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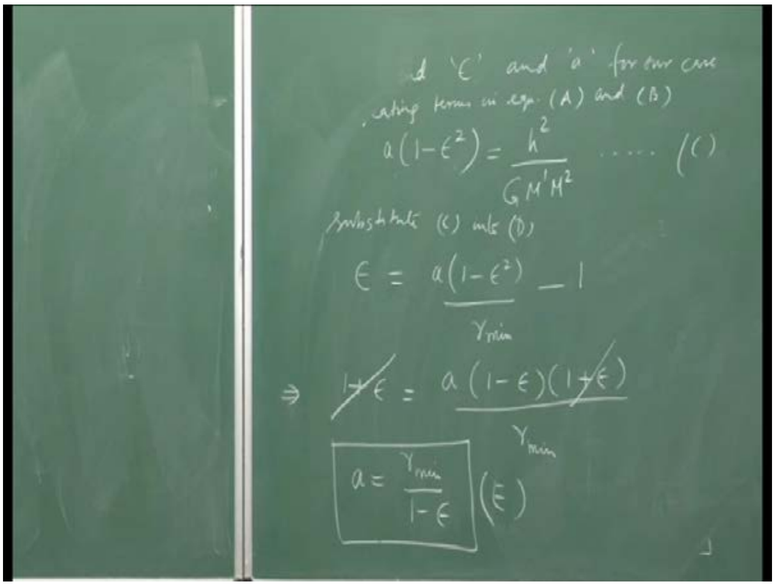

So, I will continue from here and take equation D, which I have just derived and substitute it here. We get epsilon. So, substituting C into D, we get epsilon equal to 1 minus epsilon square upon r min minus 1. Then, if I further simplify it, I get 1 plus epsilon equal to a 1 minus epsilon square. I will break into two, which is 1 minus epsilon, 1 plus epsilon. So, this is equal to a 1 minus epsilon, 1 plus epsilon divided by r min. So, epsilon plus 1 will cancel off. Now, what I have is a very simple expression for a; a is equal to r min upon 1 minus epsilon. Let me call this equation E. These equations are very important. That is why I am marking them with numbers or digits. So, this gives me an expression for a.

Now, coming back to this path then, this is a. So, now, knowing this, I can find out what will be the semi major axis. Taking this now, go back to the equation of the conic section, which was 1 by r equal to a upon something and a and epsilon – function of a and epsilon and theta. So, next… So, now, what we have done is we have derived expression for a and epsilon in terms of the performance parameters or the vehicle locations mass and angular momentum, etcetera. Now, let me go back to definition or rather the geometry expression for the conic section.

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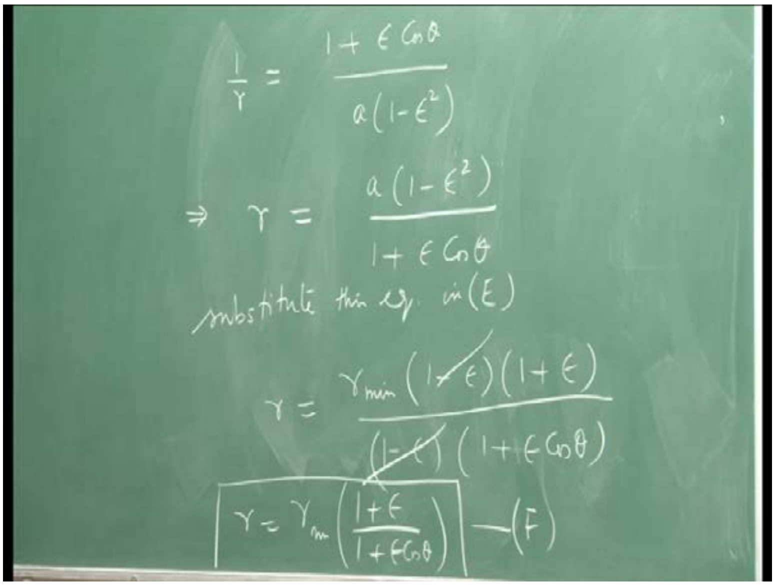

So, going back to equation B, we can write 1 by r equal to 1 plus epsilon cos theta upon a 1 minus epsilon square. This is quite simple, because if I look at the expression for r, it was 1 upon a 1 minus epsilon square plus epsilon cos theta upon a 1 minus epsilon square. So, I am just rewriting it. Now, from here then, I get an expression for r, which is the instantaneous location of the space vehicle, which is equal to a 1 minus epsilon square upon 1 plus epsilon cos theta.

Now, let us take this and substitute here. Substitute in equation E. Then, what we get is… So, here I can eliminate a from this. So, if I eliminate a, I get an expression for a from here, which I put into this equation and simplify it. So, substitute this equation in E; I will get r is equal to r min; and this term 1 minus epsilon square again I will expand as 1 minus epsilon 1 plus epsilon divided by 1 minus epsilon 1 plus epsilon cos theta. So, this I will cancel off. I get r equal to r min 1 minus epsilon upon 1 plus epsilon cos theta. So, I just write it here. Let me put it here. This is equation F.

So, what do we have got here? What we have got is the path trajectory as a function of minimum distance and epsilon, and of course, cos theta, which is the location at that particular point. I would like to point out here one thing that, this discussion that, we are carrying out for conic section is not specific to a specific conic section; it is applicable to ellipse, hyperbola, parabola, all. And, we have defined of course for the hyperbola and ellipse, how do we define the parameters. Then, we have gone back to the basic definition of conic section and we are working with that. So, this expression is again valid for the basic definition.

Now, let us go back. Remember that, I have said something that, whether the conic section that we are choosing is going to be a hyperbola or parabola or ellipse, depends on the eccentricity epsilon. Therefore, this epsilon is a very important parameter. And, depending on the choice of epsilon, then we can see that, we can form a hyperbola and parabola and ellipse. So, now, let us take a little closer look on this part or this concept. So, now, what we will do is we consider different values of epsilon and see what kind of sections we get.

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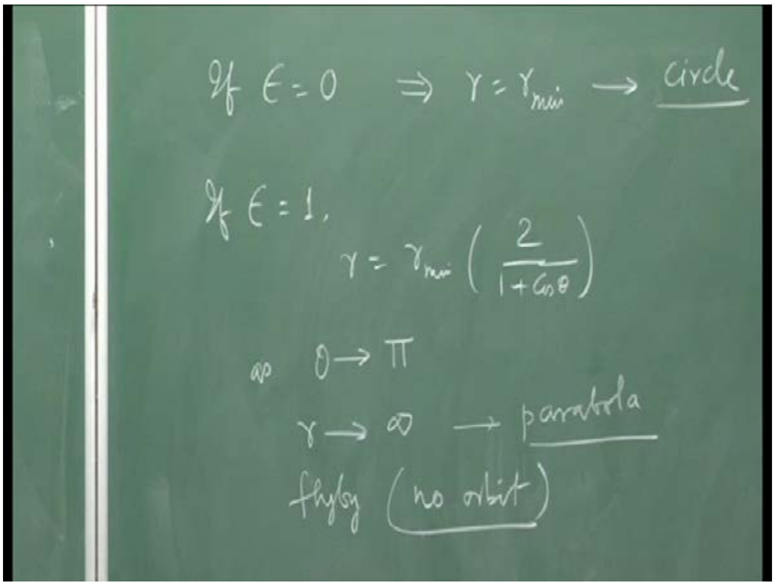

So, first, if epsilon is equal to 0… In this equation, if I put epsilon is equal to 0, what happens? This is 0; this is 0. So, r is equal to r min. This implies r is equal to r min; which means that, the radius is constant. This implies the conic section we are talking about is a circle. So, we have a circular path. So, if we have epsilon equal to 0, we have a circular orbit of radius r min. Next, if we put epsilon equal to 1; if epsilon is equal to 1, then this is 1; this is 1. So, this is 2 upon 1 plus cos theta. So, then, r is equal to r min 2 upon 1 plus cos theta.

Now, as theta tends to pi, what will this happen? When theta is equal to pi, this is equal to minus 1. So, then r tends to infinity. So, when theta tends to pi, r tends to infinity. And, that is a parabola. So, this is a parabola. So, initially, we had said that, for epsilon equal to 1, we have a parabola. Now, we are proving it that, as theta tends to pi, epsilon… r tends to infinite. That is a parabola. So, then, what is happening if you have a parabolic orbit? If r tends infinity is what? That, the vehicle is just going away; it is not being captured. So, it is fly away. So, it is essentially a fly by condition; we do not have an orbit. So, for a parabola then, we do not have an orbit. So, if the vehicle is moving in a parabolic path, it will not be captured in an orbit; it will just go away. Next, let us look at the hyperbolic path.

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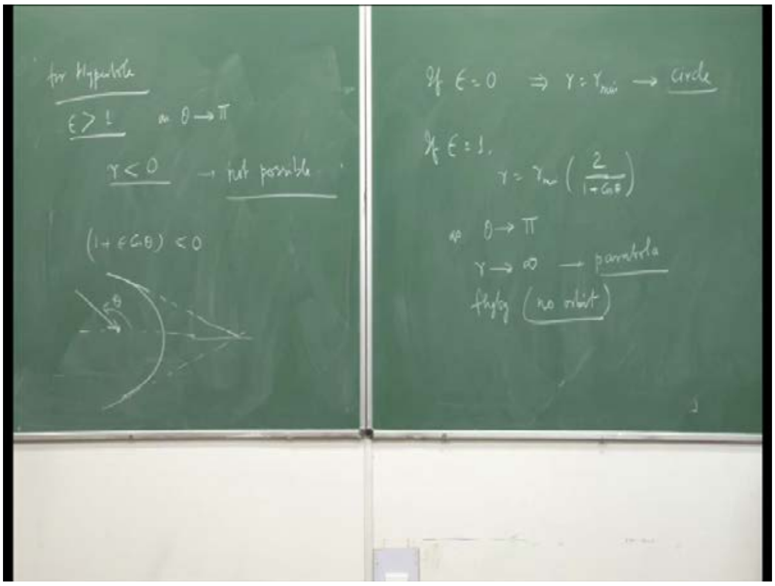

So, for hyperbola, we have considered… We have seen that, for hyperbola, epsilon is greater than 1. So, when epsilon is greater than 1, then as theta tends to pi, what will happen? Epsilon is greater than 1 means this term is greater than 1; and here we have 1 plus… At quantity greater than 1 times cos theta. Now, when theta tends to pi, cos theta becomes minus 1; minus 1 multiplied by a quantity greater than 1. Therefore, this will be greater than 1. Therefore, the denominator here becomes negative, which essentially means that, we are getting r less than 0. So, what is happening now? That the vehicle is going in; vehicle is going into the bigger body that we had talked about. Therefore, this is not possible, because the path that it is taking is essentially not possible to attain, because it has to go into the heavenly body.

However, for other application, if you are not putting a satellite into orbit; if you are launching an ICBM, this is what we want. So, the missile has to come back. Therefore, for an ICBM and all, the intended path actually better is to have a hyperbola. Therefore, this shows that, if you have a hyperbolic path, it is not possible to get an orbit. Here for a parabola, it will fly by; it is just going to go away. For the hyperbola, it is not possible to go into an orbit. But, what we can do is we can re-energize it and take it to another altitude or another path. That is a different thing. For example, typically for missions like mission to moon and all, you take a hyperbolic trajectory and then go back to the elliptic trajectory after some maneuver. But, if we have to do it in a single go, it would not be captured; hyperbola would not be captured. Therefore, hyperbola is something that is not visible.

Then, what is happening here in that case in hyperbola is 1 plus epsilon cos theta. That is the denominator of the term is less than 0, which essentially makes r less than 0. Therefore, it is not possible. So, what is happening if I draw a hyperbola; this is where it is; this is theta. So, what we see is that, there is no solution possible for theta greater than certain value. For theta greater than 2 pi and greater than pi, no solution is possible. So, what we have shown now in this lecture today; then, we have talked about ellipse, where the value of epsilon is less than 1.

We have talked about circle; epsilon is equal to 0; this gives a circle or orbit. If epsilon is equal to 1, this is a parabolic orbit; where, it is not possible to have an orbit; it will fly by. If it is a hyperbola, again it is not possible, because r becomes less than 0. So, it is going to hit the body. Other solution, other possibility only now remaining is epsilon less than 1 and between 0 and 1; which is an elliptic orbit, which is possible. And, that is why typically, we consider elliptic orbits for most of our applications. So, we have discussed various trajectories; we have discussed the force balance.

Let us now take a little step forward and try to get in the velocity increment that we are looking for; because if I look at all the equations, which I have derived so far, velocity is not present. We are talking in terms of angular momentum; we are talking about in terms of radius, etcetera; but when velocity is still not present in any of these equations. And, velocity is something that we want to have. Now, what we like to do is get the velocity back in, because that is what the rocket designer would like to know how much velocity increment we have to give. For that, what we do is we look at the energy balance; the trajectory that we had returned so far was in terms of the force balance or the conic section. Now, we write the same expressions in terms of energy balance.

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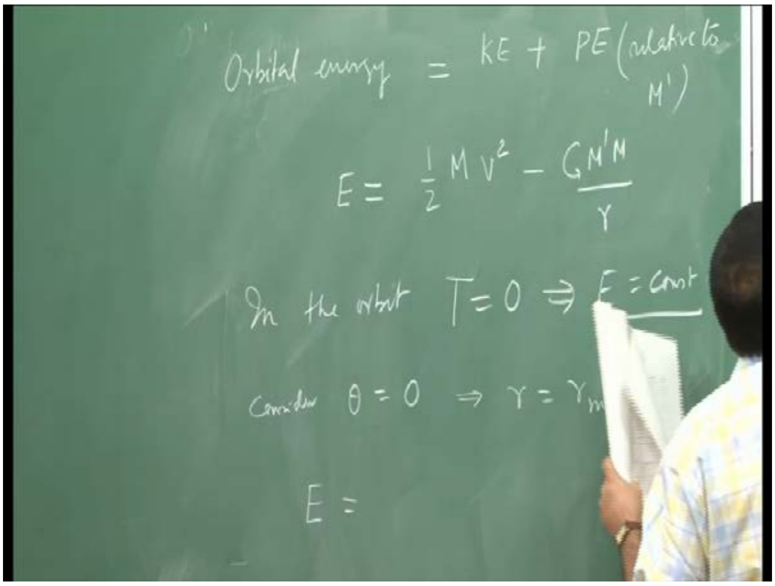

So, for that, let us look at the energy of the vehicle when it is in the orbit. So, it is called orbital energy. This will be the kinetic energy of the vehicle plus potential energy of the vehicle relative to M bar; that is, the heavenly body. Therefore, the total energy of the vehicle when it is in the orbit is this kinetic energy plus the potential energy. Now, what is kinetic energy? So, let me call this as E – total energy. Kinetic of the energy of the vehicle is half M – is the mass of the vehicle and the velocity square; where, velocity – v is the velocity of the vehicle in that orbit. And, what is the potential energy? Let us coming from law of gravitations, so the gravitational energy is the potential energy. So, this is equal to G M bar M upon r. That comes from the law of gravity. Therefore, this is the total energy of the vehicle when it is in the orbit.

I would like to point out here that, when the vehicle is in the orbit, there is no thrust. We have said that, we have given impulsive load it has is somewhere. Now, there is no thrust. So, if there is no thrust, then there is additional energy being added. There are forces acting on it, but no energy being added. Therefore, once it is in the orbit, it has no net thrust. Therefore, the energy remains constant. So, in the orbit, thrust is 0. Therefore, energy is constant. Now, if the energy is constant, then first of all I will just simplify it and consider theta equal to 0. What is theta equal to 0 condition in the trajectory? According to the definition, this corresponds to r equal to r min according to the trajectory that we have drawn, because we are measuring theta from that location. So, theta is equal to 0 means when the vehicle is at its minimum distance from the center of the heavenly body. So, consider this case when theta is equal to 0. At that point then, if I look at the total energy, energy is equal to what? First of all, this term here v square…

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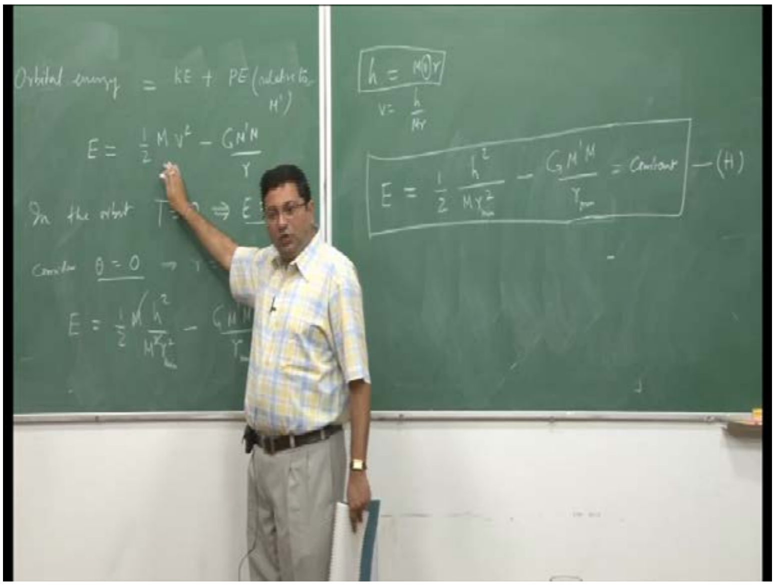

We have done earlier that, the angular momentum h is equal to M v r. Therefore, v equal to h upon M r. This is the definition of angular momentum M v r. So, v equal to h upon M r; agreed? If I take this and put back into this equation, I get half M h square upon M square r square minus G M dash M by r. And, since we are considering theta equal to 0, r is equal to r min. So, this is r min, r min. Then, I will cancel off this one of the M's and I get an expression for the total orbital energy is equal to half h square upon M r min square minus G M dash M upon r min. Let me call this equation H. So, this is another equation that we have got in terms of the… No, let me take little more time. This you have got an expression for E in terms of r min.

Now, coming back to what we have said here, this value is constant; E is constant. Now, let us look at this equation. H is the angular momentum, which we have said at the beginning – the previous lecture that, H is constant, because there is no angular acceleration. M is the mass, which is constant; G is constant, M dash is constant, and r min is also constant. It is a given quantity. Therefore, this energy at r min is equal to the energy that the vehicle will have at any location. Therefore, this is the energy in that orbit. And, that is why this energy is constant. So, I will slightly modify this equation and write that, E is equal to this; which is equal to a constant. It is given by equation H. So, what now we have is modified equation in terms of the orbital energy.

Now, before progressing further, we have to now… What we will try to do now is combine whatever we have derived so far in terms of the equations and try to simplify, so that with the minimum number of given quantities, we can estimate our requirement. But, notice one thing here is that, this equation here now contains this v; this is the velocity we are looking for, which actually appeared here also. So, what I will do now in the next lecture is that, we will take this equation; combine it with the equations we had derived for various expressions for the conic sections like epsilon, etcetera; and get a combined equation, which will be something that we can directly use for getting the performance parameters in a particular orbit. So, if the orbit is specified, we can find out what it will take to attain that orbit. So, what I will do is now I will stop here and we will continue from here in the next lecture.

Thank you.