Jet and Rocket Propulsion
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Lecture - 28

Good morning. So, in the last class, we discussed the flow through a conical nozzle. We have shown that what are the losses, how they come in because of the 3d effect, and then we said that if we kind of reduce the half cone angle, we can reduce those losses. The ultimate reduction is if you can get the flow to be parallel to the axis. And then, we discuss that why a shaped nozzle which goal is to get a parallel flow at the exit will be more at one touches so far a conical nozzle. So, today now we are going to discuss about design of the shaped nozzles, there are two ways of doing it.

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One is, so we are talking about shaped nozzles. First is if we have given a specified nozzle geometry, this actually is a roundabout way. First we choose nozzle geometry, and then we can set up a numerical solution of equations of motion for the whole nozzle. We can use a finite different scheme, a volume scheme, a finite element scheme whatever. So, the primary thing is that the shape must be specified with this specified shape, then we solve the full navies strokes. We can give proper boundary conditions through the wall, even thermal boundary conditions we can have a full structure interaction code also.

So, we can solve at present, we can solve for the full flow field, but the problem is that this is very time consuming and expensive. It will give us the full velocity field, full pressure field etcetera. First of all the solution itself is going to be very time consuming and expensive because the temperature is quite high. The flow is going to be turbulent and it is quite high temperature. It has to be compressible. So, it is a compressible turbulent flow modelling with the wall conditions which are neither isothermal nor adiabatic. There is heat transfer through the wall.

So, all this needs to be model, right. So, therefore, it is quite time consuming and expensive, and it will give only one solution for one geometry. Our goal is to design the shape. So, then what we have to do is, we have to try different shapes and then, optimize, right. So, therefore, then try different shapes and optimize. So, essentially the entire program is coming up with a design of a curved nozzle or shaped nozzle which will give us the parallel for the exit and required pressure and required velocity is going to be quite expensive. If you do it this way, the easier method, then it is something that gives us the shape as the solution which specifies the exit conditions or the required conditions. Something that gives the shape as the solution will be an easier method, much more cheaper and that is where method of characteristics comes in.

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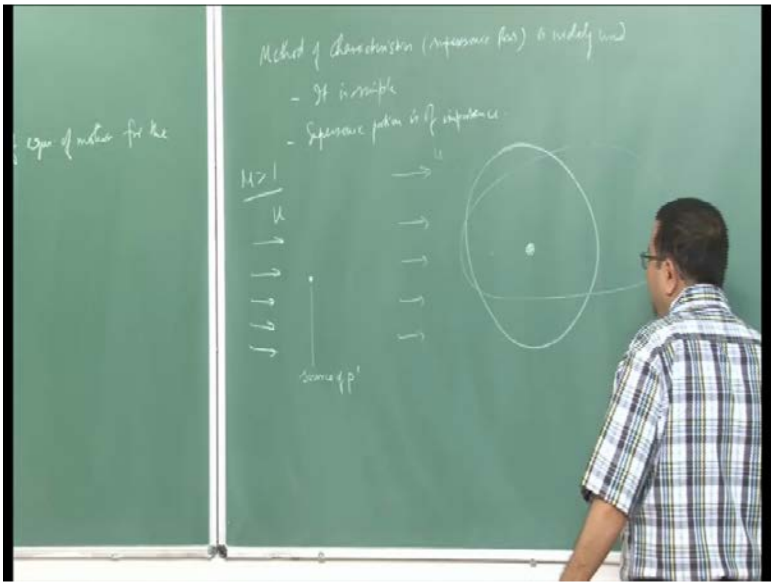

The point here is the method of characteristics is applicable only to supersonic flows. It is not applicable to subsonic flows. So, therefore, the method of characteristics of course for supersonic flows is widely used because first of all, it is simple and secondly, supersonic portion is of importance. We have discussed that in a previous class that the subsonic portion, the converging portion or subsonic portion is not that important because since we are favourable pressure gradient, we can take any simple shape and that will give us a proper flow. Supersonic portion is the important part where we need to have the design.

So, method of characteristic then will give us initial shape. At present, the practice is the initial shaped we get from method of characteristics. Then, we go to this to do the full analysis and check the validity of the initial design. That is a much improved approach and gives us much better solution. So, now, we will focus on this method of characteristics. Method of characteristics depend on the fact that in supersonic flow, the influence of a small pressure difference is limited in a specific region.

So, if you have a small pressure perturbation, it does not influence the entire flow field. It is limited to a particular region and the method of characteristic is built upon that. So, let us understand what I mean. Let us consider that I have a small source, a pressure disturbance sitting here and we have a flow coming. This is a supersonic flow at a velocity u, this is source of pressure disturbance. It can be a loud speaker; it can be a small pulse generator or something. Now, the flow is coming at supersonic speed. What happens is that we know that the disturbance propagates sound, particularly the compression waves are, particularly the pressure waves propagates through the air or the medium through sound waves, right.

So, initially if the flow is subsonic, then let me put it this way. First, let us look at a subsonic. First day is, let us look as there is no flow. We have a source here. When this source propagates, it will propagate uniformly all around the disturbance, right. So, we get a spherical disturbance going all around. It is like a spherical disturbance going all around. This is unbiased stationary flow, the stationary condition. If we give a flow here that is a biased flow, biased condition, this will change this pattern. So, now, what happens is that more will be propagating on this side; less will be propagating on this side. So, there is change in period may not be look like this. There is a change in pattern that is called biasing.

As we keep on increasing this speed, still the flow is subsonic. Some information will always propagate. There will be some effect of bias, but it will always propagate upstream. Also, we keep on increasing the flow speed when it becomes sonic. Now, this information will just be able to real because what happens is that is trying to go this side and the flow is coming on this side with the same speed. It will stop the information from propagating when it becomes supersonic. Then, it will not propagate upstream at all. It will go only in the downstream region.

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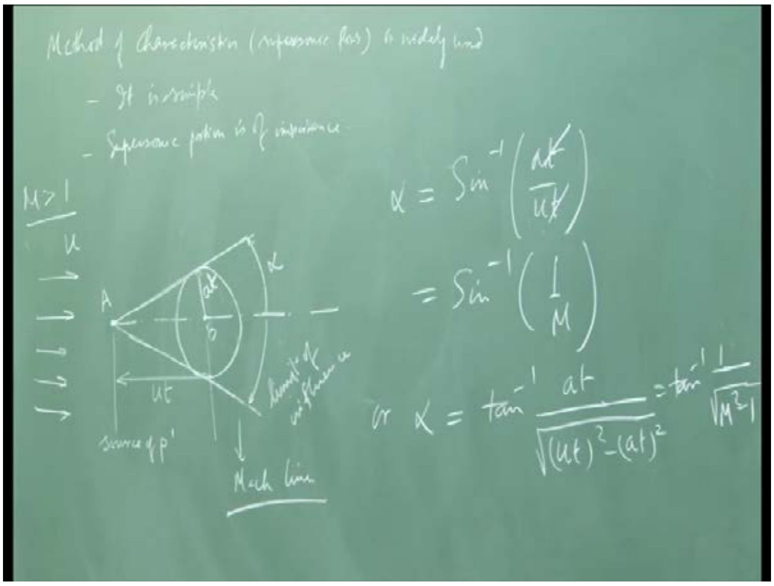

So, then if I draw the schematic of propagation of this information, it will look like this, like a cone and this is how the information is going to propagate. We call point b, this is point a. So, at point a, we have a source of disturbance. This disturbance propagates with like a spherical wave at a particular which is the speed of sound, right. Now, the centre of the spherical wave is moving downstream with a velocity u. This is a centre of the spherical wave; this is moving downstream with a velocity u. So, at time t it has moved from this location to this location and the distance between these two will be equal to u t because the flow is moving this source away.

Now, at any time if I look from time t equal to 0, it was here and time t equal to t plus 0 plus delta t, it was here. Slowly it is moving away and at every time, then depending on the time t, it is moving in a certain location and is going also. So, the influence of this that is if I put a microphone here, it will not be here. Only if the microphone is put here, it will hear this, right. So, if the domain of influence is only limited to this pole, then this cone is given by this angle alpha, right. So, q alpha, this is the limit of influence. So, the zone of influence is limited to a cone of half angle alpha here.

Now, first of all how do we get this half angle? At time t, the disturbance is emitting from here is propagating with this speed of sound a, right. At time t, this source has reached this point. This distance is u t. The disturbance that emitted from here will reach this point. The distance will be a times t, where a speed of sound, so this is a t. So, the radius of this sphere is a t. So, at every location that is how we get this radius. So, therefore, now this angle alpha I have not drawn it properly. It will be something like this. So, angle alpha will be sine inverse a t by u t, right. This is by angle alpha. This is a t by u t. Actually this will be, yeah h e by h, ok.

So, I can cancel this thing. What is u by a? This mass number, right. So, therefore, this is equal to sine inverse 1 by M, right or alpha equal to tan inverse a t upon square root of u t square minus a t square. This can also be written as tan inverse 1 upon n square minus 1. So, alpha is equal to tan inverse M square minus 1. Remember that I said at the beginning that is applicable only to supersonic flow and that is why it is coming from here. If mach number is less than 1, this becomes imaginary. You cannot define alpha, right.

So, therefore, alpha is defined only if mach number is greater than 1. Even at mach number equal to 1, this is 0 is infinite, right. So, that is something that is the limiting case. So, alpha is now the domain of influence or at the mach alpha is called the mach angle, and this is the domain of influence for this sound source. So, this is how now the measure of characteristics I will come to. First, I have defined the mach angle alpha. Now, I will come to the method of characteristics. Let us consider now and these lines by the way are called mach lines. So, the limit of influence is essentially bounded by the mach lines. So, what do we see here? For this point here, what will be the angle of the mach lines given by alpha? It depends on the incoming flow mach number, right.

So, essentially how much the mach lines will in turn depend on the incoming flow mach number? So, that is what the mach line is. So, if I look at these two mach lines, this is the upstream of this mach line. If the flow upstream of a mach line is uniform like in this case, then what we can seen here that the mach line is going to be straight line that is what in this case is, but if this flow is not uniform, then at different locations, we have different mach numbers. The mach line will be curved. So, another point here is that on this domain of influence everywhere, the flow properties are uniform, right.

So, within the limit of influence, the flow properties are uniform. Here, the flow is uniform coming in. Therefore, the mach line is straight. So, if we have a straight mach line, the flow properties are uniform downstream of the mach line. So, these are the few things that we actually use when we employ a measure of characteristics that if we have uniform flow, the mach lines are straight lines and within the limit of influence, the flow properties are uniform. These are the things that we are going to use. Now, let us look at a nozzle and see that how we use this information in the case of a nozzle.

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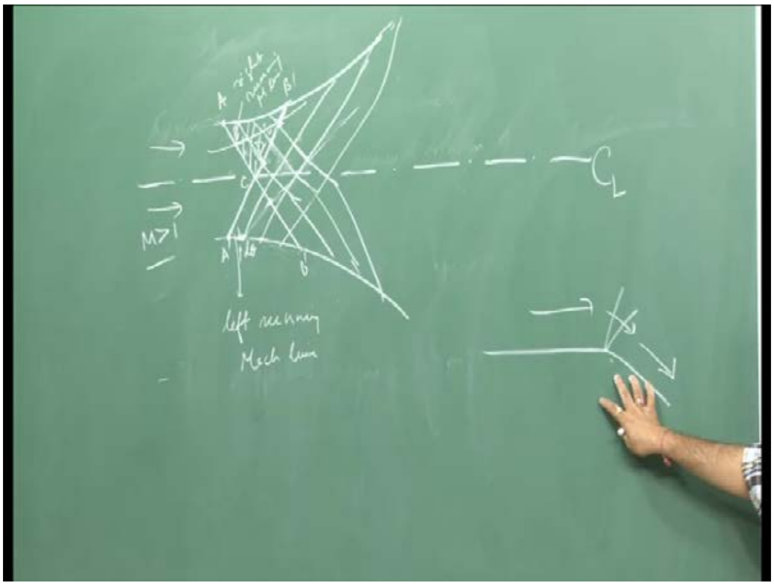

So, let me now go to a proper nozzle, proper shaped nozzle. Let me consider a shaped nozzle like this. I am not still going to the full nozzle. I am just going, just showing a part of it. This is our central line. We have let us say supersonic flow coming here into this nozzle. Now, we have a point here. From this point here, we have uniform flow. So, from this point, a mach line will emerge right go like this from since we are talking about a symmetric nozzle. Just opposite side, there is another mach line, another point a dash from where another mach line will emerge and go like this.

So, if nothing happens in between here and here, these mach lines will be straight and travel like this, but now if this was a straight section, this is what is going to happen, but here it is curved, right. As it is curved, if I look at a flow, supersonic flow over a curvature, what happens is the flow comes here because of the curvature. There is an expansion fan about which it turns and then, goes like this. So, it accelerates, right. So, the mach number here is greater than the mach number here. The flow is accelerated. So, if I now come to this point, at this point the mach number is more than at this point.

So, now another mach line will emit from here, right and for that mach number, this is higher. This mach number is higher. So, what happens to alpha? So, this is reducing. So, alpha is increasing, right because sine alpha is 0, sine sine 0, 0 is 0, sine 90 is 1. So, alpha is mach number is increasing. Alpha is reducing, sorry because sine inverse alpha is reducing, so alpha is reducing. So, alpha decreases which mean that something that was at this angle, now may be at this angle, salacious angle. So, if I drop another mach line from here, it will go like this, slight decrease in mach angle.

Now, if I look at these two, this mach line was straight with sudden property, uniform property. This is straight with some other property, but these two mach numbers are different. So, when they intersect, they will bring in a change, right which will be dependent on the mach number here and the mach number here. So, now, this mach lines will start to curve, right. It is no longer straight line because they are known uniform properties merging. So, the mach lines will start to change. Let me draw it on the one side only c d a dash. Similarly, from this side. So, we get pattern like this. We get a pattern like this, like this, we get a pattern. Now, as you can see this looks like a grid pattern where the mach lines are now intersecting at different points. So, we got a point d here, a point e here like that at different points, this mach lines are intersecting, ok.

Now, in between these two mach lines, the flow is still uniform, but now since this and this mach line are different, this at two different mach lines. So, the uniform flow will neither be this nor be this. It will be different condition. Although it is uniform, but it will be different condition and that is how the flow will propagate. When it goes along, there is going to be change in mach number. So, now, this mach number change will depend on this curvature and the incoming flow mach number. So, the initial curvature that is provided, that will dictate what kind of flow will be emerging.

So, let me just summarize. Suppose we have a uniform flow which enters the diverging portion of the nozzle here, the wall curvature initially like we have shown here will establish pressure gradients that are going to turn the stream lines. So, the stream lines are going to be turned because the fluid will be flowing along the wall like we have shown here. The stream lines are turned like this. Since, we are neglecting friction in this case, the fluid can be assumed to be sliding freely,.

So, the flow is like in this case, in the expansion fan which is flowing, sliding smoothly over the surface. So, at this point, we have a small variation in angle d theta. Let us say a small variation in angle d theta which causes a small pressure disturbance d p to be produced and now, this disturbance will propagate along this line from here to here and at what angle, it will more depend on the incoming flow mach number. So, it creates a mach line there. Similar disturbance will be propagating from this point, right. So, this small curvature here now is our source of disturbance. The flow was coming smoothly here. There is a small curvature that creates a disturbance.

So, that is the initial point which creates the initial mach lines. So, these are the two mach lines that are created. Now, we have a continuous curvature at this point beyond this. So, because of this curvature from every point, there will be mach line that will be coming up, right and by the way if I look at the mach lines here, the mach lines which are emitting from here are moving in this direction. The mach lines which are emitting from here are moving in this direction. The mach lines emitting from the lower duct valve are called left running mach lines. So, these are called left running mach lines, and emitting from upper valve are called right running mach lines. This is the nomenclature that is used.

So, we have left running mach lines and right running mach lines, and they are crossing each other in a zigzag manner as we have drawn here. Now, this designation of left running and right running refer to the direction in which the lines approximate to propagate, approximately propagates or appears to propagate downstream to an observer looking downstream. So, if I look, if say if we have an observer sitting here. Observer is standing let us say at this point and looking downstream, and that observer looks at this line. So, this line with respect to that observer is left; this line with respect to that observer is right. That is why these are called left running and these are called right running. So, observer is sitting here and looking downstream. That is why this nomenclature is used.

Now, if I look at this region a, b, c or this region a prime, b prime, c prime etcetera. If I look at this region, at that region where if I look at this region, this only one type of mach line here, right. Of course, this is bounded by the other type. Similarly, on this region, there is only one type of mach line bounded by other type here. So, it is only same type of mach line present in both of this, where as we have already discussed that if we have single type of mach line, then the fluid properties are going to be constant. Just upstream or downstream of each line.

So, if I look at this region, the fluid properties are going to be constants. Similarly, in this region, the fluid properties are constant. Of course, it is going to change from here to here because the two mach lines that are present, but within a particular region, the fluid properties are going to be constant, same type of mach number. So, specifying the properties along any stream line within this, the stream lines are moving like this. Let us say choose this stream line. We specify the property along any stream line since specify in that entire region, right. So, specifying the properties along any stream line will be sufficient to describe everything in that region. Now, what is our limiting? Stream line a wall, right.

So, if we can specify the property along the wall, this is our limiting stream line. Then, the properties all along this as specified within this zone, of course within this zone and this zone. So, specifying the wall properties will specify the properties in between everywhere. Now, let us see that what is happening in this region more closely. This region let us say that this point there is a curvature of d theta. The flow is turning by amount d theta which is given here. Let us see that how do we get the property variations. Now, when the flow is turning by this amount d theta, so for that I will draw this diagram, bigger diagram to represent the flow.

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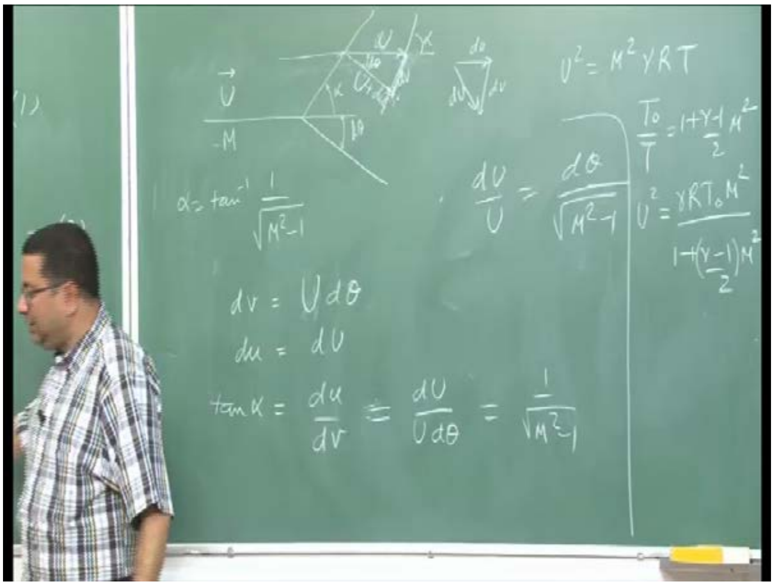

Let us consider this that we have a velocity coming in this direction u, and the flow is turning by a small angle d theta. Because of that there is a mach line that is emitting. So, upstream mach number here is M. This is the upstream mach number M, the disturbance propagate at an angle alpha. So, this is my angle alpha, this is a mach angle that we have already discussed, and we can estimate alpha as tan inverse 1 upon M square minus 1. Now, any fluid crossing this mach line will have a change in its direction, right. So, the stream line, any stream line crossing the mach line must have its direction change, so that it continues to be parallel to this wall, right. So, there is only change in the direction.

So, let me consider this. This is the stream line. It is coming here and its direction get's changed. So, now, it moves in a manner parallel to this, right. So, this is what is the changed direction stream line is going to move and now, just draw an analogy. This is like a Pendal Mayer Flow expansion fan. So, we know that when the supersonic flow changes direction because of the increase in angle, it accelerates. So, its velocity is going to increase. So, let say the velocity now is u plus d u, where as this flow velocity was u, this is the same flow I am just drawing it on this side. Velocity is u, then for this I can have a, there is a change in direction now to flow and this angle is alpha.

Let me draw it properly. This is better. So, this is u, this is u plus d u. This term is u which is the increasing actual velocity u plus d u and d v velocity is now changing. We have a velocity in the v direction as well this angle is d theta. So, if I draw this triangle just for the increased d u d v, this is the overall increase. This is the vector diagram only for the changed part. So, the actual velocity increases by the amount d u, there is a d v time. So, the total velocity increase is d capital d u.

So, there is if we look at this, there is no change in momentum component parallel to mach line. Parallel to the mach line, it does not change. So, the change in velocity is particularly let me list this diagram as yeah. So, the change in velocity is primarily in the v direction, not in the u direction. From this diagram, we can get d v is u d theta. This is my u, right. This is u d theta, d v is u d theta and d u from here is equal to this is the change in the x component.

Let me write a different notation capital u for the actual velocity, otherwise because I am representing the change also with this small u. So, I can write a different notation here. So, d v is equal to capital u d theta and d u is equal to d u which is the change in velocity. Therefore, from there we can get tan alpha is equal to d u by d v, right and then is equal to. Therefore, now from here what do we see tan alpha in place of d u, I write this here. I write u d theta. So, this is equal to d u by u d theta, right and tan alpha I know is equal to this value. So, tan alpha is equal to 1 upon square root of M square minus 1, right.

So, this is equal to 1 upon square root of M square minus 1. So, I can get now an expression for d theta, that is the change in angle which is equal to square root of rather let me write it like this. Therefore, d u by u equal to d theta upon 1 square root of M square minus 1. So, this is the change in velocity. Because of this change in angle or else, we can see the change in velocity is function of the change in angle as well as the incoming flow mach number. So, based on that, we get this expression. Now, from the definition of mach number, u square is equal to M square gamma r t, right and from isentropic flow assumption, this is actually applicable to any adiabatic flow we called the isentropic. We get the isentropic relationship in word by t given like this. I can get t coming here.

So, let me first write it here. This gives me u square equal to gamma r t naught M square divided by 1 plus gamma minus 1 by 2 M square. So, the velocity now can be written as a function of t naught which is the stagnation temperature mach number. That is more important. This can be written in terms of mach number. So, we have represented velocity in terms of the mach number. Now, what we can do is, we can differentiate this.

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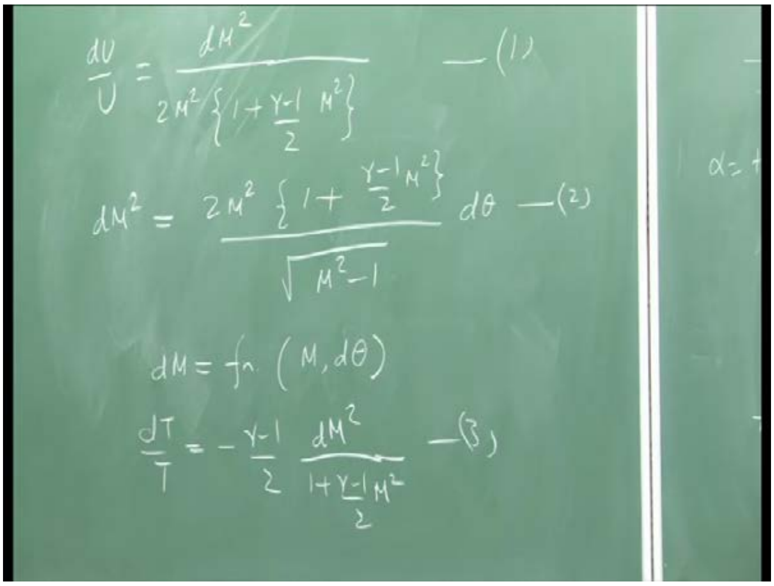

So, if we differentiate this, differentiating that expression, we get d u by u is equal to d M square upon 2 M square 1 plus gamma minus 1 by 2 M square. Let me call this equation 1. What actually what we are trying to do is to change, get an expression for change in mach number. When there is a curvature, what we got is a change in velocity. We want to represent it as change in mach number. So, we get this expression from this then. Now, if I combine this, this and that expression if I combine, then I will get d M square is equal to 2 M square 1 plus gamma minus 1 by 2 M square by M square minus 1 d theta. Let me call this equation 2.

So, let me again see what we have done. We have derived this expression here which is getting the velocity change to the angle change and of course, it is a function of mach number. The velocity we have expressed in terms of the mach number, then we differentiated that and got an expression for d u by u in terms of mach number, and here we have an expression for d u by u in terms theta and mach number. When we combine these two, we get this expression. So, what is this expression representing the change in mach number as a function of incoming flow mach number and change in angle theta d theta, right.

So, this is what we wanted to get because we have the curvature. We want to know when there is a curvature, how much change in mach number we can expect or we can estimate from that from this equation? So, the change in mach number here relate to a change in direction of stream line for isentropic flow because d theta is the change in direction of stream line, and we are considering isentropic flow completely here. This is something that you have to keep in mind that all this derivation isentropic flow is inherent. So, what we are doing now is completely for isentropic flow, it is applicable only to isentropic flow, ok.

So, therefore, because this was isentropic, this description is isentropic. So, we are doing it for an isentropic flow. So, this is the variation in mach number because of change in stream line which is brought about by changing this wall angle. So, now, once we have this mach number change, we can estimate the change in other properties. Also, d t by t will be equal to just differentiating the isentropic relationships, we can get this. Let me call this equation 3. This is a change in temperature. So, once again differentiating that isentropic relationship, we get this.

Since, we are talking about an isentropic flow without any work done, there is no work done here an isentropic flow. Therefore, stagnation temperature remains constant. So, t naught is constant in this case. Therefore, when we differentiate this t naught goes away. Similarly, stagnation pressure is constant.

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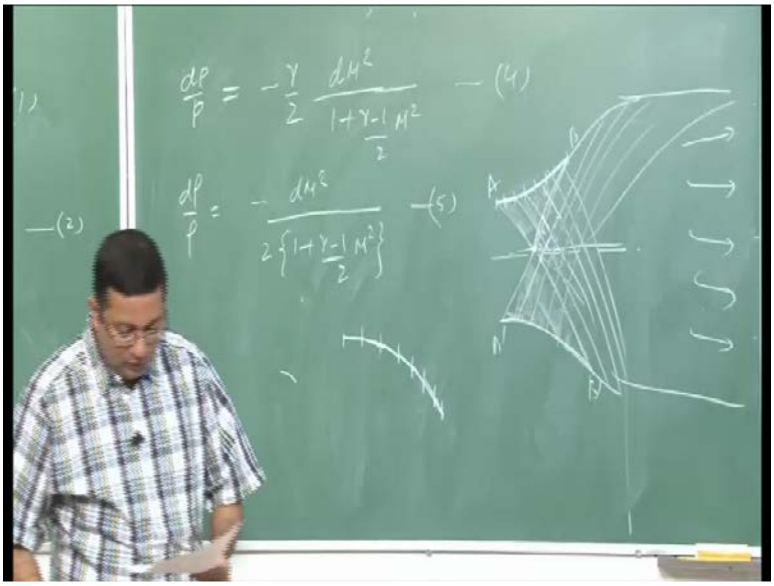

So, we can take the isentropic relationship for pressure, differentiate it, we can get the relationship for change in pressure, so that will be given by change in pressure relationship will be given as d t by t is equal to minus gamma by 2 t M square 1 plus gamma minus 1 by 2 M square. Let me call this equation 4. Similarly, we can get the change in density d rho by rho is minus d M square upon 2, 1 plus gamma minus 1 by 2 M square. Let me call this equation 5.

So, we have 5 equations listed here 1, 2, 3, 4, 5. All these equations actually represent the change in flow properties when the angle theta or the wall curvature is changed for a supersonic flow with a mach number M, right. We are now getting a change in velocity, change in mach number, change in temperature, pressure and density and this is what we wanted to estimate how the properties change when we have the variation in angle. So, now, the wall curvature then is giving us all the required changes that we are looking for.

So, the wall cum curvature now can be replaced here. We have in this case, what we have done is you have considered a small curvature, right. So, using this small curvature, we calculate these changes, right. So, then what we can do is, this is our finite curvature. We can break it into many small curvatures, right. So, for this we know how much d theta is. We guess get all this, then come to this. When we come to this point, now this mach number has come from that solution and then, we have another change in theta. So, another change will occur, we can get the conditions here.

So, we move downstream, we keep on changing the wall curvature and we keep on changing, getting the new flow properties because of change in wall curvature till we come to this point at the end. So, the wall curvature can be replaced by finite number of straight line segments, right. So, this is the curved wall. What we have done is, we have removed, replaced it by many straight line segments and thus, the flow properties in the flow field corresponding to each infinite small turned can be calculated now.

So, if we break this curvature into many small straight line curvatures, we can calculate the entire property variation here during this curved portion now. So, what we have done so far once again go back to our original try. We are now focusing on this portion only. This is the straight line c a c b a dash b dash. Remember what I said is in this portion either in this region or in this region; there is only one type of mach line. On this side, we have all right running mach lines here; we have all left running mach lines. So, therefore, the property variation essentially if I get the variation in the wall, that is good enough. So, from here to here, then knowing the wall curvature here, I can get all the property variations and that is what we are doing. So, what you are doing is, from here to here, we are breaking this wall into many small segments and calculating the variation here and since, for each of them we have the same property, nothing changes. So, we can get the full property variation within this zone.

So, this is the first part of our problem. Our problem actually has multiple parts. First part is when we have only single type of mach lines that we have now explained how to get the property variation. One thing for this portion we need to know theta variation, right. So, this curvature for this portion must be specified. So, for this portion, it is not designed. It's initial curvature is specified. So, we get all this from there. So, this theta variation is specified on this wall, we get all this.

Now, coming back to this. Now, when we come to this zone, so here like this. So, we have seen here, this portion and this portion we have only single type of mach lines, when we come to this portion. Now, we see that the mach lines are crossing left running, and right running are crossing, right and not only that, if I look at this domain, it has come from here and here, but if I look at this one, it has come from some other location, right. So, therefore, this mach line up to this is when it crosses this, it has also crossed this mach line. So, the properties have changed here.

So, at this point, there is a different property. When it comes here, again is a different property where as this has seen only one change, right. So, there is a continuous change in the properties. So, now, we have mach lines crossing each other or the flow crossing each other with different properties and now, that makes it little more complex. So, now we have is intersection of two mach lines. First of all, it can be an intersection of one left running essentially always a left running and right running mach line will intersect, but the point is that these two mach lines may not have emitted from the same disturbance. So, therefore, the mach number and the flow properties corresponding to these two mach lines may not be same, they can be same also.

For example, if I look at this point, this mach line and this mach line are same. So, when we come to this domain, it has the same property, but this and this are going across. They may not be coming from the same mach line or emitted from the same property. In that case, one mach line as say emitted from a flow with mach M 1. Other has emitted from a flow with mach M 2 and then, the mach angles are going to be different for these two.

So, that needs to be accounted for. Then, the next thing what we are going to see is that when the two mach line intersects, how do the properties change and this is one part.

Finally, what is our goal is to make the flow straight. Our design is not complete. So, next thing what we will do is, we will look at how the mach lines intersect, what will be the property variation and then, so far here the wall itself was enough to give us the conditions. Now, we will design the wall curvature that what kind of d theta should be given here, d theta was specified. Now, we will change the theta in such a way that we get finally a uniform flow at the exit. Let us say this is our nozzle. We will get a flow which is uniform like this. This is what we want to do for that. First, we have to understand how the properties will change when the mach lines intersect and the next is to complete the design.

So, I think we have spent all, most used up the time today. So, what I will do is, I will stop here today. In the next class, we first talk about the intersection of mach lines and then, continue with the design process using method of characteristics and finish this discussion. After that we talk little bit about plug nozzles which are other type of shaped nozzles and then, the effect of friction, will talk about the effect of heat transfer. We will talk little bit there will be just small descriptions. Therefore, main focus in the next class will be completing this measure of characteristics, particularly when the mach lines cross each other what happens. So, I will stop here now. In the next class, we will continue from here.

Thank you.