Jet and Rocket Propulsion
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Lecture - 19

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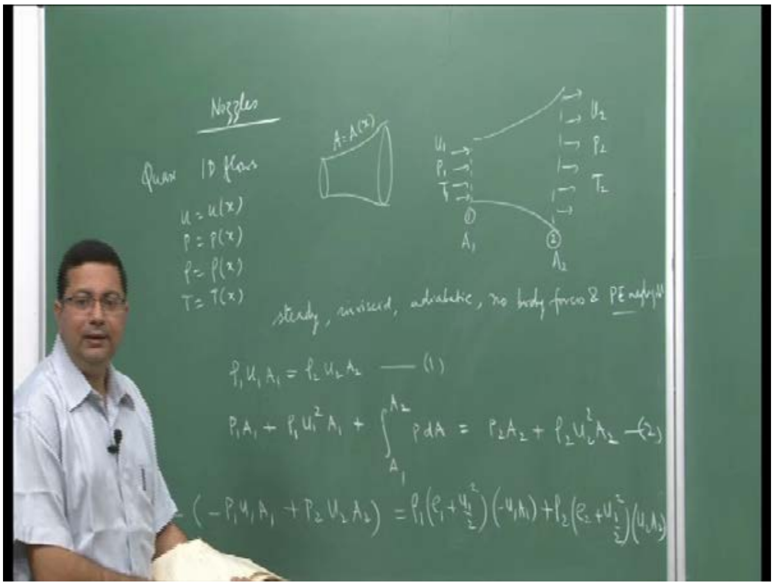

So, welcome to this lecture on rocket propulsion. In the last class we had started discussing on nozzles, we are discussing quasi 1 d flows which are essentially flow through variable area ducts, we have said that if the area variation is small then the flow properties are going to vary on the along the direction of variation of area. In that case even though the flow 3 dimensional, we can consider them to be 1 dimensional flows which are called quasi 1 dimensions flows.

So, for those cases all the flow properties will be function of x only for the 1 dimension T T x, etcetera. Then we started discussing the flow through a stream tube with variable area with area A 1 at section 1, and A 2 at section 2, and this is the stream tube. We considered the flow to be quasi 1 d, let say the properties at the inlet of the stream tube are p 1, T 1, u 1, etcetera, and the exit the velocity is uniform u 2 pressure is p 2, T temperature is T 2, etcetera. So, this is the problem we are solving last time. For that we had made certain assumptions that the flow is steady inviscid adiabatic, then no body forces and potential energy negligible.

So, I made this assumptions with this assumptions then we derive the continuity equation, momentum equation, and energy equation. So, the continuity equation was rho 1 u 1 A 1 equal to rho 2 u 2 A 2 we call this equation 1. Then we derived the momentum equation, which was p 1 A 1 plus rho 1 u 1 square a 1 plus integral A 1 to A 2 p d A equal to p 2 A 2 plus rho 2 u 2 square A 2, we call this equation 2. We have said that this term here the presence of this term, which is the integral from area A 1 to A 2 p d A, which is essentially the contribution of a special forces acting on this curved control surface, makes this equation non algebraic, then we started discussing our energy equation. So, till the end of last class we had derived the energy equation for this system for this control volume, and we have shown that the energy equation will be plus p 2 u 2 a 2 equal to rho 1 u 1 plus u 1 square by 2 minus u 1 A 1 plus rho 2 u 2 plus u 2 square by 2 u 2 A2. We have proved up to this till the end of last lecture.

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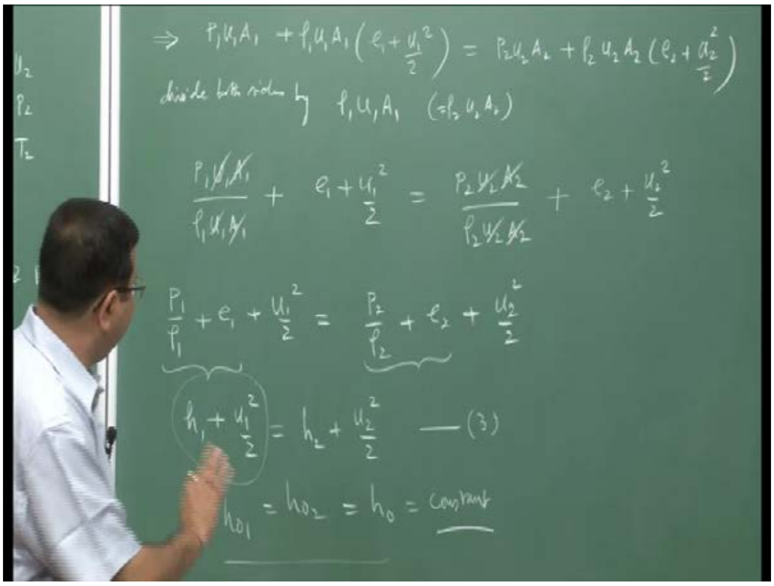

Now, let us continue from here. Let us simplify t his energy equation little more. So, from this energy equation we can rewrite this as p 1 A 1 a 1 plus rho 1 u 1 A 1 times u 1 plus u 1 square by 2 equal to p 2 A 2 u 2 plus rho 2 u 2 A 2 times e 2 plus u 2 square by 2, we can write it like this. Now, what we do is let us divide both sides by rho 1 u 1 A 1, and we know that rho 1 u 1 A 1 is equal to rho 2 u 2 A 2, right. So, what we will do is the left hand side we divide by rho 1 u 1 A 1, right hand side we divide by rho 2 u 2 A 2. While doing so we will get p 1 u 1 A 1 by rho 1 u 1 A 1 plus e 1 plus u 1 square by 2 is equal to p 2 u 2 A 2 by rho 2 u 2 A 2 plus e 2 plus u 2 square by 2, we get this.

Now as we can see here we can cancel this of, we can cancel this of. So, now what we are left with is p 1 upon rho 1 plus e 1 plus u 1 square by 2 is equal to p 2 upon rho 2 plus u e 2 plus u 2 square by 2. So, here pressure p is pressure, rho is density, e is internal energy. Then from the thermo dynamic definition internal energy plus p by rho is enthalpy. So, therefore, this term is h 1, which is specific enthalpy at 1 plus u 1 square by 2 is equal to, this is h 2 plus u 2 square by 2. So, what we have is h 1 plus u 1 square equal to h 2 plus u 2 square, let me call this equation 3, so this is the energy equation. As we can see that area is not appearing anywhere, area has been eliminated completely. This is the energy equation for a steady, adiabatic, quasi, 1 d flow.

Now, if we take it further we had not defined a specific location for 1 and 2 right, still we have shown that at 1 and 2 this relationship is valid. So, if I consider this as a property at any particular location 1, then this is the stagnation property right, if I consider that the flow is bought to 0 velocity adiabatically then this becomes a stagnation property h naught. So, what this is showing is that h 0 1 is equal to h 0 2, which essentially means that that the stagnation enthalpy is constant everywhere in the flow field, this is of course, valid only with the assumptions that we had made.

So, therefore this is something that we get for a quasi 1 d flow, that the stagnation enthalpy is constant everywhere in the flow field. So, this completes our discussion on the conservation equations in the control volume approach integral form we started with the integral form of conservation equations, in order to get this term here integral form is not enough. So, now what we will do is let us look at the differential form, integral form can very easily handle our continuity equation and the energy equation, but 1 parameter remains here which unless we specify A - variation of A, we cannot solve for this. So, now let us look at the differential form of the conservation equations. So, let me clean this part and we took at the look at the same problem, but little differently.

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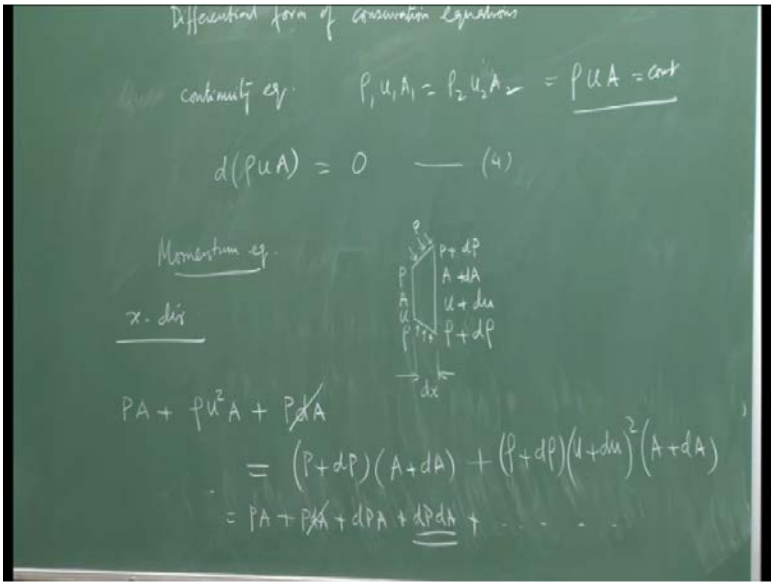

Next, what we do is we look at differential form of conservation equations; first of all we start with our integral forms. Starting with the continuity equation, in continuity equation what we have shown for this problem is that rho 1 u 1 A 1 equal to rho 2 u 2 A 2, that we have just shown. We had now as we had not specified any specific location for A 1 and A 2, therefore this shows that anywhere if I pick any location then rho u A is constant, right.

So, the integral form of the conservation equation for this problem says that rho u A is constant, we use this that if rho u A is constant, if we differentiate this d rho u A is going to be equal to 0, let me call this equation 4, that d of rho u A is equal to 0. Next, let us look at the momentum equation, we consider a small portion of the control volume that we had considered earlier. Let us say at this section pressure equal to p area is A, velocity is u, density is rho; at this section pressure is increased little bit to say p plus d p area has increased little bit like A plus d A. The velocity has changed little bit by among d u and the density has changed little by among d rho. Let us consider that the length of this section is d x, and we have this pressure forces acting on this side.

Now, we have in the previous cases we are considering a quasi 1 d flow, we got the momentum equation in x direction. So, let us do this again here that momentum equation we write in x direction. And the expression that we got earlier from the control volume approach, we use that only, only for this control volume. So, in that case if I use consider this as the control volume, we get p A plus rho u square A plus, now if the pressure acting on this side is p, area variation is d A, then the pressure forces acting from the side we have seen in our integral form is p d A, right. So, this is p d A equal to the pressure the the term on this side, this is our section 1, this is our section 2. So, now we write this for the section 2 for section 2 we have pressure is p plus d p, area is a plus d A, density is rho plus d rho, velocity is u plus d u square A plus d A. So, this is the right hand side of the momentum equation which was the values at section 2.

Now, what we do is let us expand this right hand side, when we expand it we will get equal to p A plus p d A plus d p A plus d p d A plus with expansion term you can write, I am not going into a details of that. Then what we do is this, p d A term will cancel of as you can see from here. And all the second order terms we will drop, because they are going to be small. So, dropping the second order term we will get a simplified equation which will be... So, here what we are doing is first we are expanding then dropping the second order term.

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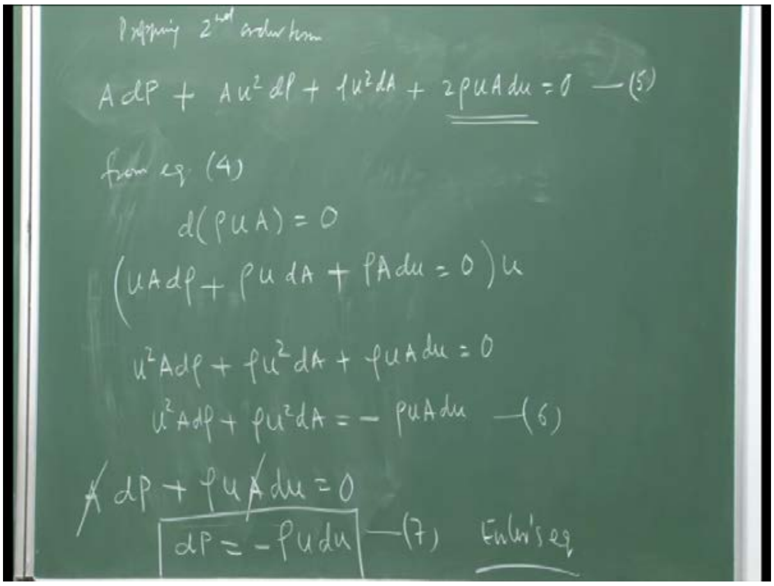

So, dropping second order terms, second and higher order of course that we will be a spare here also. So, second and higher order terms we are dropping then we are redoing this calculation, we get A d p like here again this p A term will cancel of, similarly this rho u square a term will cancel of as you can expand you will see. So, finally what we will be left with is A d p plus A u square d rho plus rho u square d A plus 2 rho u A d u equal to 0, let me call this equation 5.

Now, let us come back to this equation 4, that is our momentum equation in terms of in the differential form, let us come back to the continuity equation, equation four. From equation 4, we have d rho u A is equal to 0. So, now if I differentiate this I get 0 equal to 0, let me multiply both sides by u. Then I will get u square A d rho plus rho u square d A plus rho u A d u equal to 0. Now this term I will take to the right hand side, so I will get u square A d rho plus rho u square d A equal to minus rho u A d u, let me call this equation 6. We will further simplify it by putting this term now into this equation. As you can see here, we have this term 2 rho u A d u, which is also present in the right hand side of this. So, I can take this back and put into this equation, and then we will get A we had two of them I will just replace 1 right, because there were 2 of this I just replace one of them.

Then after doing that because it has a minus sign, we will see that this term and this term will cancel of, this term and this term will cancel of. So, we will left with A d p plus rho u A d u equal to 0. So, A d p plus rho u A d u equal to 0. Now, area can be cancelled of. So, what we have is d p plus rho u d u equal to 0 or d p equal to minus rho u d u, let me call this equation 7. This is a very important equation in fluid mechanics, this is called Euler’s equation, this equation is valid with the assumptions that we had made what about our assumptions that the flow is steady, in visit, adiabatic, no body forces, potential energy negligible. And quasi 1 d with that this equation we have derived is a very important fluid mechanics equation called Euler’s equation. Now, so this is the differential form of our momentum equation. Next let us get the differential form of the energy equation.

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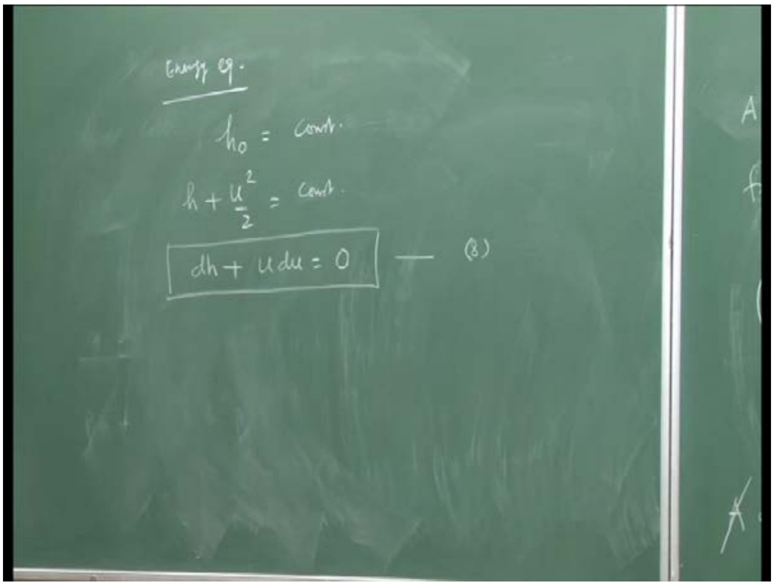

So, coming back to the energy equation, we have proved that the energy equation can be written as h naught is equal to a constant, and h naught is equal to h plus u square by 2. So, that is equal to constant for the quasi 1 d flow we are studying. So, h plus u square by 2 equal to a constant, if I differentiate this, I will get d h plus u d u equal to 0, let me call this equation 8. So, this is the differential form of my energy equation. So, therefore, equation 4 was the differential form of the continuity equation, differential 7 is the differential form equation 7 is the differential form of momentum equation, and equation 8 is the differential form of the energy equation. Now, after doing that let us try to combine this two, and get and expression for the relationship between area and velocity, because that is what primarily we would like to find out. So, the next after having done the derived the differential form of equations, next we look at the area velocity relations.

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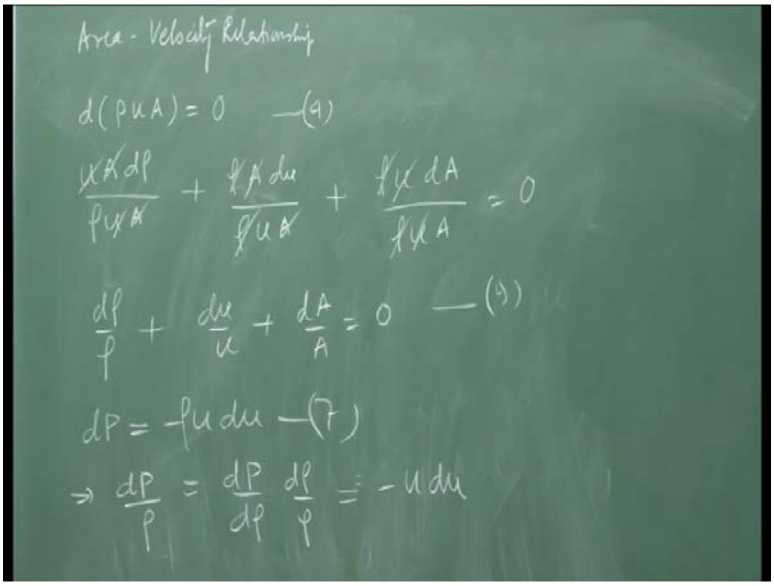

So, the next topic is area velocity relationship. This tells us that when the area varies in the manner that we have considered how the velocity is going to change. We see that velocity is changing from u 1 to u 2 how it is changing in between, that is what we try to find out. So, starting from our equation 4, which was the differential form of the continuity equation d rho u A equal to 0. If I differentiate this, and then divide both sides by rho u A, I will get u A d rho upon rho u A plus rho A d u upon rho u A plus rho u d A upon rho u A equal to 0. Then if I cancel this, I get a simplified form that d rho by rho plus d u by u plus d A by A equal to 0. Let me call this equation 9, which is another form of differential form of continuity equation.

Now, let us go back to Euler's equation or equation 7, which we have proved that d p equal to minus u d u, this was our Euler's equation. What we will do is we will write it as d p by rho equal to d p by d rho d rho upon rho equal to minus u d u, there was rho here right, d p was equal to minus rho u d u. So, we just take the rho to this side. So, therefore, this is the form of Euler's equation. Now, we had assume the flow to be adiabatic right, and we have also assumed it is inviscid or friction less. So, therefore, it is reversible also. So, if it is adiabatic and reversible it is isentropic, right. So, we have assumed the flow to be isentropic.

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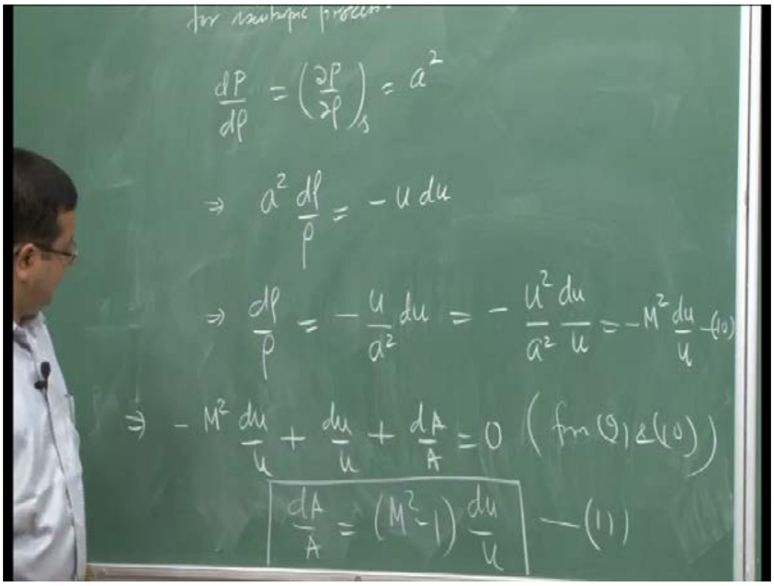

Then let us go to the isentropic relationships. Since the flow is isentropic for isentropic process, we know these relationships that the speed of sound is defined as d p d rho for a isentropic process, this is a definition of speed of sound. So, we use this now. If we define this like this, I take this and put it back into this equation. So, this term here is d p d rho. So, then this becomes A square d rho by rho equal to minus u d u, so let me write it here, therefore A square d rho by rho equal to minus u d u.

Now this I can simplify it as d rho by rho equal to minus u by a square d u, this can be written as minus u square d u upon a square by u, and by definition u by a is mach number. So, this is equal to nothing but minus m square d u by u. Let me call this equation 10. So, now we have bought brought in the mach number into picture, our through using by using the isentropic relationship and considering the speed of sound. Now, we take this and put it back into this equation - equation 9. So, you can see here we have this term d rho by rho. So, this d rho by rho term I replace by this, then we get minus m square d u by u plus d u by u plus d A by A equal to 0, this we are getting from 9 and 10.

Now, if I simplify this I will get d A by A equal to m square minus 1 d u by u, let me call this equation 11. So, if I look at this equation once again this involves my continuity equation, and momentum equation and the definition of speed of sound. Energy equation is not there, but energy equation we had already included when we derived the other equations. So, what we are seen here now in this equation is relationship between the area and velocity. So, this is called area velocity relationship, it is a very important relationship and tells us a lot of stuffs. Now, let us look take a closer look at this area velocity relationship, and see how we can infer from this. So, let us look at the significance of this relationship. First what is the area velocity relationship telling us how they reaccelerated to velocity for a given mach number, mach number is also present here.

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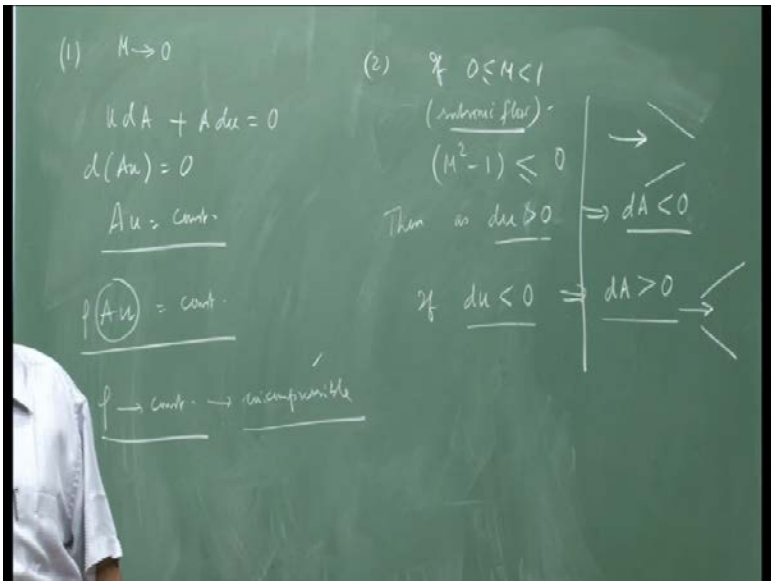

Now, first let us consider a case when mach number tends to 0, when mach number tends to 0, this terms here, this term here tends to 0 which essentially means that the coming back to this, this term tends to 0. So, what we will have is u d A plus A d u equal to 0. So, u d A plus A d u equal to 0, we essentially means that d A u equal to 0, which means that A u is a constant area times velocity is a constant. So, when mach number tends to 0, area time velocity is a constant. Now from continuity equation we know that rho A u is a constant, which is generally true for any mach number, that we have shown for mach number tending to 0, we are showing that A u is a constant. Therefore, this is a constant and this is a constant, which means that density is constant right, which means that the flow is incompressible. So, what we have done here is starting from the general equations, where we had not made anywhere that the incompressible flow assumption right we started with the general equations, we have shown here that when mach number is small tends to 0, density is constant therefore is incompressible.

So, here now we are proving that is the mach number is low, then density can be considered to be constant that the flow can be considered to be incompressible. So, far it is better at the statement right, that if density is constant mach number is low or other ways. So, when we define incompressible flow we say that mach number is low. Now we are proving it from the first principles.

So, this is a big significance of this. Next case 2, if mach number is bounded between 0 and 1, the mach number is between 0 and 1, what kind of flow do we have we have subsonic flow right by definition. So, for subsonic flow now what is happening? Looking back at this equation, we have this term m square minus 1. So, if mach number is bounded between 0 and 1 m square minus 1 is going to be less than 0 right. So, this term is going to be negative less than equal to 0 right, it is negative if this term is negative then as d u increases or d u greater than 0 d u greater than 0 means velocity is increasing.

So, this term is positive this is of course positive, this term is negative. So, positive negative is negative. So, therefore, d A is less than 0. So, this implies d A less than 0 which essentially means that for a subsonic flow, if you have to accelerate the flow the area must reduce. So, we should have a converging passage right, that is a nozzle because when we have to accelerate the flow we have a nozzle. On the other hand if d u is less than 0, that is we want to reduce the velocity, if the velocity reduces d u is less than 0, but m square minus 1 is also less than 0. So, there are two negatives. So, product of this 2 should be positive. So, therefore, if d u is less than 0, this implies d A must be greater than 0. So, what we can say is that for a subsonic flow once again, if you have to reduce the velocity the area must be increased and reducing velocity is a diffuser. So, in a subsonic diffuser the area must increase only then it follows or they confirms to the conservation laws.

So, therefore you have shown here that for a subsonic flow, if you have a nozzle then the area must decrease, if you have a diffuser - the area must increase. other way around I can say that for a subsonic flow area is decreasing d u must increase, which the velocity should be increased. So, there must be an acceleration if area is reduced. On the other hand if area is increased like in the case of a diffuser d u must decrease, which means the d u must be negative. So, u must decrease. So, therefore the velocity should decrease if you have a diverging area. So, for a subsonic flow converging area means nozzle flow accelerates, diverging area means a diffuser flow decelerates which again is proved from the area relationship that we have. on the other hand, so let us look at the third possibility now, where the mach number is greater than 1 supersonic flow.

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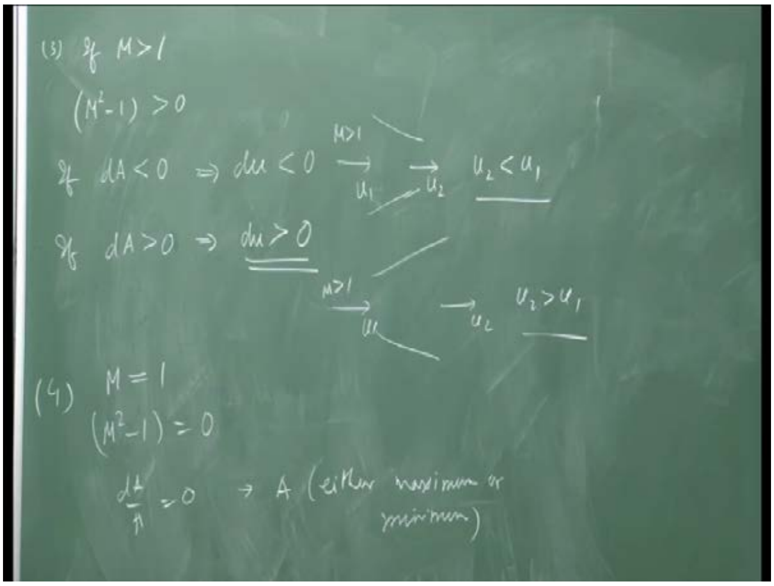

So, third if mach number is greater than 1, you have a supersonic flow. In that case again this term m square minus 1 is now greater than 0 right, because m square is greater than 1 say m square 1 minus 1 is greater than 0. In this case then if d A is less than 0, if d A is less than 0, is this term is negative, this is positive. Therefore, this must be less than 0. So, implies d u less than 0, which means that for a supersonic flow if the area decreases - velocity decreases right. So, therefore if mach number is greater than 1, then the velocity this is u 1 u 2; u 2 is less than u 1. So, from the area relationship we have shown here now that if the flow is supersonic as the area is decreases, the velocity must decrease; that means, for a supersonic diffuser the area must be converging.

On the other hand if d A is greater than 0 then once again this is positive, this is positive. So, therefore, this must be positive. So, d u greater than 0 means, d A greater than 0 means d u should also be greater than 0; that means, the velocity should increase. So, for a supersonic flow once again, if the area increase the velocity must increase. So, we have a supersonic flow coming here u 1 u 2 u 2 greater than u 1. So, once again for a supersonic flow the nozzle should be increasing area. So, this is from the area rule what we have shown is that for a supersonic flow velocity decreasing in a converging area, velocity increases in a diverging area. So, this discussions are very, very important in coming to de laval nozzle now.

So, what we have if I can summarize what we have discussed today one more case is remaining, which is the limiting case before I summarize I will go to the limiting case 4, when mach number equal to 1. What happens when mach number equal to 1 that m square minus 1 equal to 0, which means that from this equation d A equal to 0 d A by A equal to 0, which means that the area is either maximum or minimum, right. So, when the mach number is equal to 1 the area is either a maximum or a minimum area, right. So, let me now summarize what we have discussed from the area velocity relationship.

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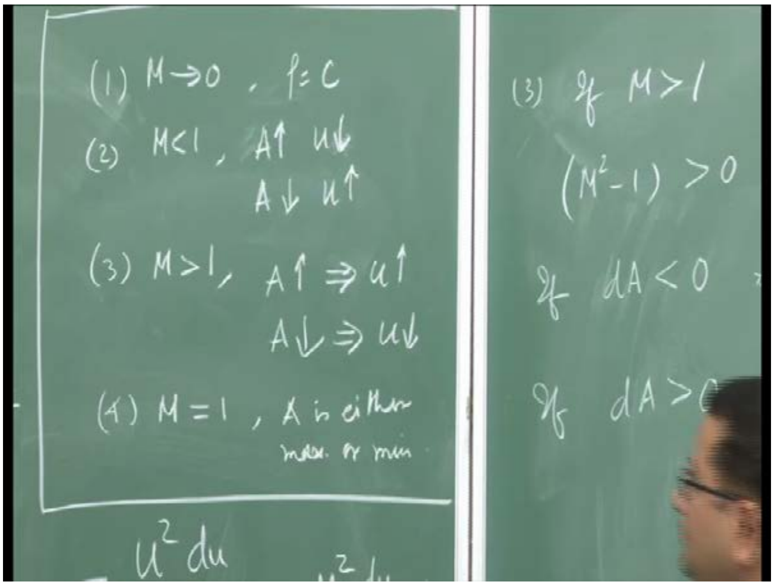

We have proved first when mach number is small, we have density equal to constant which means we have incompressible flow. We have proved that when mach number is less than 1 subsonic flow, then as area increases - velocity decreases sorry velocity yeah, velocity decreases as area decreases - velocity increases. We have proved that if mach number is greater than 1 for supersonic flow as area increases - if velocity increases, as area decrease - velocity decreases. And we have also proved that for mach number equal to 1, area is either maximum or a minimum. So, these are the things that we have proved from the area velocity relationship.

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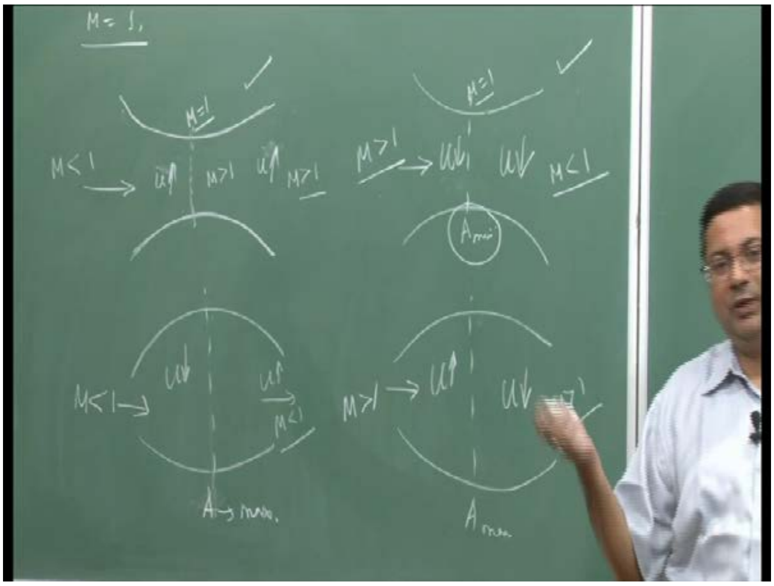

So, what we have seen then that for mach number is equal to 1, we have a maximum area or a minimum area. Now let us find out which one is the realistic physical solution, should we have a maximum area or a minimum area. So, let us consider various options; 1, 2, 3, and 4. First let us look at a subsonic flow. So, incoming flow is less than 1 and it is going through a converging diverging area, therefore at this point there is a minimum area that is present.

Now, let us say let us look at this flow, since the flow is subsonic here, the area is decreasing then according to our description so far, the velocity will increase. So, the velocity increases in this direction, it is possible then let us since we have a minimum minimum area here that the mach number becomes equal to 1, right. So, it is possible that mach number is equal to 1 at this area, because we would like to have either a maximum or minimum for mach number 1. So, mach number becomes one here when we go beyond that now the flow is still accelerating, and then here the area is diverging. So, therefore, as the area is diverging flow once it crosses mark 1 becomes supersonic. So, here mach number is greater than 1, and then flow will further accelerate. So, it continues to velocity continues to increase, till we get at the exit a supersonic flow, Therefore, in this case the velocity is continuously increasing from the inlet to exit. So, there is no discontinuity in the process.

So, therefore, this is physically possible that we have a subsonic flow, we take it through a converging diverging area then it will accelerate from subsonic to supersonic without any problem, still remain maintaining an isentropic nature. Next let us look at the same flow, but for a diverging converging nozzle. So, that here the area is maximum. So, once again the flow is coming from this side a subsonic flow as it enters this duct, the area is increasing therefore here the velocity will decrease. So, now its slows down, as it slows down it is becoming slower. So, mach number is not approaching 1, and if mach number does not approach 1 we do not have this maxima corresponding to mark 1, because according to our area rule at the maximum area mach number should be equal to 1, which is it will not attain then on this side. Now, it becomes more subsonic till it comes here, then in this side this subsonic flow it is accelerating. So, it will again accelerate here on this side, and then reach some mach number, but still remain mark less than 1.

So, therefore in this case the flow will first decelerate then accelerate, it is neither it is not either nozzle or a diffusal, and it will not attain mark 1 or beyond. Therefore, physically we cannot accelerate a subsonic flow to a supersonic flow, if we go through a diverging converging nozzle. Next let us look at this possibility where mach number is greater than 1. So, we have a sub supersonic flow coming in, and we have a minimum area. So, since the incoming flow is supersonic and we have converging passage velocity will decrease. So, the velocity decreases. So, here velocity increases, increases, here velocity will decrease; velocity decreases till it reaches this point at area is minimum, as the velocity decreasing mach number is also decreasing. So, at the minimum it can reach mark 1. So, it is possible that we have mach number equal to 1 here. After this when we come to this side of the mark 1 line, now the flow has become subsonic and the area is increasing.

So, subsonic flow increasing area velocity will further decrease. So, at the end we can get a supersonic flow, sorry subsonic flow. So, starting from a supersonic flow we can get a subsonic flow by going through a converging diverging area. So, and we have a minimum area, where mach number is equal to 1. So, this is also physically possible, when we look at the fourth option, that a supersonic flow entering an area is diverging converging area, initially mach number is greater than 1 and is diverging area, so velocity increases. So, it becomes mores supersonic. So, it will not reach mark 1 till the maximum area. So, at the maximum area mach number is not 1, it is actually much more than 1, then it comes to diverging area sorry converging area. So, it will slow down. So, the velocity will decrease here on this portion. So, it can come up come out still supersonic, but depending on the area some other value, but it will not become subsonic. So, bottom line is if I look at this two options a diverging converging section.

We are going through a maximum area, but it is neither a nozzle nor a diffuser, because the direction or variation of velocity is not monotonic, is in decreasing increasing or increasing decreasing. Whereas, when we go through a converging diverging area then for this case a subsonic flow going through a converging diverging area can be expanded to a supersonic flow, so this is supersonic nozzle. A sub supersonic flow when through a converging diverging area can be slowed down to a subsonic flow. So, this is a subsonic diffuser supersonic diffuser. So, therefore a converging diverging passage confirms to all the discussion that we had for the area velocity relationship and gives us a physically realizable solution. Therefore, for practical proposes for either a supersonic nozzle or a supersonic diffuser, we need to have converging diverging passages.

So, that and another point is that at the minimum area, which is called the throat of the nozzle or diffuse the mach number is equal to 1. So, this minimum area is a physically realizable solution is called the throat where mach number is equal to 1. So, therefore, we see that out of the four possible conditions, two are feasible where we have converging diverging geometry. So, therefore next what we will do is we have shown that from the area velocity relationship converging diverging is the feasible solution. Next, we will look at how to get the actual parameter. So, far it is all qualitative, we talk about increase or decrease. Next, now we will use an isentropic relationship for variable area duct or a converging diverging nozzle, and get the actual variables. So, we will stop here today.

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