Introduction to Finite Volume Methods-I

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Lecture – 11

So welcome to the lecture of this Finite Volume Method.

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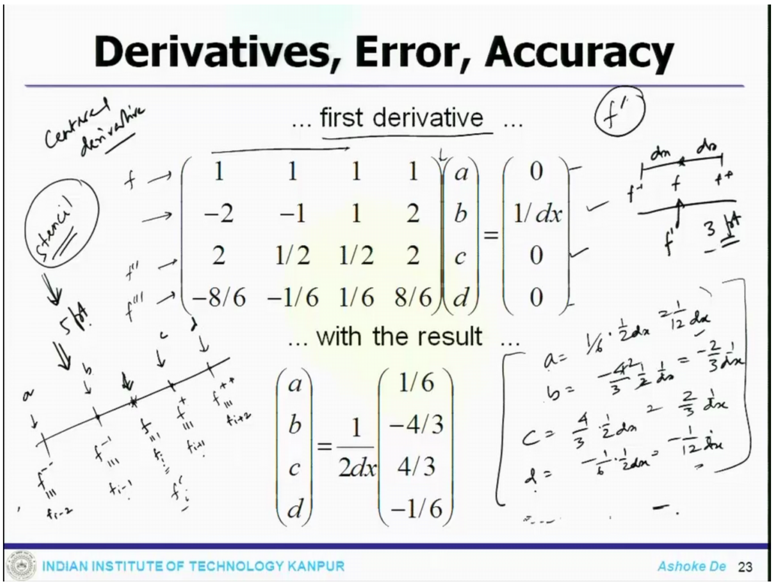

So, where we have stopped in that finding the derivatives; whether it is a interpolated or first order derivative or the higher order derivative through the waiting procedure. So, if you come back or recall to that particular things what we are trying to do, is that finding those points around a particular point. If you have this point, and ahead of this point is f plus, and the second point is f double plus f minus f double minus. And when you find out those values you got this interpolated values of a, b, c, d. And what we are trying ? Here we are trying to find out a derivative at this locations using 2 point ahead of it 2 point downstream of it.

So, it uses some sort of a symmetry. Because it uses 2 points ahead 2 points behind. So, this symmetry because of that and when it uses this kind of points it is called the Stencil. And this will be one term which will be using very often to define the numerical systems or the discretization. And the points if you are distributed around that interested point symmetrically, these are called the Centered derivative of a particular order. And the similarly same first derivative could be also obtained using one-point head of it and one point downstream of it.

That means, using only a f i; f i plus 1 or f minus 1 the; or f f plus and f minus. So, this is also another kind of Stencil where you can find out the derivative at this location. And if you find out that this is called 3 points Stencil to find out the first derivative this particular one called 5-point Stencil to finding the derivative. But the important thing is that the points are symmetrically distributed around this point. And that is why they do carry equal weights. Whether it is a 3 points Stencil or it is a 5-point Stencil the weights would be similar, ok.

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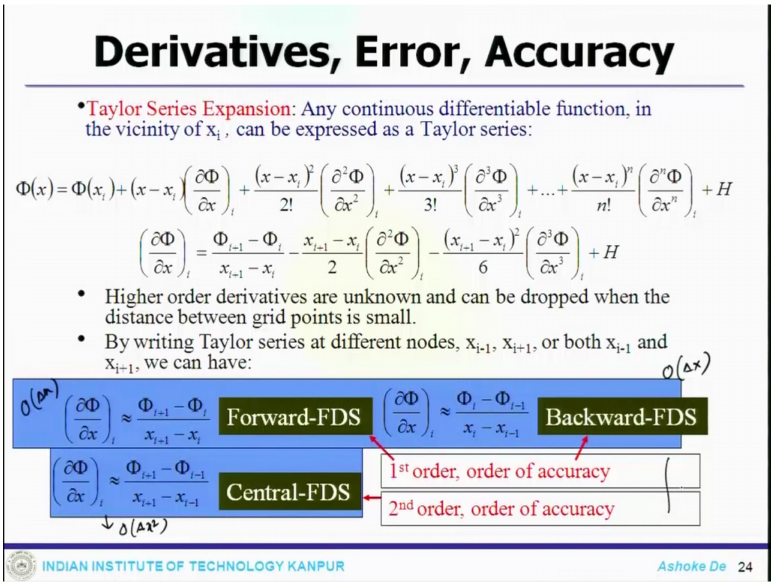

So now moving ahead when you look at the derivatives; so what we have talked so for if you put them together? So, what Taylor series done at any function for a given point, and you try to find out the function at any given location. And using a point i, you find out this is the point i and this is a point where it is at x. So, it is in a generic format. So, for what we have been discussing in that case, x minus x i is equivalent to d x that is what we have been doing.

So, any given function at this f x using point x i is represented that function, distance between the points with first derivative evaluated at I, distance square factorial 2 second derivative at i, distance cube third derivative and at the higher order derivative and if you have any other. So, if you write del phi by del x at point i, it can be written phi i plus 1 minus phi i divided by x i plus 1 by x i x plus 1 minus x i by 2 del 2 phi by this. And so, essentially taking everything else from this side to the other side, you write del phi by del x.

Now, when you are trying to find out the higher derivatives and, if you drop the higher terms, that will lead to the reduced order system, if you return the higher order terms it will give you the higher order system. So, depending on that so, as we have seen if you have done the forward one. So, you can write this is always first order accurate. That means, the truncation error is order of first order system. So, forward difference is this should be order of delta x backward difference is also first order accurate. So, it would be also delta x, central this would be order of delta x square. So, the highest order term which is written in the truncated system they will lead to the order of accuracy.

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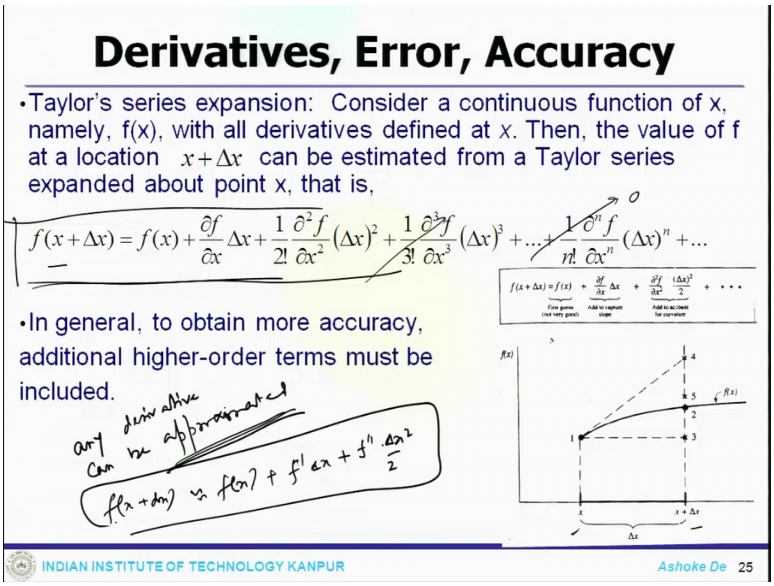

Now, when you consider this Taylor series system, and this is what we have been talking if you have a point Ax and x plus del x. And we have a function, you can define the generic Taylor series expression like that. And using this any derivative can be approximated. So, note it all this term what we are saying that it is not equal they are approximated, because as soon as you drop certain terms in the system it is only approximation. So, if I write like this x plus d x equivalent to f x plus f prime delta x plus f double prime delta x square by 2 like that. So, this is an approximation, this is not equal. And then you can find out the different derivative.

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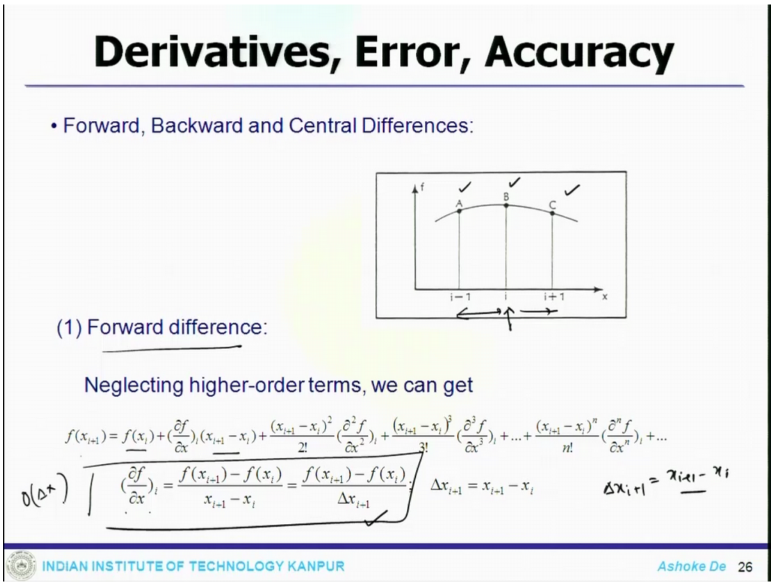

Now, this we have been doing you have a point here at i, you have a point ahead of it is i plus 1 and you have a point i minus 1. Now if you do forward differencing. So, I am trying to find out the value at i plus 1 using a value at f x i del f x del x i at i the distance between these 2. Now if you say, any delta x i plus 1 is x i plus 1 minus x i. Now this could be uniform, then this lead to the uniform spacing of the points. Then they will be equal, ok. And I can write down del f by del x at I is f i plus 1 f x i plus 1 minus f x i by I plus 1. So, this is again my order of accuracy is first order.

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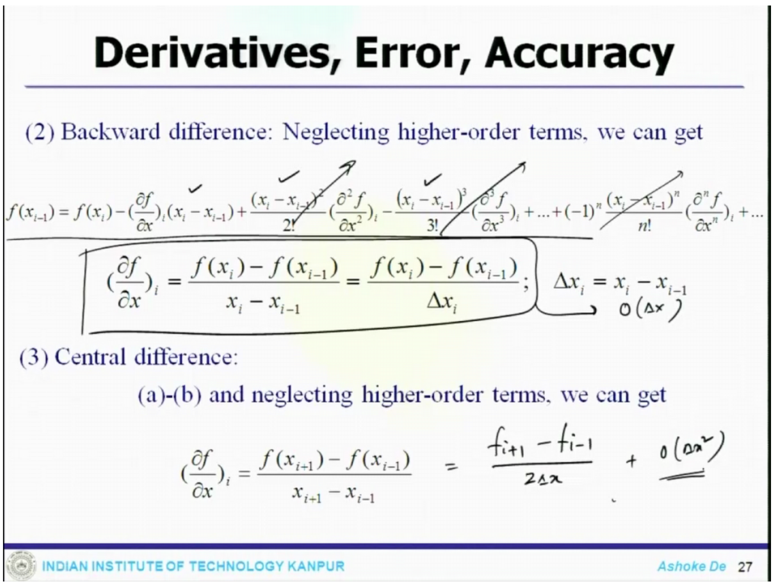

Similarly, if you write the backward system; so you remove the higher order terms, and this is x minus i you write, that is why the difference between the points comes x minus x minus x i minus 1 x i minus x minus 1. And if you neglect the higher order terms like, this term, this term and this term, you get this is the expression. Again the order of accuracy is delta x. Similarly, if you take both the systems together, and if you write this is f minus 1 by 2 delta x.

So, this is second order accurate. So, this we have been talking for a while that, when you are doing forward difference or backward difference, and when you move to second order difference. This actually gives back to a second order accurate system.

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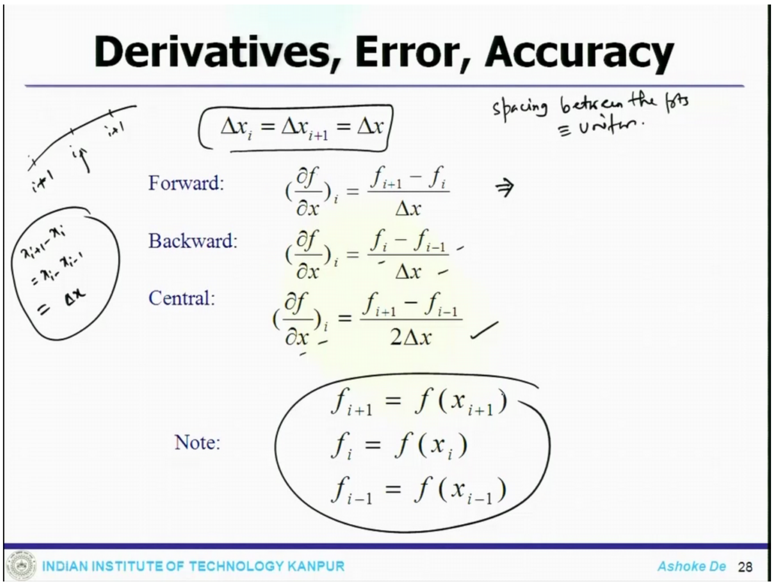

So, if you go to centre difference this gives back to. So, if you put them together the spacing between the points they are uniform. So, that is why they get back to this, then my forward difference can be written like that. Del f by del x at i i plus 1 and my points are distributed like this i i plus 1 i minus 1 and x i plus 1 minus x i equals to x i minus x i minus 1 equals to delta x.

So, they are uniform spacing system; backward, I write f i minus f i minus 1 by delta x central del f x by dell x at i. Everything is evaluated at i by 2 delta x. So now, all these are representative of the particular system.

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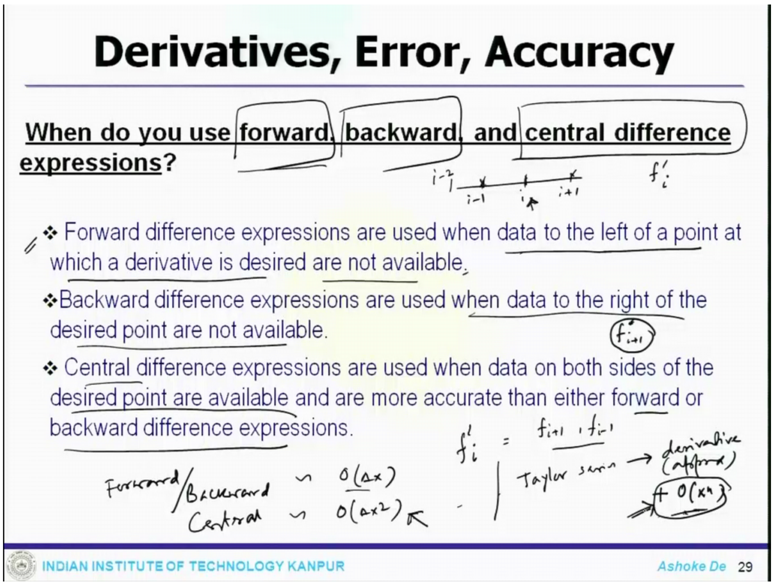

Now, point comes when you use forward system, when do you use backward system or rather when do you use central system. So, it all depends of the kind of problem that you are dealing with. And when you use these different kind of forward backward or central difference.

So, essentially as an user or the numerical methodologies, you need to have an idea when use what kind of system. So, the forward difference expressions are used essential when the data to the left of a point at which a derivative is desired are not available. It is a very important statement here. Forward difference expression are used when a data to the left of a point at which the derivative is desired are not available. So, if I have i, I have i plus 1 and i minus 1; now I am trying to find out a derivative at i if I am trying to find out a derivative at i, which derivative is desired not available then only I do that.

Backward difference expressions are used when the data to the right of the desired point are not available; that means, I am trying to find out a derivative at i; that means, what I am trying to find out is f i. And I do not have information of f i plus 1, then I have only information before i; that means, i minus 1 or i minus 2, I used backward difference. And when, but the other case in the forward differencing case when i minus 1 is not available, but I plus 1 is available, I use forward difference.

Now, central difference expression contrarily used when data on both sides of the desired points are available; that means, if I am trying to find out f i prime I need both f i plus 1 I need i minus 1. So, these points are available or the information at these points are available, and that time only I can use the central differensive scheme. And as we have mentioned, that the central differensive scheme would be always higher order accurate compared to forward or backward difference system; because forward scheme or backward scheme is first order accurate system. This is the highest order truncation term sitting when you do the Taylor series expression, and get the approximated derivative; whether the central scheme is second order accurate.

So, obviously, one would like to have a higher order accurate system compared to a lower order accurate system. And this is what it is preferred in a CFD when you move from problem to problem or rather, one to one another problem, you tried to prefer or prefer to use as higher as possible in the order of accuracy. Now here when you talk the order of accuracy that is only tell you from the Taylor series expression. So, essentially from my Taylor series expression, I get the derivative or approximation of the derivative.

Once I do the approximation of derivative, these are having some order, ok. So, it could be first order, it could be second order, it would be higher order. Now these order of accuracy or the term which are the higher order term which are actually truncated these are sometimes called the truncated error. Now in the numerical system there are different kinds of error which are associated with that.

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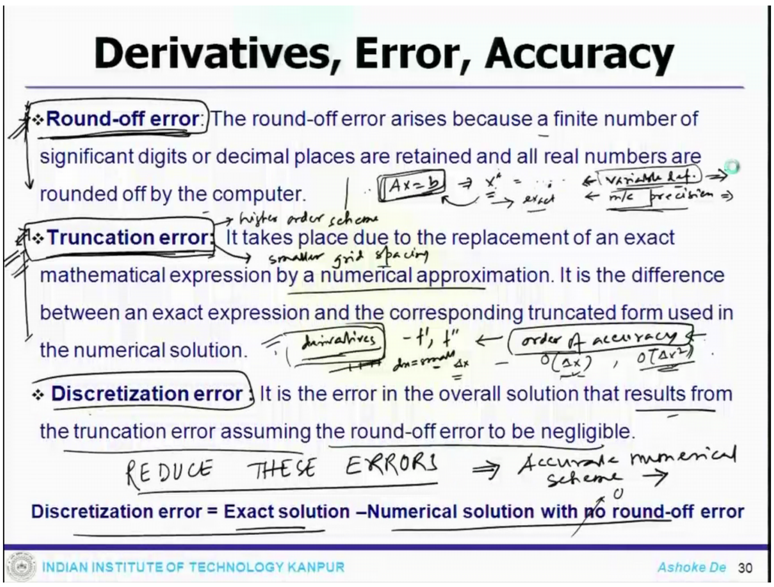

One important error is called the round off error. And what does it mean? The round off error if one has to state the round off error essentially arises, because a finite number of significant digit or the decimal places are retained and all others are rounded off by the computer.

So now what happens; essentially solving a problem of linear system Ax equals to b. Now when you are solving a linear system in a computer, computer get backs you some sort of a solution of x star. Now the solutions what you are getting it also depends what would be the digit that you retained after decimal. And that depends on 2 things. What is the variable definition you have. That means, what kind of variable you have defined whether it is a single precision or double precision. And then top of that what is the precision of the machine so, the machine precision. So, it may possible that I may define a double precision variable, but my hardware or the computer the precision of the computer is less.

If that happens then even then I have defined a I mean double precision variable, it will round of the values after decimal after certain digit of the decimal. So, in certain problems these, could be a problem in certain applications this may not be a that kind of problem, but typically if the round of error is more and more this gets accumulated, because every instant of time or physical solution I am getting a solution of x star. And this x star is updated to get the next level of solution. In a physical problem that if you are think about when your linear solver actually works, it works in a time domain. So, go from one physical time instant to another physical time instant, and you update the solution, and then you approach towards the exact solution.

Now, while doing that in every step or every stage if you rounding of the values, whether it depends on the variable definition or depends on the precision of the hardware, it gets accumulated and finally, the solution that you get there you have lot of error. And that error is called the round off error. So, to the best way to avoid that when you define the variable typically, you define as a double precision variable. And then as of today's hardware or the architecture that we have in our hand, that is good enough to handle those double precision variables. And you do not have this kind of errors anymore.

But keep in mind, if you do not define your variables properly most of the times you get the round off errors in that nature. And problems of certain class we will have a huge impact due to this round off error. But it is essentially look at that, if you are looking at a real engineering solution then you may not have a problem with this rounding off error, but if you are looking at the real scientific value or the physics of it, then this rounding off errors would be of concern. And one needs to reduce or minimize as much as possible of this. So, best way to define the variables properly; that means, you define double precision variable, and use proper machine precision so that your programming takes care of the things and your rounding of error is as minimum as possible. That is one kind of error; that there is another kind of error which is called the truncation error.

And that comes from the numerical approximation. And numerical approximation means, we are getting all these derivatives f prime f double prime at certain values and you have a order of accuracy of the approximation. So, here if you look at that; if your delta x which is the spacing of your numerical grid if that is higher; and if you are using forward or backward difference in scheme your order of accuracy is order of delta x so, your error is too high. At the same time if you use central scheme your order of accuracy is delta x square so, this would be less. So, the truncation error comes from 2 different aspect. Or one is that how you have approximated your derivatives that means, the numerical approximation.

So, how well you have done that from there? And what kind of higher order terms you have truncated off? So, if you have written the higher order term, then the leading order derivatives or the leading order errors would be less. If you have retain the I mean remove the higher order terms, then your order of the error should be high and now. So, that is one way you can actually avoid rather reduce this truncation error; that means, uses the derivatives with higher order of accurate system. That means, your numerical approximation must get you back the higher order accurate system that is one way. Other approximation way is that; you have refined grid system. If your delta x is really small.

If it is really small, then the delta first order term or the delta square would be much smaller. Once, that is there then the truncation error will be also reduced; so in order to minimize that. So, truncation error is one of the inherent properties of your numerical scheme. It does not matter whether you are using finite volume final difference or finite (Refer Time: 22:35). This kind of error is going to be there. So, that is also true for your round off error, ok. So, it does not depend on what kind of numerical methods one is using. So, best way to minimize the truncation error is that uses higher order scheme that is one way to do that.

And second approach is that you use a smaller grid spacing. So, that means, any higher order term that you have truncated off that would be order of delta x or square or cube, they will not lead to too much of truncation error. So, this is something is in users, hand and this is something one should be able to handle it very carefully, and when you look at any CFD code weather it is commercial or in house code, or you develop your own code. This is one term which will be people keep on talking, what is the order of accuracy of your code, because, that means a lot.

As soon as you say order of accuracy of the system you know how you have actually evaluated your derivatives. And the way you are evaluated your derivatives, that will have the order of accuracy, and once you have the order of accuracy you know that truncation order is of error is of that order. So, as I keep on refining my grid my truncation error will be less. Third is the discretization error. So, that is the error in the overall solution that results from the truncation error assuming the round off error to be negligible.

So; that means, if someone has handle the round off error very nicely or the round off error is quite minimal, then the primarily the discretization error is the contribution which comes from the truncation error. But overall the discretization error should be the total error of; so, discretization error would be the exact solution minus the numerical solution with no round off errors, ok. So, theoretically if there is no round off errors, if this guy is 0, then all the discretization error should be equal to exact solution. Otherwise, you will have the discretization error and that primarily comes from the truncation error.

So, you have essentially 2 broad category of error. One is round off error another one is a truncation error. And no matter what kind of numerical approach you use. So, you have to essentially reduce these errors. Once you reduce these errors, you will have a accurate numerical scheme. And that is what one desires to have, ok. So, as much as possible you should reduce this errors, and through higher order scheme through smaller grid spacing through higher variable proper variable definitions and all this things.

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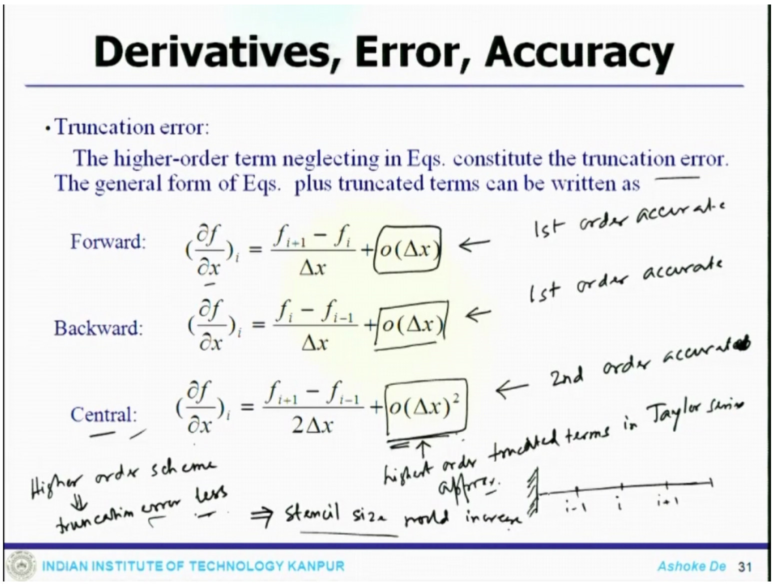

Now, as we keep on coming back to the calculation of the derivatives. So now, if you look at, the Taylor series expression actually gets you back the different derivatives, whether through forward scheme or backward scheme or the central scheme. And all these different scheme they do have certain errors. Forward scheme once you do it is higher order term is order of delta x. So, that is the truncation error or the higher order error, or rather we say this scheme is first order accurate. Similarly, that holds good for the backwards scheme, because the higher order term or the truncated term are order of delta x; and once that is there, you also term that one as a first order accurate scheme.

At the same time if you look at the central your higher order term or the higher order term which has been truncated off there of the second order. So, this scheme is second order accurate. So, as I have mentioning that order of accuracy primarily depends on the highest order truncation truncated term in Taylor series approximation.

So, essentially the kind of terms that you retain, that will dictate the higher order, I mean the order of accuracy. More and more higher order if you use you get less and less truncation error. But at the same time so, I can have higher order scheme and which immediately means my truncation error would be less. But at the same time, one has to note that as you move higher order your number of points required to find a derivative will be also more. Like, if you look at central to just get a first derivative it requires 3 points. If you go to higher order accurate it will be requiring more and more points.

So, the Stencil size would also Stencil size would increase. Now if that increases, then that has a significant impact on the data structure and the programming point of view. And not only that, also you will have lot of impact when you go down to the application of the boundary condition. So, when you come down to boundary, you have more points or you have difficulties to apply that kind of thing. So, inside the domain you can always maintain these, but when you come down to boundary you have to again come down to the lower level of system.

So, we will stop here today.

Thank you.