Aircraft Dynamic Stability & Design of Stability Augmentation System

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Module 7
Lecture No 40
Numericals: Transfer Functions and Response.

Hello friends. So far in this lecture, we have seen about spring mass damper system, equation of motion, longitudinal mode, Phugoid mode, lateral directional mode, different modes- spiral, roll, Dutch roll mode and introduction to SAS, stability augmentation system. Today we will be focusing on transfer function. As you know, transfer function is given by the Laplace transform of input and assuming that all the initial conditions are 0.

Or you can say that the system was in equilibrium. We will be seeing what will be the response of a system if it is exposed to different it would signals. We will be dealing in this course with major 2 signals, that is impulse and a unit step signal.

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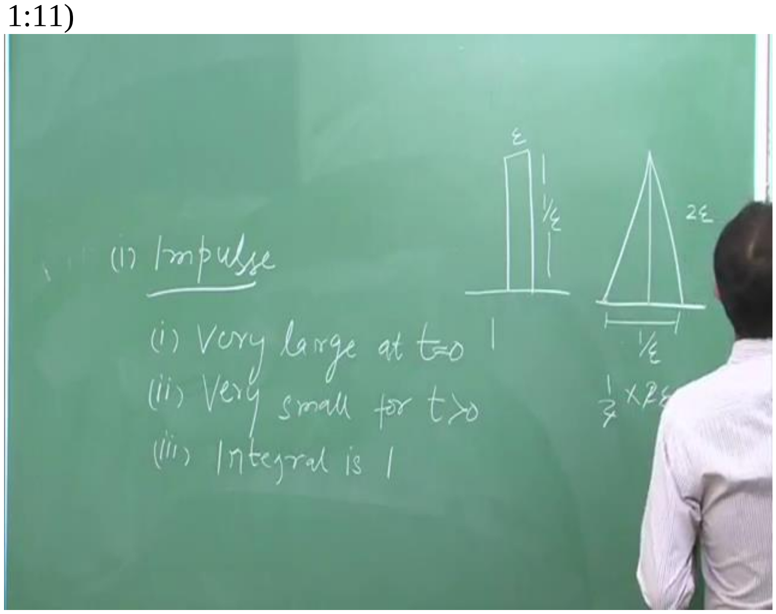

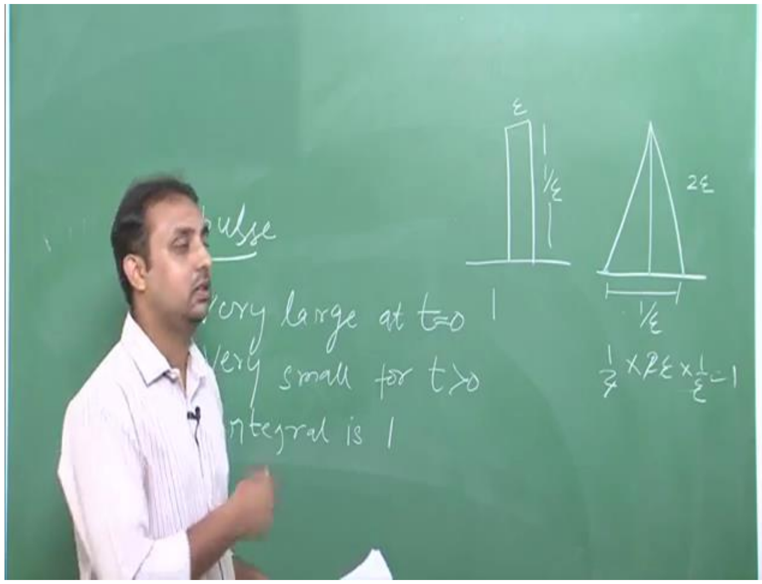

Now impulse signal. Mathematically, it is defined as it is very large at T equal to 0. 2nd, it is very small for T greater than 0. And 3rd, integral is 1. These are the basic properties of an impulse signal. Now the shape of the signal does not matter. It can be either a rectangular shape or triangular shape but it must satisfy these following conditions.

Now practically, a signal with a very large magnitude, it is practically impossible. So we are using that a signal for a short duration which is having an area of unit or you can integral is 1. So assuming this is a step signal, your duration will be Zeta and its amplitude is 1 by Zeta. So overall area is 1. Or in case of the signal, my amplitude will be 2 Zeta and my duration will be 1 by Zeta.

So area is half into 2 Zeta into 1 by Zeta which will be 1. We have seen about impulse signal. Now we will be seeing about unit step signal. Now where this impulse signal is, physically you can see this impulse signal during some collision or when an object hits for example when you hit a nail with a hammer, the force is known as impulse. Even for Spring Mass system, if you apply a force for a very short duration, the response of that system will be due to impulse. That response is called as impulse response.

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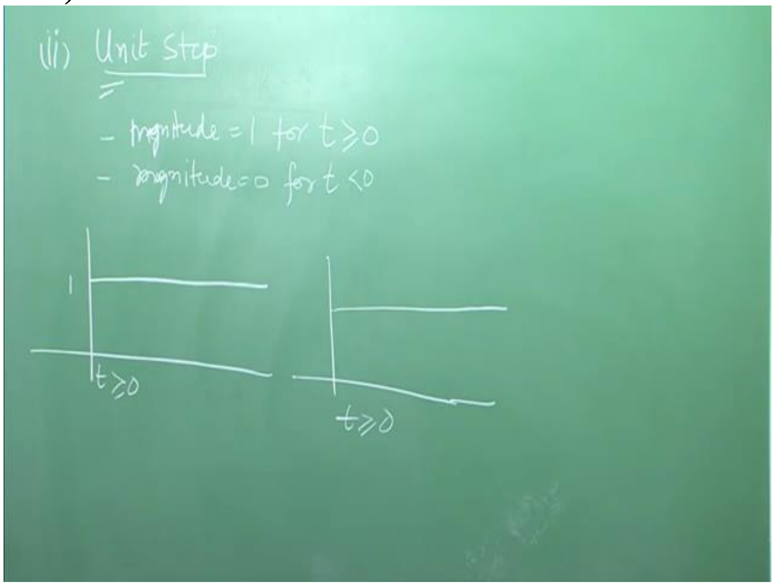

Now the 2nd signal is unit step signal. As the name suggests, unit step signal means it will have magnitude equal to 1 for T greater than equal to 0. And magnitude is 0 for T less than 0. That means, a signal exists for time which is greater than 0. That is, if I draw this signal, it will be a constant magnitude signal for T greater than 0. And it will be same magnitude at T equal to 0 also.

Since unit is mentioned in this unit step signal, so my magnitude will be 1. If unit step or in general if you call it a step signal, the general definition is, it will have a constant magnitude for T greater than equal to 0. Now we will be seeing how the system responds when these input signals are given to its transfer function. So best will be, we will be taking a numerical and that will explain how the system will respond when it is at (inaudible 5:39) unit step signal.

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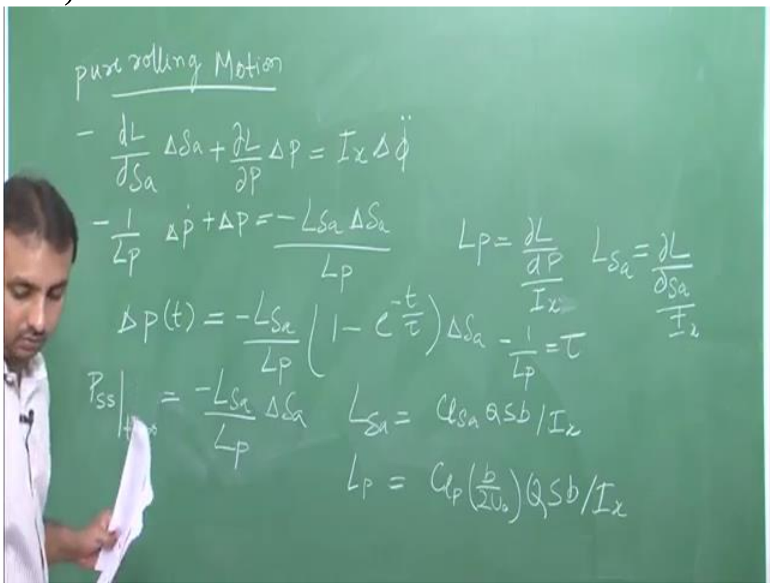

So let us consider the case of pure rolling motion. The differential equation given for pure rolling motion is - DL by D Delta A into Delta A + DL upon DP into Delta of P equals to IX Delta of I double dot. Now further solving this differential equation, we will get - of 1 by LP into Delta P dot + Delta P equals to - L Delta A divided by LP. Now these equations have already been solved in previous lectures. I am not solving step-by-step solution of these equations.

You will get these final equations substituting the suitable values in the 2 differential equations. Now where this LP is defined as DL by DP upon IX L Delta A is defined as DL by D Delta A upon IX. And we will be taking 1 by LP - Omega LP is Tao. Substituting this value in this differential equation, we will get finally Delta P in time domain equals to - L S Delta A upon LP 1 - E to the power - T by Tao into Delta of Delta A.

So, solving for P at, solving for steady-state of roll rate, we will get P steady-state at T equals to infinity equals to - L Delta A upon LP into Delta of Delta A. As you know, steady-state case, IT equals to infinity, what will be the value? That is known as steady-state. Now since we know, we can write L Delta A as CL Delta A QSB by IX. And LP as CLP B by 2U not QSB divided by IX.

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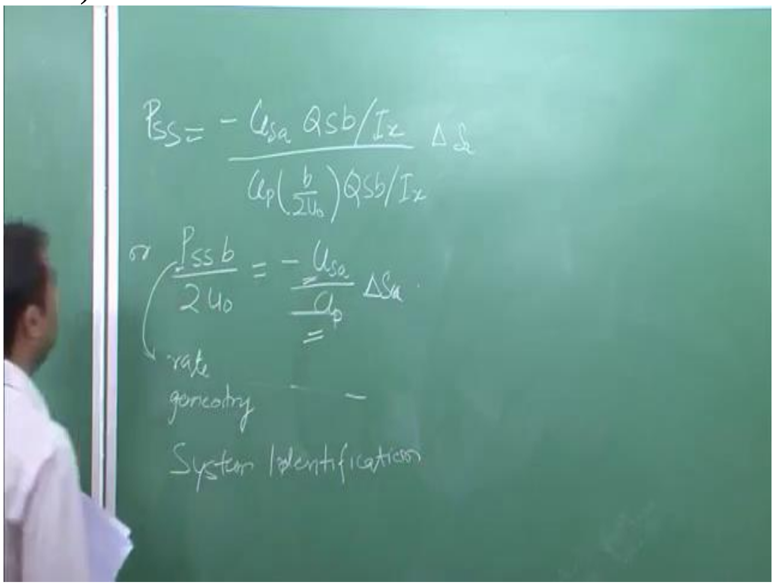

Now substituting these 2 values in this equation we will get, BSS equals to - CL Delta A QSB by IX. CLP by 2U not QSB by IX into Delta of Delta A. Or PSSB by 2U not equals to - CL Delta A by CLP into Delta of Delta A. For this differential equation, we have seen, the steady-state given by this particular equation. Now here we had to find the steady-state for roll rate.

Now suppose we measure the value of this roll rate from instrumentation for an aircraft which can be easily measured using rate gyro. Suppose that if we measure these values, so using the measured values, if we know the geometric dimensions of my aircraft, I can easily get the value of CL Delta A and CL Delta P. So this is a very good method of finding the derivatives of an aircraft using the measured value of steady-state.

This technique of deriving the aerodynamic derivatives using the measured values is known as identification system. Based on the previous derivation, let us see a numerical, based on that steady-state value derived.

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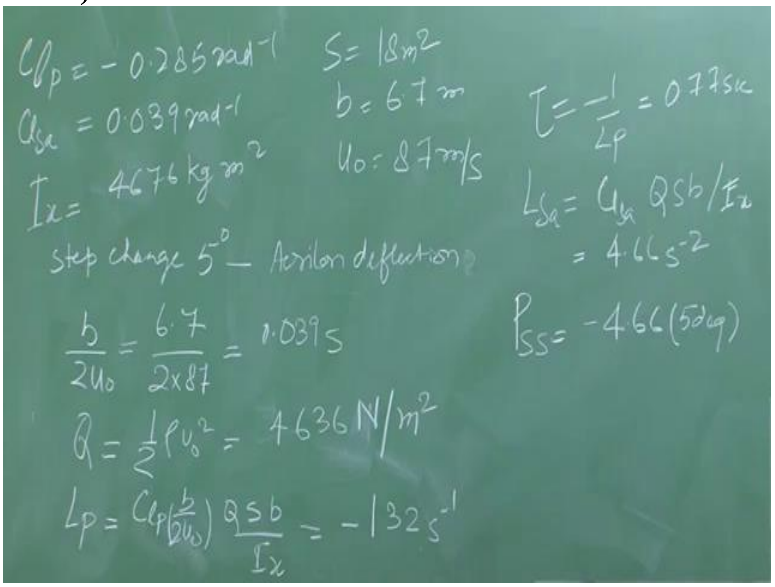

Suppose that for an aircraft, CLP is given by - 0.285 rad inverse. CL Delta A equals 0.039 rad inverse. Area is given as 18 m^2. Value of B is given as 6.7 m. Inertia is given about X axis as 4676 kg metre square, step of change of 5 degree, that aileron deflection of 5 degree. So using these values, B by 2 U not, sorry U not is given as 87 metre per second. B by 2U not is given by 6.7 by 2 into 87 equals to 0.039 seconds.

Then pressure Q equals to half a Rho V not square equals to 4636 newton per metre square. LP will be given as CLP B by 2 U not QSB upon IX which will be put to - 1.32 second inverse. Tao, as we know it was - 1 by LP, so Tao will be equal to 0.77 seconds. L Delta A will be called to CL Delta A QSB upon IX which is equal to 4.66 second inverse.

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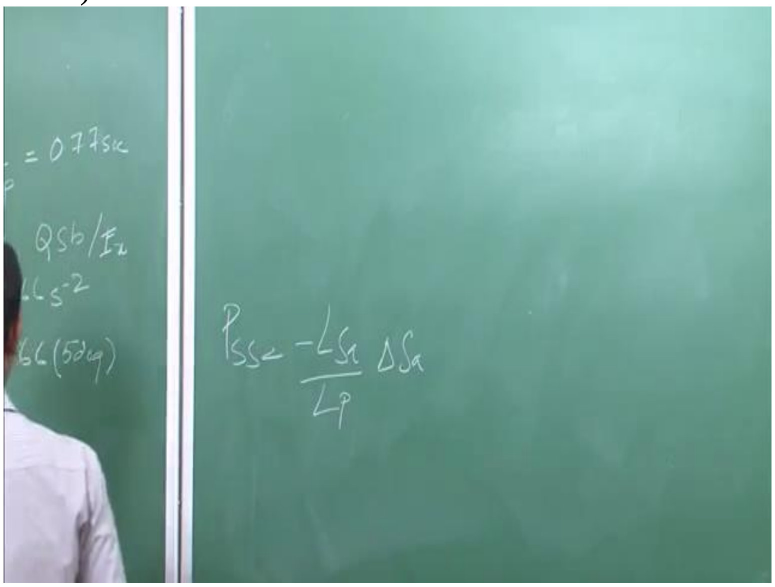

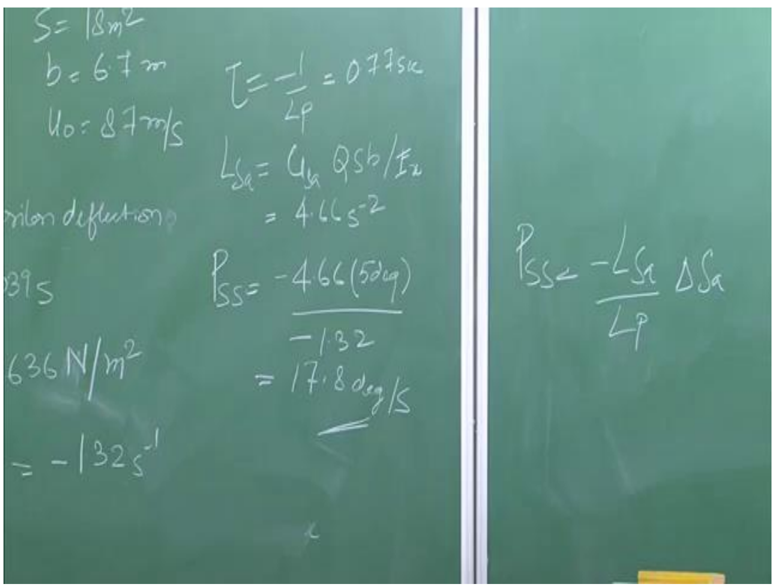

So steady-state value of roll rate will be - 4.66 into 5 degree since the equation for steady-state value of roll rate was given by PSS equals to - of L Delta A by LP into Delta of Delta A. So, steady-state roll, value of roll rate at steady-state will be - 4.66 into 5 divided by - 1.32 which will give me 17.8 degree per second. So this is my roll rate, value of roll rate at steady state.

As I already told you, instant of calculating steady-state value of roll rate, if we measure the value of this roll rate from rate gyro, we will be easily able to derive the values of your aerodynamic coefficient, CLP, CL Delta A sense these geometric values are easy to determine for an aircraft. Let us take another problem for a second-degree equation or 2nd order derivative.

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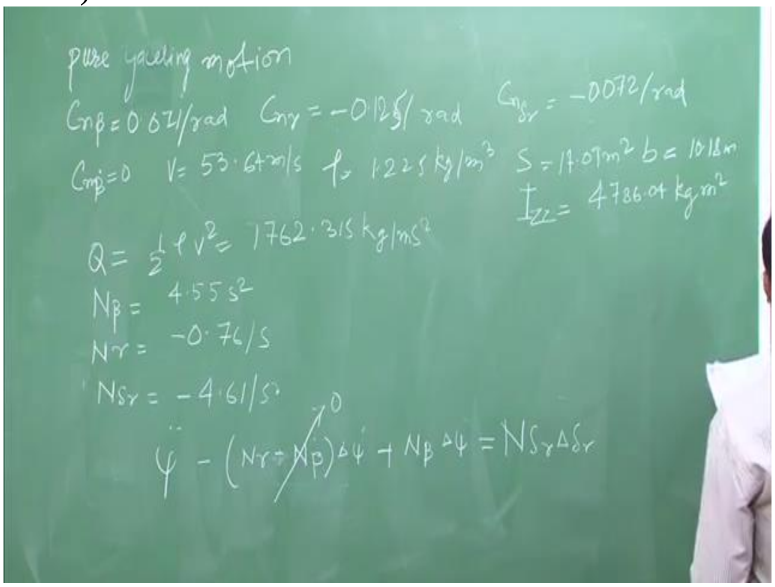

Suppose for an aircraft, the following details are given as CN beta is equal to 0.0715 per rad, CNR is equal to - 0.125 per rad, CN Delta R is equal to - 0.0752 per rad, CN beta dot is taken to be 0 for velocity of 53.64 m/s, density at sealevel is given by 1.225 kg per m cube, area for aircraft is 17.09 metre square, span is 10.18 m and inertia moment of inertia about Z axis is - 4786.04 kg m square.

Now you can do the calculation. The formula for these derivatives are already been told. And you can find it in any book. So I am just writing the values which will come upon substituting these values in that equation is are, your dynamic pressure will be half Rho V square equals to 1762.315 kg per m second square. And beta will be 4.55 second square and R equal to - 0.76 per second, N Delta R equals to - 4.61 per second square.

Now for pure rolling motion, the differential equation is given by psi double dot - NR - N beta dot into Delta of psi dot + N beta Delta of psi equals to N Delta R into Delta of Delta R. Now since we have already mentioned, CN beta dot is 0, so this term will be 0.

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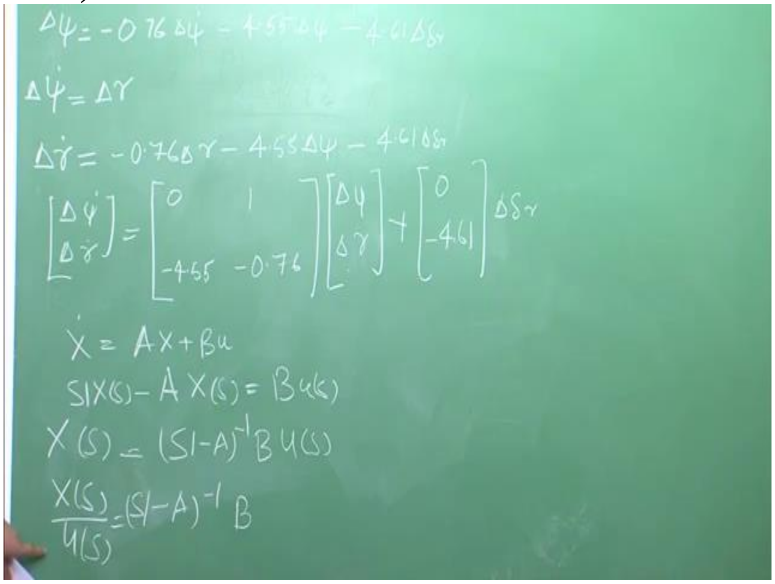

So rewriting this differential equation, we get, I am substituting the values for N beta, NR and N Delta R, I will get differential equation as Delta of psi double dot equals to - 0.76 Delta psi dot - 4.55 Delta psi - 4.61 Delta of Delta R. Now since psi dot, Delta psi dot can be written as Delta R, so we will get Delta dot equals to - 0.76 Delta of R - 4.55 Delta psi - 4.61 Delta of Delta R.

Now as you can see, using these 2 differential equations, we can form a state matrix of this differential equation represented by Delta of psi dot equals to 0, 1, - 4.55, - 0.76 into Delta psi, Delta R + 0, - 4.61 into Delta of Delta R. So this state matrix representation of differential equation is in the standard form of X dot is equal to AX + BU.

Now in such case, if you want to form the transfer function of this state matrix, we can use either Kramer's rule or simply you can use the formula as, this will be S of IX, taking Laplace transform of this state matrix equation, SI - A of X of S equals to BU. So X of S will be SI - A universe into BU of S. So from this transfer function, we will get X of S by U of S equals to SI - A inverse into B.

So you can find transfer function using this formula or using Kramer soon. We will be seen deriving the transfer function using both methods. So 1st using this formula, less derive the transfer function.

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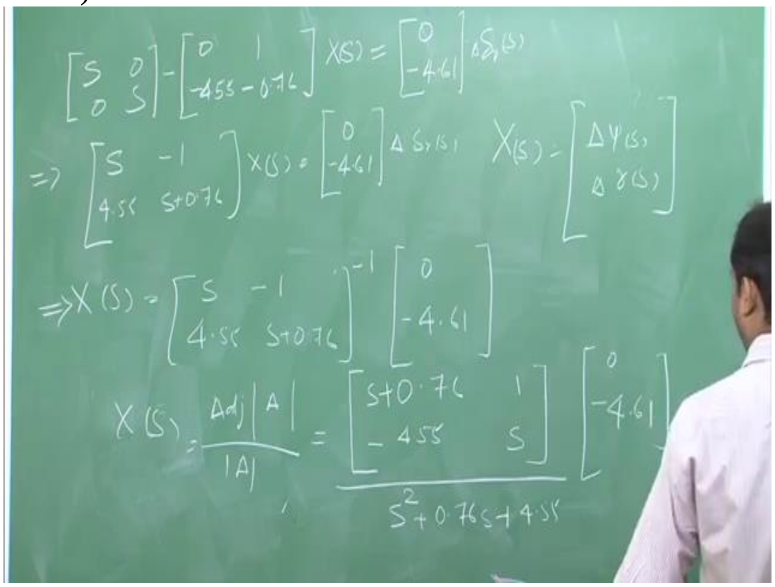

S, 0, 0, S - my matrix A which is 0, 1, - 4.55, - 0.76 2X of S equals to B of U which is 0, - 4.61 into Delta R. Okay? Further solving this, S, - 1, 4.55, S + 0.76 X of S equals to 0, - 4.61 Delta into Delta R S. Here my XS is nothing but matrix involving Delta psi and Delta R.

Now, so my XS will be S, - 1, 4.55, S + 0.76 inverse into 0, - 4.61. Now calculating inverse of 2 by 2 matrix is very simple which will be a joint of this matrix. Our A matrix divided by determinant of this matrix which comes to S + 0.76, 1, - 4.55 and S. And determinant is S square + 0.76S + 0.55. This whole matrix multiplied by 0 and - 4.61 will give me my value of XS.

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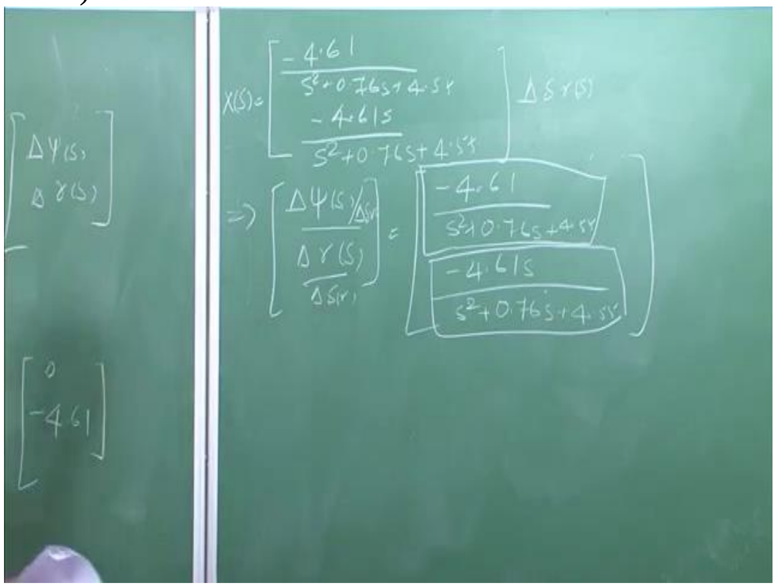

Now this multiplication is very simple. Since 1st element in 0, so we will get X of S as - 4.61 by your value of determinant which is S square + 0.76S + 4.55. And multiplying this Rho by this column, you will get - of 4.61 upon S square + 0.76S + 4.55 into Delta R of S.

So transfer function of this matrix will be, since you know that X of S is nothing but Delta psi of S and Delta R of S of this whole with respect to Delta R of S will be, Delta R of S divided by Delta R of S equals to 1 value of this matrix, - 4.61 by S square + 0.76S + 0.55 and - 4.61 S upon S square + 0.76S + 4.55.

So this is the transfer function. For Delta psi upon Delta R, the transfer function is given by this equation. And for Delta R upon Delta R, rudder deflection is given by this particular equation, set of transfer functions we have derived using your SI - A inverse into B formula. Now let us see using Kramer’s rule.

Why we will be needing Kramer’s rule? Because this is a 2 by 2 matrix, finding inverse is very easy. But if you go for higher order differential equations or matrix which is 4 cross 4 or 5 cross 4, finding inverse is a little bit tough. So in that case, Kramer’s rule is very beneficial.

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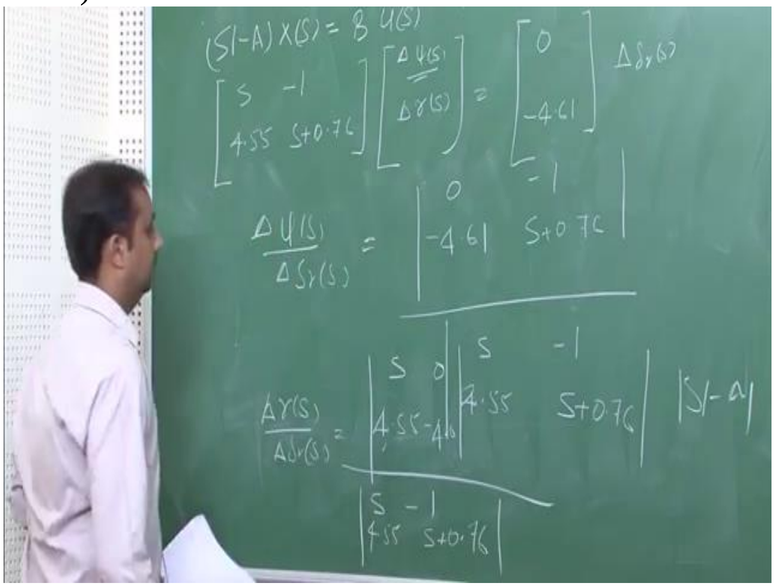

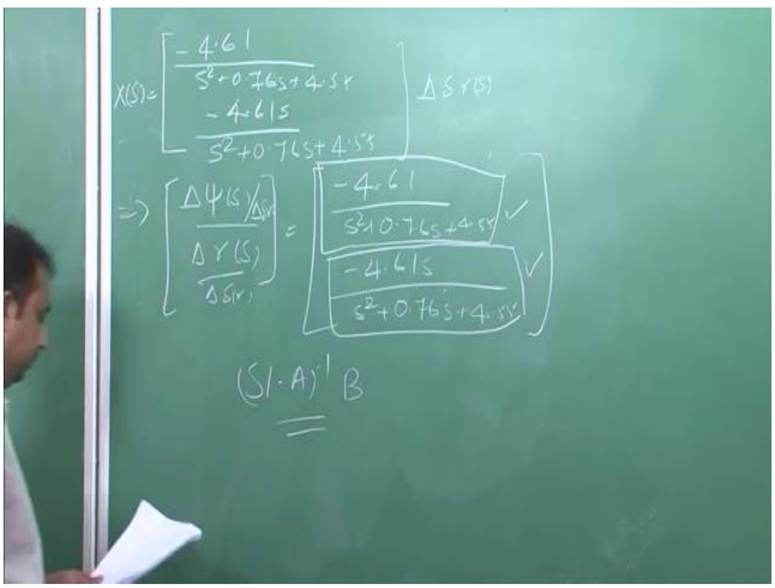

You know that SI - A X of S equal to B of U of S. This U we got from substituting, finding characteristic equation. So now, the value of SI - A was S, - 1, 4.55 and S + 0.76. Value of XS was Delta psi S and Delta R of S. Value of B was 0 and - 4.61. U of S of Delta of Delta R of S.

So while deriving the transfer function using Kramer's rule, Delta psi of S by Delta of Delta R of S will be just substitute the value of B matrix in the 1st column of 12 derived by transfer function of Delta psi of S. So it will be determinant of 0, - 4.61, - 1, S + 0.76 by determinant of S, 4.55, - 1, S + 0.76.

This is determinant of SI - A. And similarly for finding transfer function of Delta R of S upon Delta of Delta R of S, we substitute the value of this column by the value of B matrix. So Delta R of S by Delta of Delta R of S will be determinant of the matrix S, 4.55, 0, - 4.6 divided by determinant of this matrix.

To solve this equation, we eventually get the same transfer function as derived from using the formula, SI - A inverse of B. Okay?

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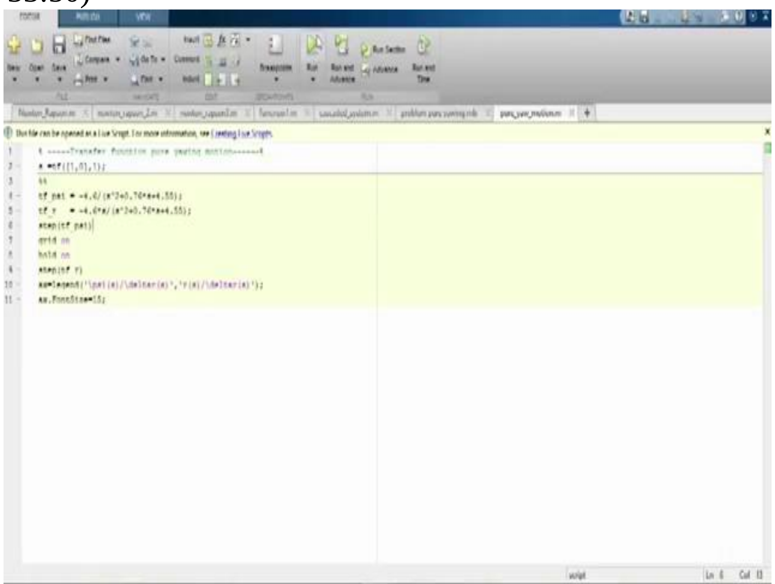

Now the problem related to pure yawing motion. The transfer function for psi S upon Delta RS will be - 4.6 by S square + 0.76S + 4.55. And transfer function for RS upon Delta RS will be - 4.6S upon S square + 0.76S + 4.55. Now when this transfer function input, step input signal will be applied, let me run this command and you can see the output for step input signal.

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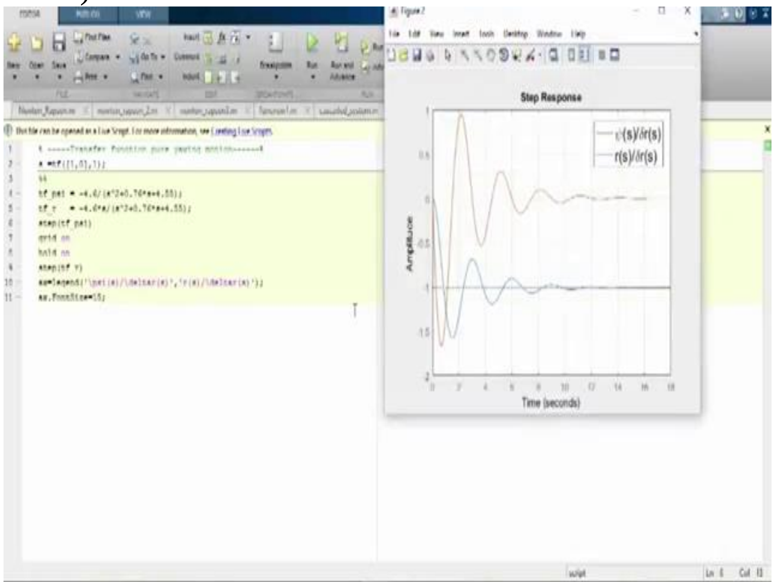

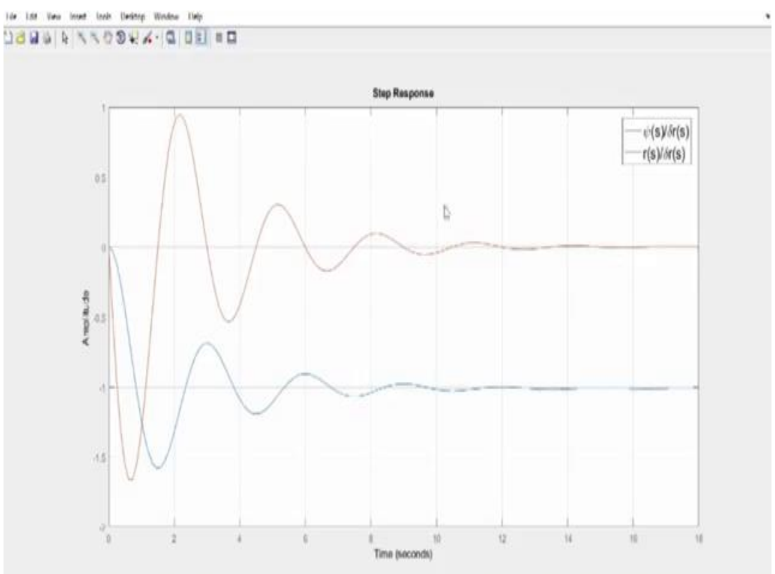

So these are the responses for transfer function for unit step signal. Now on the plot tab, you can see, these steady-state value for RS upon Delta RS is 0. Let us see the characteristic of this transfer function steady-state and yaw settling time.

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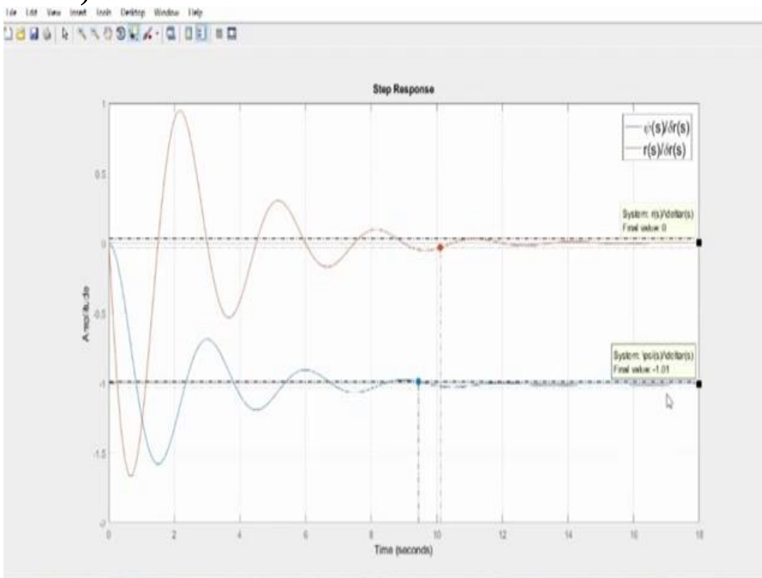

Now the value of settling time for Delta R upon Delta R of S, final value for Delta psi upon Delta RS is - 1.01.

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Final Value Theorem

This theorem states the final value of the function.

In time domain final value is given by: $Final\ Value = \lim_{t\to\infty} f(t)$

Where $f(t)$ is the function in time domain

In S-domain $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s.F(s)$

Where $F(s)$ is the function in Laplace domain

In our problem

$$\frac{\psi(s)}{\delta_r(s)} = \frac{-4.6}{s^2 + 0.76s + 4.55}$$

$$Final\ Value = \lim_{s\to 0} s.\psi(s) = s.\left(\frac{-4.6}{s^2 + 0.76s + 4.55}.\frac{1}{s}\right)$$

$$\boldsymbol{Final\ Value = \frac{-4.6}{4.55} = -1.011}$$

$$\frac{r(s)}{\delta_r(s)} = \frac{-4.6s}{s^2 + 0.76s + 4.55}$$

$$Final\ Value = \lim_{s\to 0} s.\psi(s) = s.\left(\frac{-4.6s}{s^2 + 0.76s + 4.55}.\frac{1}{s}\right)$$

$$\boldsymbol{Final\ Value = \frac{0}{4.55} = 0}$$

From final value theorem, as you can see, the value for transfer function Delta psi upon Delta RS will be - 1.011 and for Delta R upon Delta RS, will be 0. Now during our analysis, we saw a 1st order equation, a 2nd order equation and what will be the response of that when subjected to a step input. Now I want to discuss what will be the response of a higher order system when it is subjected to a step input. So in that case, the concept of dominant pole comes into.

So I will be explaining what are dominant poles and how system responses to different input signals. So for that, let us take a system which is a 2nd order system cascaded with a 1st order system.

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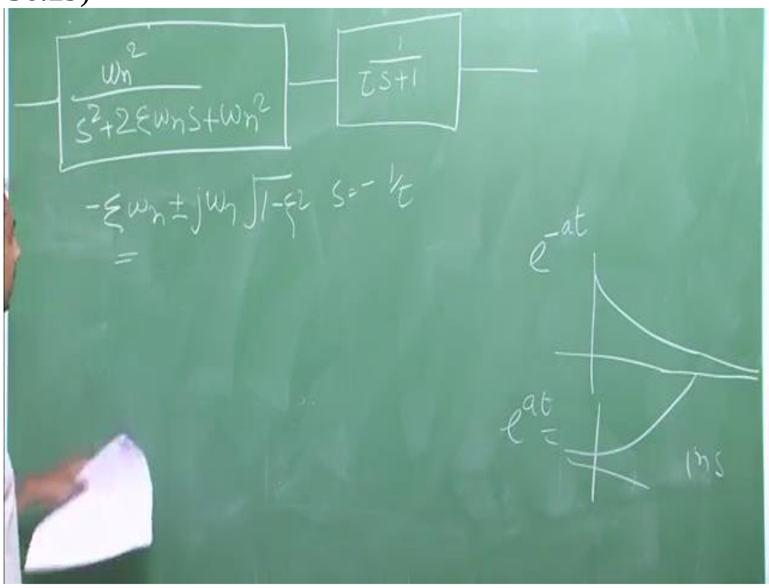

So my 2nd order system, as you know a standard 2nd order system is represented by Omega N squared by S square + 2 Zeta oOmega N S + oOmega N square. This is my 2nd order system. Now it is cascaded with a 1st order system which is 1 upon Tao S + Tao. So overall, this will be a 3rd order system. What will be the response when this is subjected to a unit step or impulse signal?

We are talking about dominant poles. So let me explain what is meant by dominant poles. In this equation, as you know your poles will be - Zeta Omega N + - J Omega N root over 1 - Zeta square. And for this case, as you know, it will be - 1 by Tao. As you know, in Laplace domain, every root, the real part of the root in exponential term corresponding to that in time domain which results in decaying function or increasing function.

That is why we go for real part to be negative because for each term, there will be exponential term in time domain. So if my real part is negative, my function will be a decaying function. If my real part is positive, my function will be an increasing function. This will result in instability. This is stable because at some finite amount of time, this will reach a very small value so you can say 0.

So we were discussing about dominant poles. So in this case, in this scenario, you can have different cases. Now 1st case will be, suppose this is a 2nd order under damped system and this is dominant pole. Now what do you mean by dominant pole? As I already told you, for each root in Laplace domain, there is a corresponding exponential term in time domain.

And if that is a root in negative half of S plane, you will be getting a decaying function as I explained in this (())(39:11).

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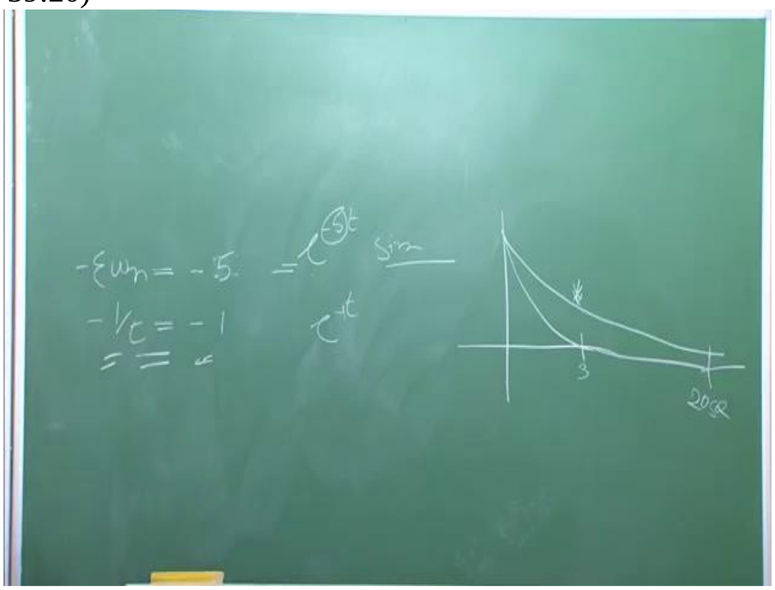

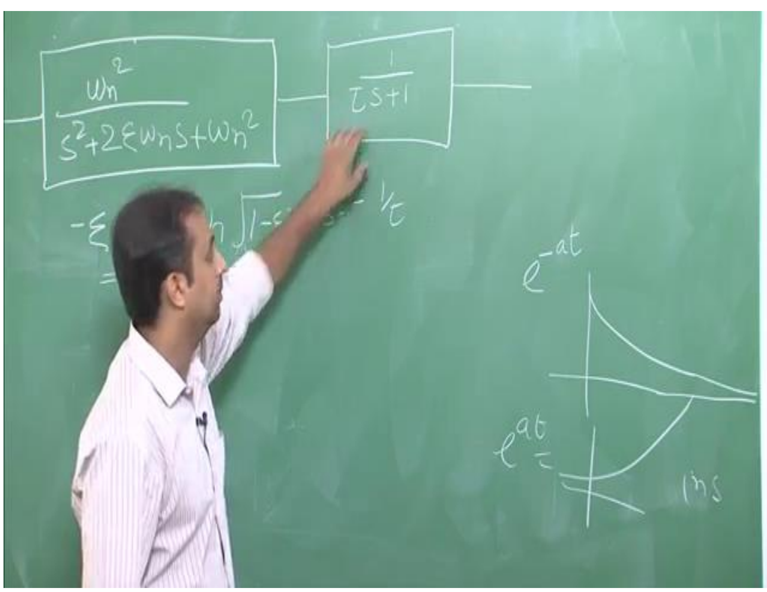

So for this case, as you know, decaying response will be due to - Zeta of Omega N and for this particular case, your decaying function or decaying response will be due to - 1 by Zeta. So, dominant means, suppose value of my Zeta Omega N comes to be - 2 and the value of - 1 by Zeta comes - 1. Let us take a larger value. Let us take that - Zeta Omega N equal to - of phi and this equal to - 1.

So corresponding to this term, we will be getting function, something E to the power of - phi T + 2nd order sin term something will come. And corresponding to this, this value will be getting, E to the power - T. Now since you can see the coefficient of this function is - of phi T, so the function will decay rapidly. Suppose this function decays rapidly.

Say I take some random value must say about 3 seconds. This function reaches 0. And you can say, the power of this particular function is 1 or - 1 you can say. So this function will be decreasing but not as fast as compared to this function. So suppose, it will be decaying and after T equals to say about 20 seconds, this function reaches 0.

So in this case, my whole response depends on the root of first-order function. Since for collective system, your whole response should reach 0. So because of that value of this particular root, my whole transient response will be 0 only when this particular response due to - 1 is 0.

So I can say that my response of my this particular system depends surely on the response of the 1st order system. So I will call this as a dominant pole. Let us see what will be the different cases and what will be the response of that system. Let us discuss case 1.

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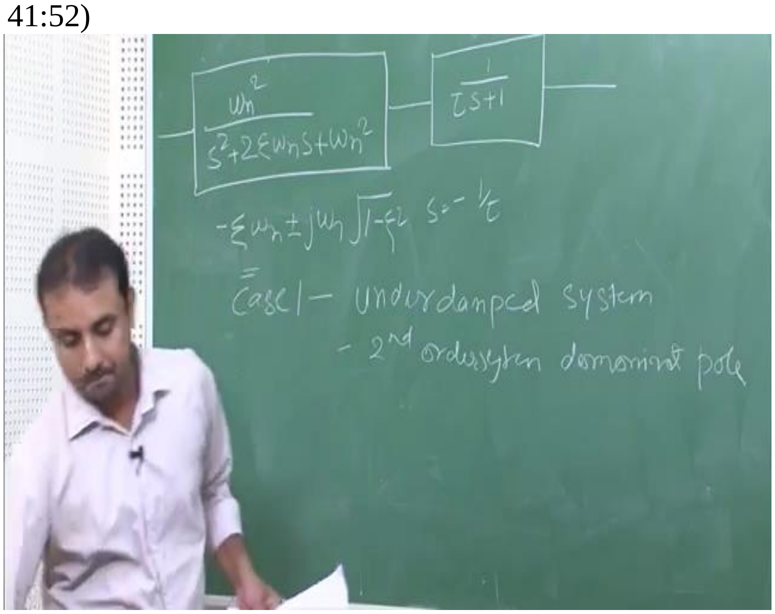

This is, my 2nd order system is an under damped system with second-order poles as dominant poles. Now in this scenario, when this scenario is there, percentage over shoot will be less as compared to a purely 2nd order system. Your settling time will be less as compared to second-order system. But your rise time will be more as compared to your second-order system.

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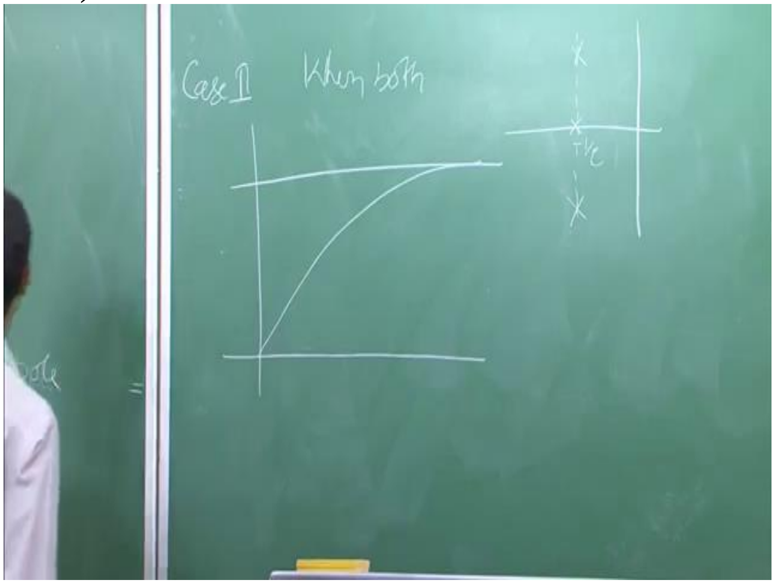

Now 2nd case, when both 1st order pole and 2nd order poles are dominant. This is my Laplace domain. My 1st order pole corresponds to some value, say - 1 by Zeta. 1 by Tao and your 2nd order poles will be assume there will be 2 conjugate poles. So it lies somewhere over here. So that is, all coincide in the same particular line. Then in this case, both 1st order and 2nd order are dominant poles.

The result in this case will be, you will get a critically damped system which will be very similar to a 2nd order system. The response will be similar to, up to 2nd order, critically damped system.

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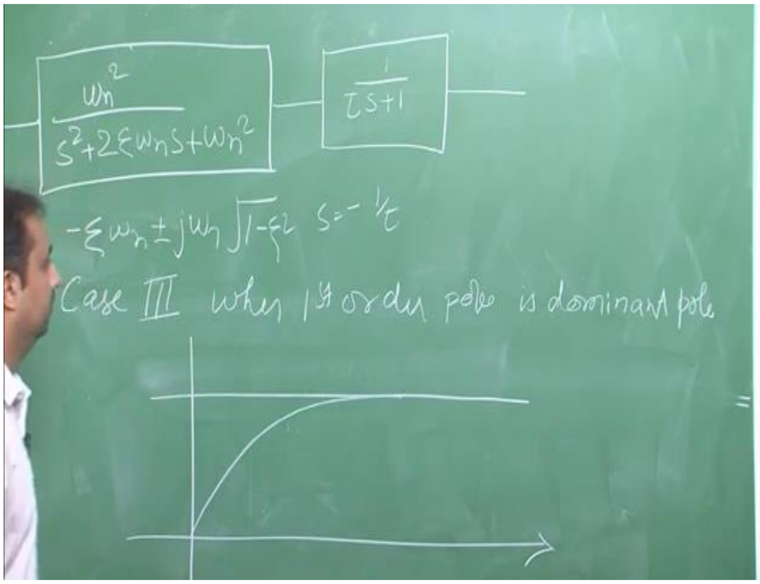

Case 3-when 1st order pole is dominant pole. When this is the scenario, the response will be similar to 2nd order over damped system.

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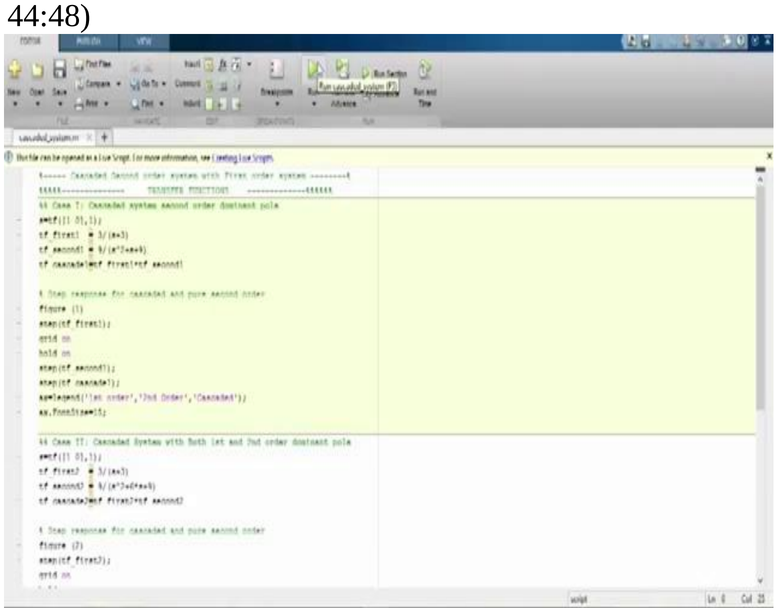
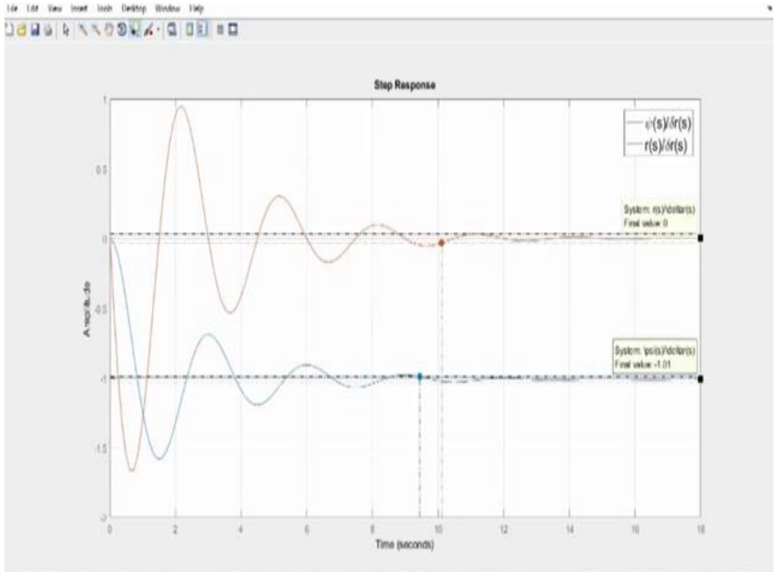
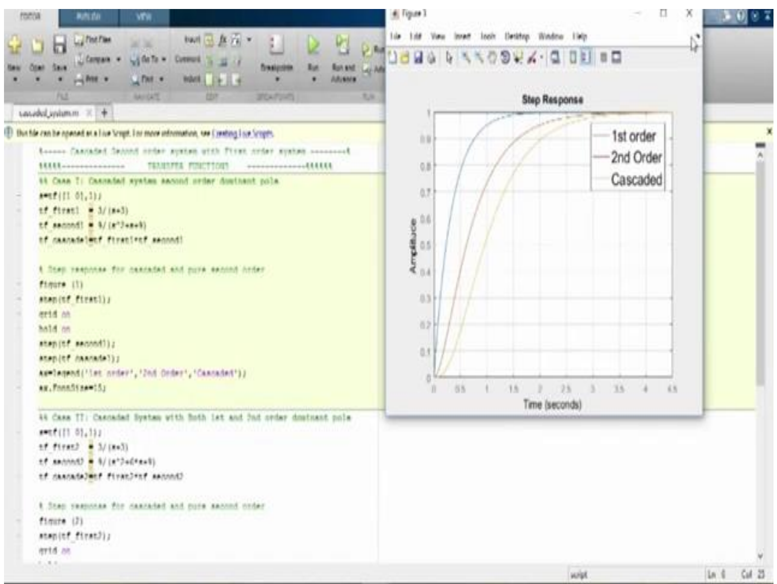
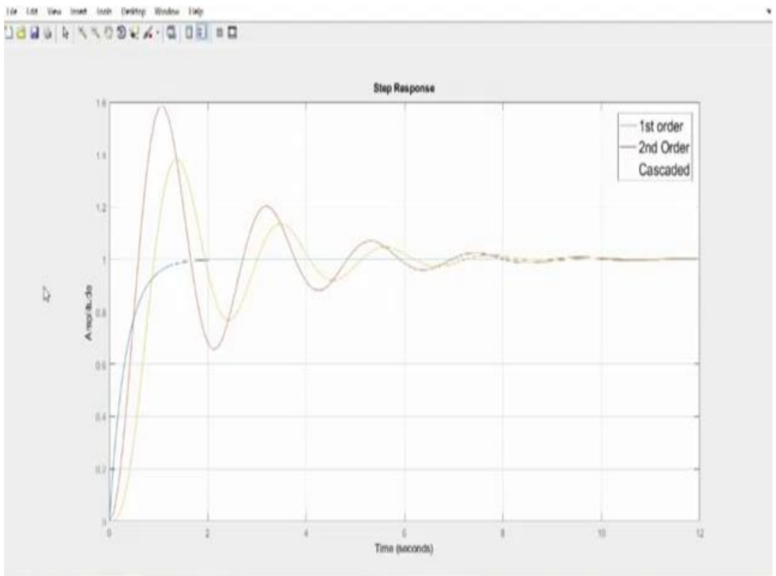



Step Response
ψ(s)/δr(s)
r(s)/δr(s)
Amplitude
Time (seconds)


Step Response
1st order
2nd Order
Cascaded
Amplitude
Time (seconds)



We saw higher order system, that is a cascaded system in which a 2nd order system was cascaded with a 1st order system. So, we will be seeing the response of that cascaded system to a step input signal. Let me take a 1st order transfer function as 3 upon S + 3 and a 2nd order transfer function as 9 upon S square + S + 9. Now the case 1 which we discussed was a 2nd order poles will be the dominant pole.

The response of this cascaded transfer function with respect to a 2nd order system is, the response of this cascaded system to step input signal will be for case 1. Yes, for case 1, as you can see, these are the responses of proper function with respect to step input signal, 1st order is represented by blue colour, 2nd order, pure 2nd order system is represented by red and your cascaded system is represented by this orange colour.

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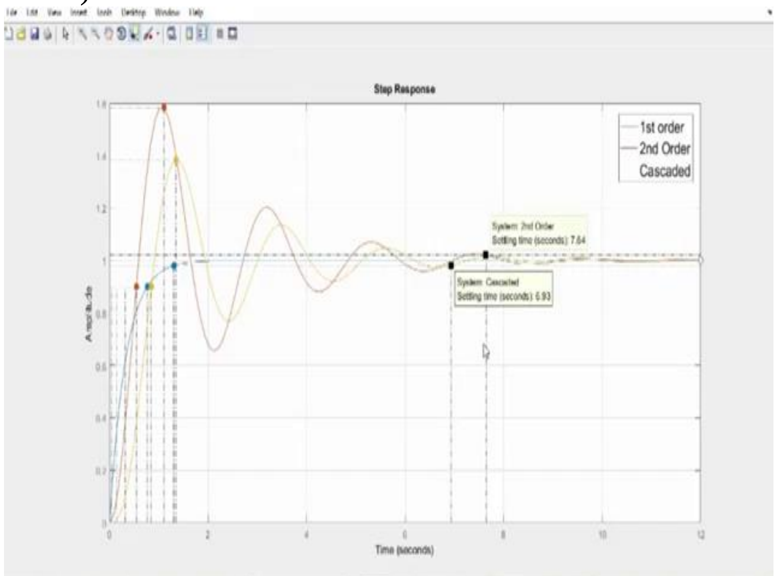

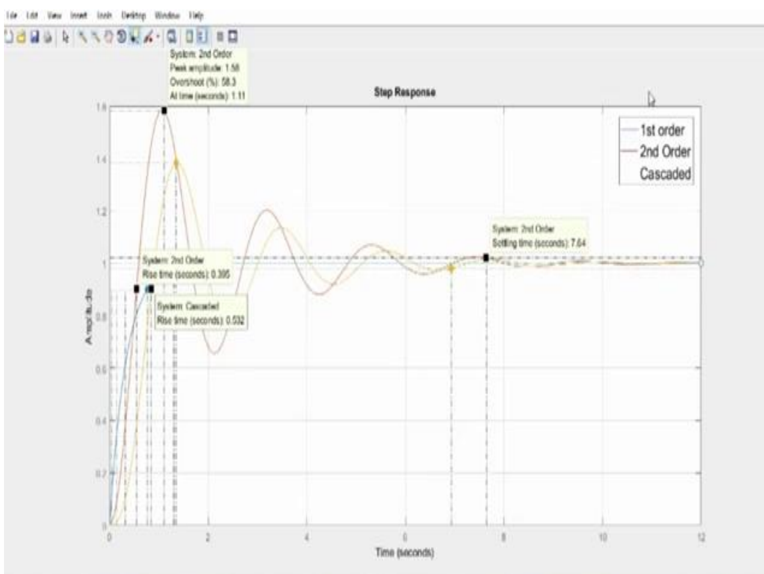

Now, as per the properties of the cascaded system, since the roots of the 2nd order system are the dominant poles, so the characteristic of this will be as discussed in the class, the settling time will be less than the settling time of your pure 2nd order system. So your settling time for 2nd order system is 7.64 seconds. While for cascaded system, it is 6.93. You can see the settling time for cascaded system is less than the settling time of pure 2nd order system.

And your percentage over shoot for the cascaded system will be less as compared to your 2nd order system. Now this is a value of percentage over shoot. For pure 2nd order system, you can see peak amplitude is 1.58 and for cascaded system, the value of amplitude, peak amplitude is

1.38. So percentage over shoot has decreased from 58.3 to 38.3 percentage stop similarly, we told that for a cascaded system, your rise time will be greater than the rise time of your 2nd order system. The rise time for pure 2nd order system is 0.395 seconds whereas for cascaded system, it is 0.532.

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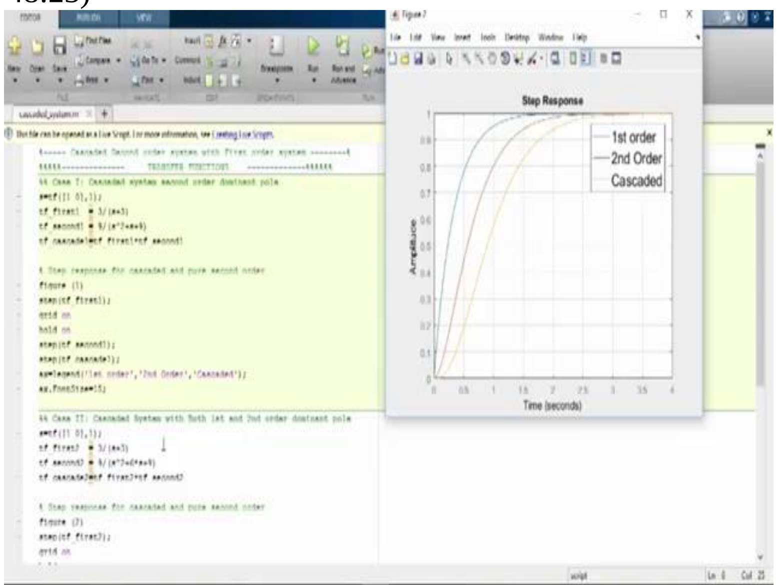

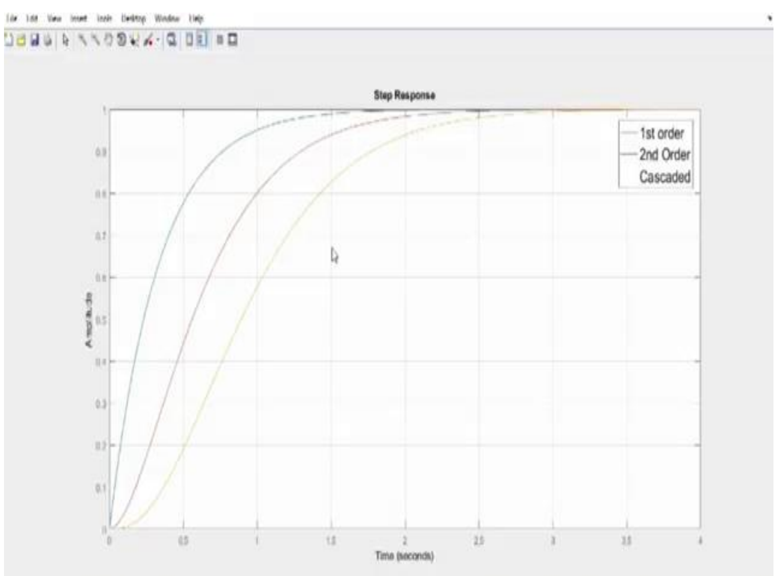

Now as for case 2, where both 1st order transfer function and your 2nd order transfer function have dominant roots, in that case, the transfer function I have taken as S upon S + 3. While the 2nd order transfer function is at S square + 6S + 9. The roots will be - 3 for 1st order system and for 2nd order system, it will be - 2 + some complex part. So in that case, the response to the step input signal will be as described with these plots.

There is a step input, it will be same since I have not changed the toughest function on case 1. Now this is a pure 2nd order system represented by red plot and your cascaded system represented by orange plot. As I told you in the class, the response for this curve will be close to or very similar to a critically damped system for a pure 2nd order system.

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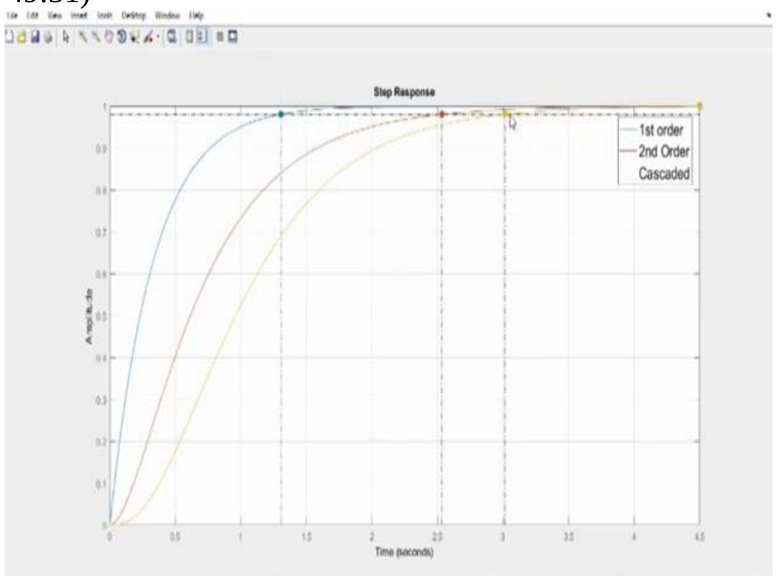

As for the 3rd case, the transfer function which have taken for 1st order system is same, S upon S + 3. While for 2nd order system, it is 9 upon S square + 7S + 9. So that makes my Theta greater than 1. So in this case, my dominant pole will be due to 1st order transfer function. In that case, my response to step input will be as follows, represented by this plot.

In this case, the response of the cascaded system is similar to the response of an over damped system due to pure second-order system. So far we have discussed about second-order system, 1st order system, your cascaded systems. What will be the response of that when exposed to step input or impulse signal.

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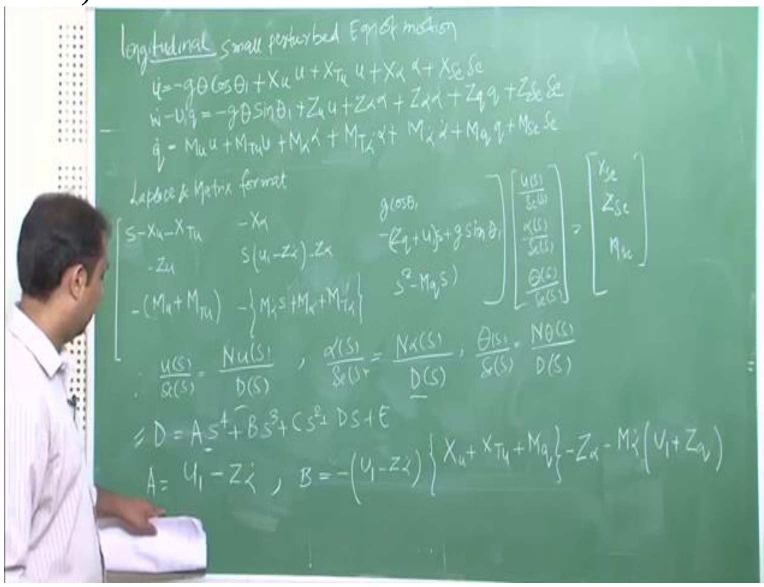

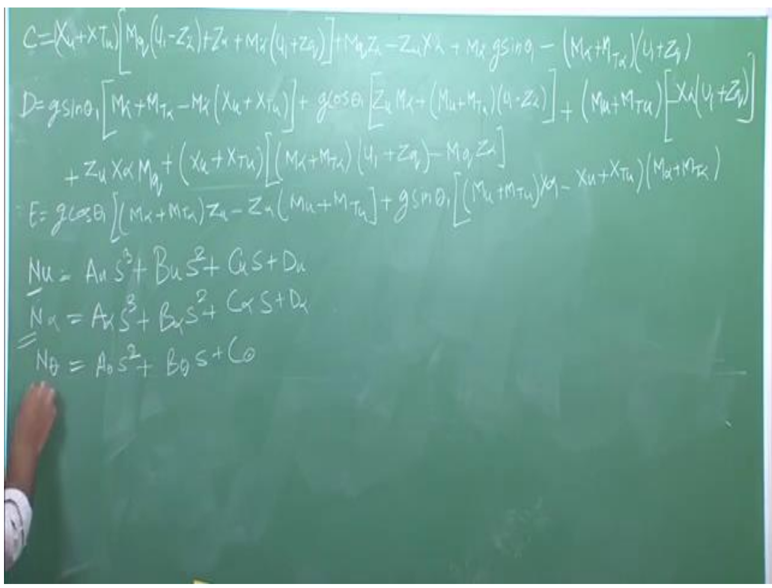

Now as you have studied in previous lectures, this is a longitudinal small perturbed equation of motion. So what will be the transfer function of that? So that will be derived using, this is a differential equation. Take Laplace transform and were ranging matrix format. And as you know to derive transfer function, the best method will be using Cramer's rule.

Use Cramer's rule to derive the toughest function for O of S upon Delta E of S, alpha of S upon Delta E of S and Theta of S upon Delta E of S. You will be getting a transfer function of the format, something numerator in terms of capital as divided by a denominator and as you saw in previous lectures, the denominator was same but the numerator quantity differed from equation to equation. So here also, the transfer function will be derived.

The denominator is same. The numerator will have different coefficients. So this is a pretty long equation and I would like to do it yourself because there might be some errors while equating this lengthy equation. But I have still tried to solve this equation, this matrix and this coefficients, this D of S or your denominator I have got was of 4^{th} order and the coefficient of A, B, C are respectively given as mentioned on the blackboard. This is A1 or A, your B, C, D and E. This is very big equation.

I would like to say try to equate. No need to memorise this big equation. Try to equate and see. Because substituting the value for a generic aircraft, this will be a much simpler matrix to solve. Similarly for the numerator part, for U of S upon Delta E of S you will get 3^{rd} order equation.Similarly for alpha of S, you will get 3^{rd} order equation. Theta of capitalist much will get 2^{nd} order equation.

I will be posting this on the forum what will be the coefficients of AU, A alpha, A Theta. So no need to worry about that or no need to memorise this big equation.

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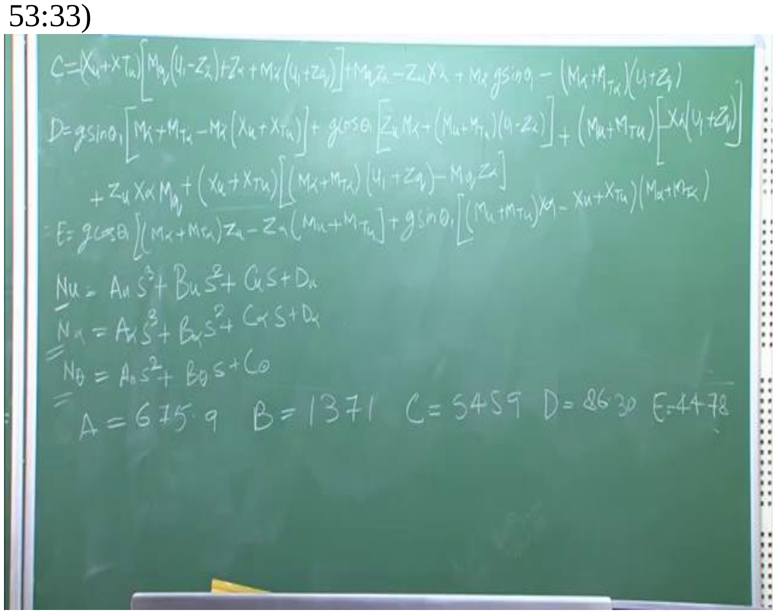

So, taking this equation of motion for longitudinal motion, for a general aircraft, the following data if I take, the value of A equals to 675.9, B as 1371, C equals to 25459, D equals to 86.03 and E equals to 84.78.

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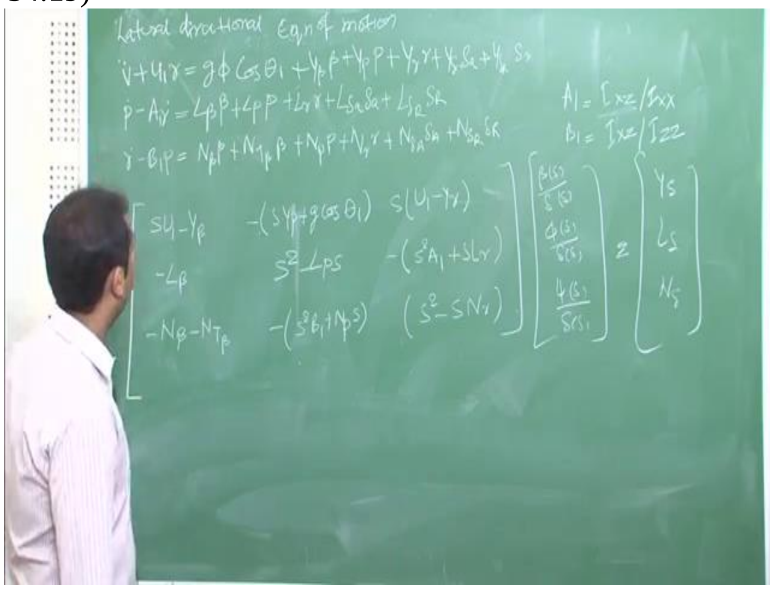

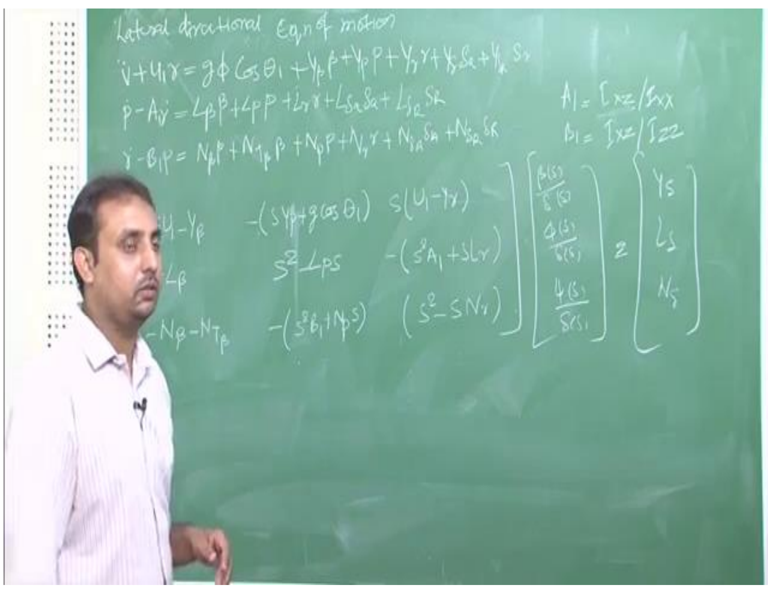

Now value of A to E if you take for a general aircraft, the transfer function for each of these will come to be U of S upon Delta E of S will be - 6.312 S square + 4927S + 2302 by 675.9 S to the power 4 + 1371 to the power of 3 + 5459 S square + 86.31S + 44.78. If I write alpha of S upon Delta E of S will be - 0.746S cube - 208.3S square - 2.665S - 1.9 divided by 625.94 + 1371S cube + 5459 S square + 86.31S + 44.78.

And similarly, my value of Theta is upon Delta E of S - 208.1 S square - 126.9S - 2.380 divided by same denominator. I will not be writing that. As we have seen in previous section, we derive the transfer function for longitudinal perturbed equation of motion. Similarly we can derive transfer function for lateral directional equation of motion. The set of equation or differential equation is given by this formula.

Again we have to apply Laplace transform and write it in matrix form. And as you know, using Cramer's rule, you can derive transfer function for different variables of lateral directional equation of motion. Now, I have not described the full equation. What I will do? I will put that full transfer function on the forum and you can check.

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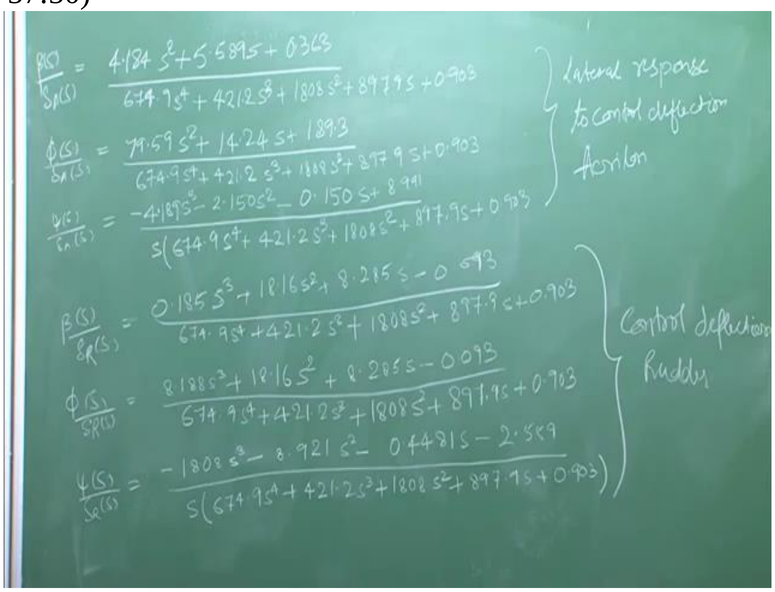

Now based on this transfer function, and using data for the aircraft, my transfer function for different variables will be given by this transfer function where since you know that lateral directions are controlled by 2 deflection surfaces that is aileron and rudder. So transfer function related to ailerons are given by this particular set of situations. While for rudder, it is given where this particular set of equations. I would like you to practice more problems on transfer functions. Thank you.