Introduction to Finite Volume Methods - II

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Lecture – 15
Convection term discretisation - VII (Private)

So welcome back to the lecture series of Finite Volume. And what we are in the middle of the discussion of the discretization of convection diffusion system. And what we have completed so far we started with the central difference scheme, then we have seen the problem with the central difference scheme, because this is going away from the exact solution which has to do with some sort of an stability; otherwise central difference scheme provides you second order accurate.

Then from there to get out of that problem we discuss the first order upwind scheme which was obviously, the order of accuracy was less, but that provided the solution which follows the proper trend. So, the accuracy was compromised, but the solution become stable. And it provides you the solution which follows exact trend. Then we discuss the downwind scheme which is the reverse of the first order upwind. And from there when we moved forward we discussed second order upwind scheme which was accuracy wise it is also second order in space and it was also stable and provides better solution.

Then we discuss the quick scheme which is also third order accurate in space. And then for particular condition it is stable some other condition, one has to be careful regarding the boundedness of the quick scheme. And finally, we discuss the form scheme which is also second order accurate. And for certain cases that is stable and some other cases again the boundedness needs to be taken care of. And then finally, we looked at the functional relationship between your uniform and non-uniform grid. And when you move down to non-uniform grid actually instead of your regular coordinate system there you use the curvilinear system or the transform system in psi or eta. And accordingly your interpolation distances would be calculated, where we have stopped in the last class

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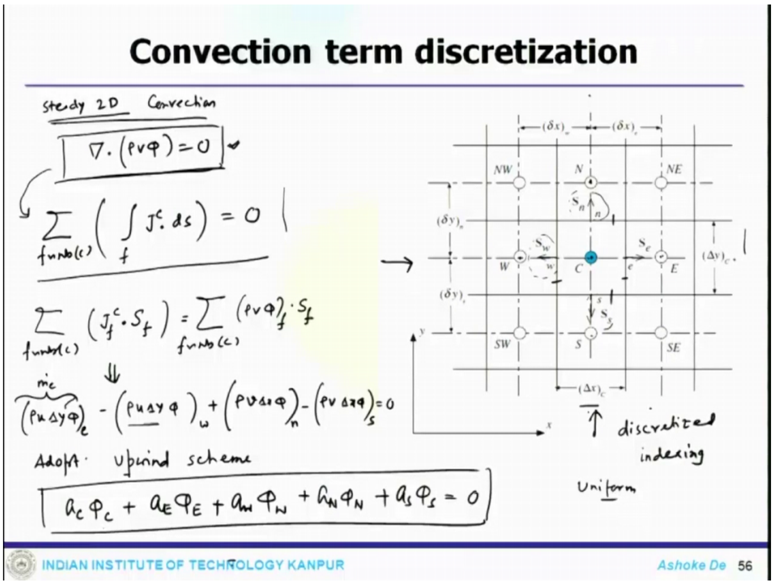

Now, we will move ahead and look at the two-dimensional steady 2D convection system. So, now, our discussion so far which we have talked about all the schemes, there we have taken convection diffusion equation. Now, this here we will start with the 2D steady convection only which will be represented as del dot rho v phi equals to 0, so that is the equation which will govern the two two-dimensional steady convection system without having any external source term.

Now, if you look at this particular stencil or you can call it is the discretized indexing, these are the terminology that we have used in our lecture series from the beginning. So, here again the grid is uniform and the cell which are interested is the middle the cell C, and then you have upstream cell which is E, downstream W, north N, south S. And the surrounding phases like e stress west where your surface vector is Se; then you have north phase, surface vector is S n; then you get west phase, where surface vector is this and the south phase. So, these are and the distances between the cells. So, these are standard information or standard grid related information that we have been using since beginning.

Now, once you look at these particular equation, how do you get the equation discretized. Now, once you do the integration over cells, this will get you back the system over all the phases and this should be phase J dot d s equals to 0. So, it is a volume integral then the volume integral gets transferred to surface integral these are the thing that we have been doing every now and then. So, now onwards I will not go in that much of finer details, but you know from this equation if you take an volume integral then the volume integral gets converted to this particular or this is where you end up getting an algebraic system.

Now, if you expand these terms, all these terms this should be looking like summation over f n b c J f for convection dot S f equals to summation of f rho v phi f dot S f, so that is what you get across the services. Now, if you expand from there, you get rho u delta y phi at the east phase which is nothing but my m dot e minus rho u delta y phi. So, this should be the mass flux up to this west. So, this represents my m dot w then rho v delta x phi at north phase minus rho v delta x phi at south phase which is 0. So, once you expand this is what you get across those phases. So, these are the phases, east phase, north phase, west phase, south phase.

Now, what scheme that one has to use. So, once you use different scheme and so far we have discussed lot of different variants of schemes for the convection calculation. So, let us adopt something the simplest one is the simple upwind or upwind scheme or other first order upwind scheme. So, then getting the flow to be locally one-dimensional what you get the discretized system is a C phi C plus a E phi E plus a W phi W a N phi N plus a S phi S equals to 0, so that sort you get for two-dimensional system.

Now, since we have used only upwind scheme that is the first order upwind scheme for each direction, this is x-direction, we just need this cell, this cell and this cell which will lead to this coefficients. And y-direction we need this, this and this, so that lead to this, this. If at the same time or in the other hand someone uses second order upwind, he might end up getting one more coefficient from this cell one more coefficient from this cell and then y direction one more. So, there will be few more coefficients which will appear.

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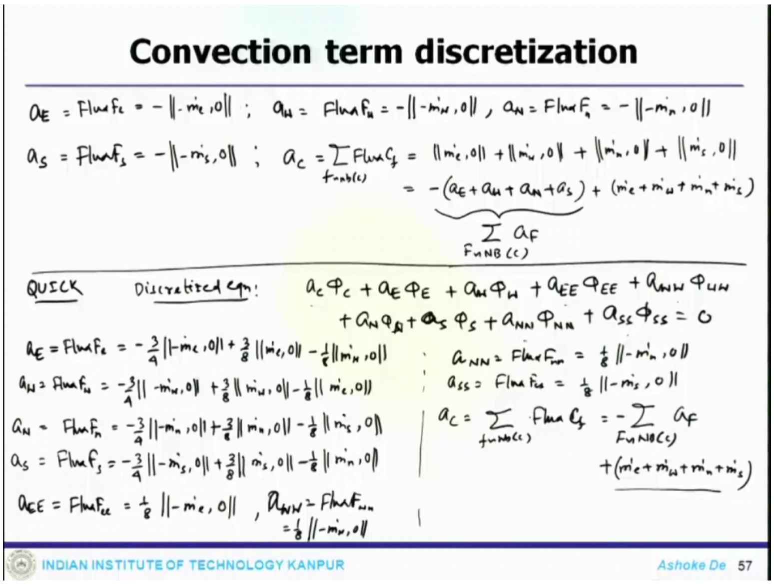

So, if you look at the coefficients here, a E would be flux F e so that is flux F e which is minus m dot e 0. And a W is flux F w which is minus m dot w 0. Then you get a N which is flax F n which is minus m dot n 0. Then a S which is flux F s minus m dot s 0. And finally, you get a C which is summation over phases go over all the phases flux C f that will be m dot e 0 plus m dot w 0 plus m dot n 0 plus m dot s 0, so that is with minus sign a E plus a W plus a N plus a S plus m dot e m dot w m dot n plus m dot s. So, this essentially get to back a coefficient which you can think about NB cell capital a F, so that is how you get when you use the upwind scheme.

Now, for example, if someone uses quick scheme, so what happens to that. Now, this is for your first order upwind scheme. If someone uses quick scheme which is third order accurate in the system, then the discretized equation will look like a C phi C plus a E phi E plus a W phi W plus a E E phi E E plus a W W phi W W a N phi N, a S phi S, a NN phi NN plus a SS phi SS which I just said few minutes back since it is first order upwind in every direction, you just need three cells x and y cell.

If it is second order scheme or third order scheme like the quick, you need five cells in each direction. And if you have that, then you end up getting apart from these coefficients you get these two more and the y-directions these two more. So, you end up getting so many coefficients. And the coefficients would be computed like a E equals to flux F e which is minus 3 by 4 minus m dot e 0 plus 3 by 8 m dot e 0 minus 1 by 8 m dot w 0.

Then a W is similarly flux F w which you can write as minus 3 by 4 m w 0 plus 3 by 8 m w 0 minus 1 by 8 m e 0. And a N which will be flux F n which is again minus 3 by 4 m dot n 0 plus 3 by 8 m dot n 0 minus 1 by 8 m dot s 0. And a S is flux F s which is minus 3 by 4 minus m dot s 0 plus 3 by 8 m dot s 0 minus 1 by 8 m dot n 0.

And other terms like a E E which is going to be flux if ee which is 1 by 8 minus m dot e 0 then a WW which is flux F w w which is also 1 by 8 minus m dot w 0. And other coefficients like a NN which is flux f n n that is 1 by 8 minus m n n 0 and a SS flux F s which is 1 by 8 minus m dot s 0. So, finally, you get the term which is a c which will be summation of the fluxes of c f. So, a C is the summation over all the phases which is flux C f that is minus summation over F NB cell a F phi F or rather one can write a F plus m dot e plus m dot w plus m dot n plus m dot s. So, this is the term comes from the mass conservation. So, if you get these things and equation system if you look at it from the standard first order upwind scheme to the quick scheme, you get this excess term.

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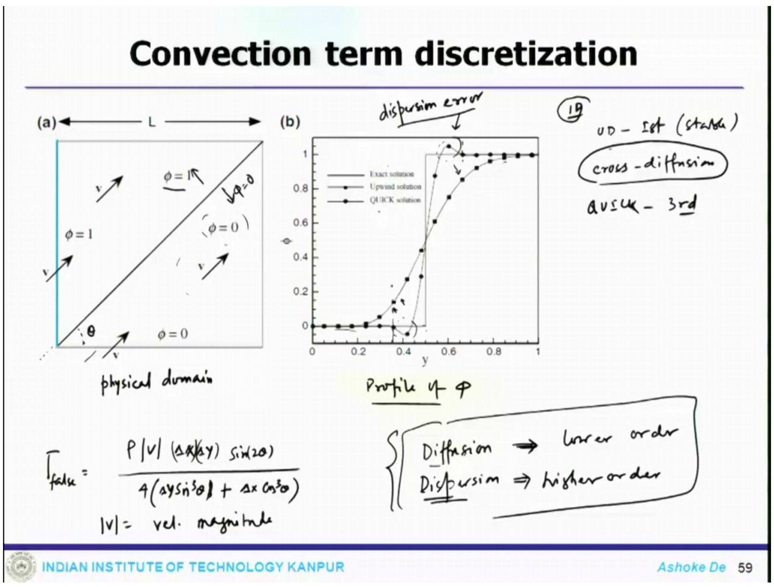

Now, if you look at both the discretized equation, there used for pure advection problem. So, the physical domain one can look at so this is the let us say the physical domain which is shown here, this is the physical domain ok. And this side is the profile of phi along the vertical centre line of the domain for pure advection problem. This is along central line in the y-directions.

So, physical problem if you see the velocity was oblique in the physical domain. Now, which represents that in this particular square domain the scalar phi which is being convicted in the field with. So, there is a velocity field underline flow field which is actually convecting the phi. So, the phi value one on the left hand side and the 0 is on the right hand side of the domain. If you do not have any diffusion, so then the phi equals to 1 would always be along the diagonal or along this line, because you do not have any diffusion, so from scalar value of these would not diffuse only it will convect along this line.

Now, here when you look at the profile and you compare with the exact solution along with the upwind scheme and the quick scheme. So, this comparison shows or lot of information which are very very important. So, due to the absence of diffusion, the solution exact solution phi equals to 1 above the diagonal and phi equals to 0 would be the below the diagonal. So, this is if you think about in the square domain along the diagonal as there is no diffusion so above that the all the solution will have phi equals to 1 in this portion and this portion all the solution will have phi equals to 0. And this is only convected from this side to this side due to these velocity that is why if you look at the exact solution phi 1 suddenly along the diagonal or the middle of the system it drops to 0 and then its stays 0. Because the lower side is all phi 0, upper side is all phi 1. Now, and some where it would be 0.5 and something.

Now, if you compare the exact solution with the profile generated by the upwind scheme which is essentially shows by this line, this is the line. And what you can see because of that accuracy because the upwind scheme is 1st order accurate though it is stable it is 1st order accurate. So, because of this accuracy and the solution shows such a smooth profile, but which is far away from the exact solution. So, it is also at the same time quite in accurate. And this inaccuracy is arising due to the cross diffusion term cross diffusion which is caused by the interpolation profile that is used.

So, now actually this cross diffusion term can be also applicable or it is always also applicable in multidimensional system and from we are looked at it though in the one dimensional, but whatever is happening here there is a one to one correspondence at the multidimensional system. Now, this will happen this cross diffusion when the velocity field is not aligned with the grid.

So, one can have an approximate expression for the cross diffusion term which was given by the I mean given in the literature in way such that you can write the false diffusion term can be estimated as rho v magnitude delta x delta y sin 2 theta divided by 4 delta y sin cube theta plus delta x cos theta cube. So, one can have an estimate like that where magnitude of v is the velocity magnitude and theta is the velocity vector is the angle made by the velocity vector with the x coordinate. So, essentially this one can think about theta so that is the theta.

Now, the error can be reduced by using higher order interpellation scheme which we have also demonstrated like quick; quick is third order scheme. So, we have also obtained the discretization using quick scheme. Now, the quick scheme profile what if you see is it shows quite accurate or rather quite close to the exact profile. So, except from this point and this point, the solutions are exactly matching with the analytical solution of the exact solution, but it is some locations here where you can see some over shoot at in the some where you see some under shoots.

And this is happening where the gradient actually exist because they are if you look at the exact solution this is this sharp gradient. So, when there is a gradient, this scheme actually showing some sort of an riddling or oscillations over shoot or under shoot or whatever. So, this is the error one can think about the dispersion error. Now, this dispersion error is coming now which causes this local maxima or minima in the solution. And this is one of the characteristics of the higher order resolution scheme.

Now, so that is what if you clearly see the differences which arises because of these different, different scheme. And this is a very good example when there is a sharp gradient what kind of error you obtained. So, there are two different errors which are sort of dominated in the system; one is the diffusion error or false diffusion another is the dispersion. So, one end this is more prominent in lower order scheme and the other end this is prominent in the higher order scheme.

So, higher order scheme actually reduces the diffusion, but it also becomes prone to the dispersion; lower order scheme actually increases the diffusion or rather reduces the dispersion, but it becomes highly diffusive in nature. So, this is always an problem that is one has to make some trade off between scheme this is related to stability; this is related to accuracy.

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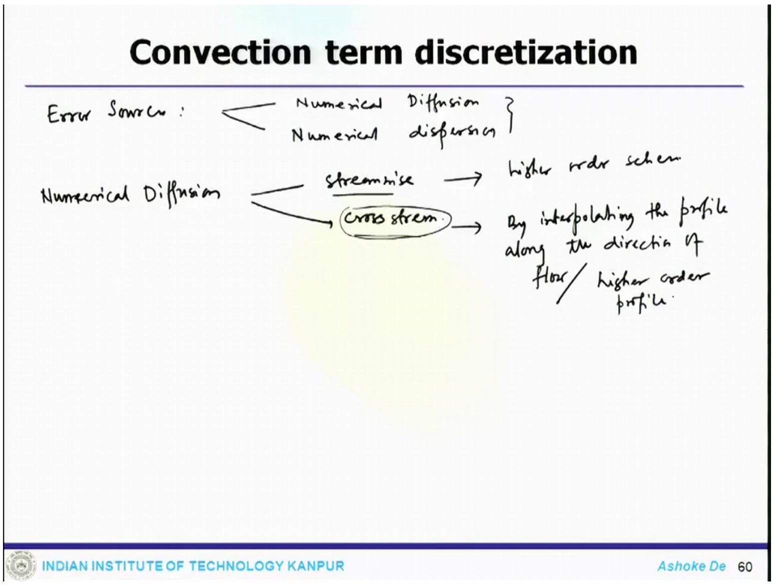

Now, if you look at the error sources, so the error sources if you want to look at it these are the two primary sources; one is the numerical diffusion; other case is the numerical dispersion. So, two source of errors which are now numerical diffusion. What it does is that it can be also divided into two systems so or two ways. Numerical diffusion one can think about in two ways it can possibly arise in the system. One is the stream wise other will be cross stream.

So, stream wise diffusion which will be along the normal direction; cross stream is along the. And this is the term which probably if you recall from your previous lectures when you are talking about the diffusion equation in non-orthogonal system unstructured system, we had some cross diffusion term and that essentially some sort of an correlation you can think about the cross stream diffusion. Now, stream wise numerical diffusion can be reduced by. So, this can be reduced by increasing the higher order scheme interpolation profile.

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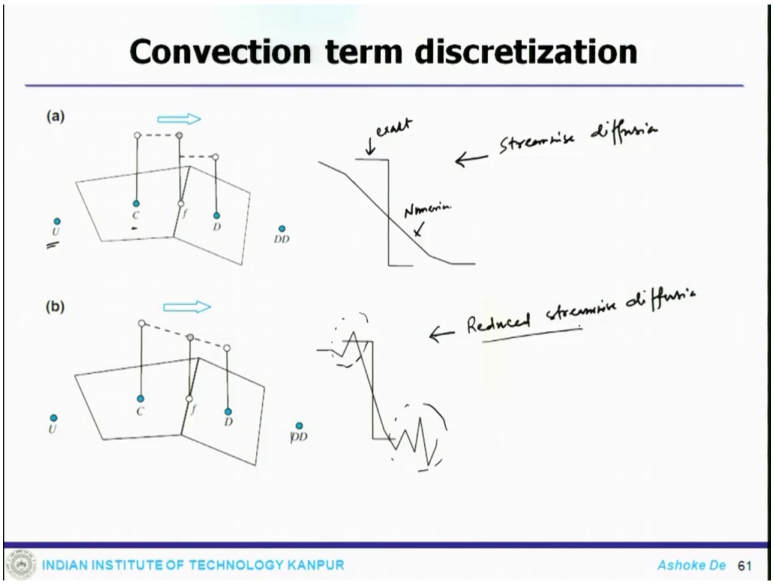

So, but what would happen that once you do that so one can see this is what happens, this is where the prime example of numerical stream wise diffusion. So, you have this flow field going in this direction, this is upstream, this is the cell centre. And exact this is the exact solution and this is the numerical solution. So, the stream wise numerical diffusion can be reduced by higher order scheme. So, this is where in this profile if you look at it, this is the reduced stream wise diffusion by using some sort of a higher order profile.

But as soon as you use a higher order profile you come across this kind of riggles or oscillation low amplitude oscillations in the solution, but with higher order profile the stream wise diffusion can be reduced. Now, the cross stream numerical diffusion which actually arises in term where you just we discussed in the this kind of a problem where the velocity gradient is aligned or with an angle theta. So, the cross stream diffusion actually by caused by the one-dimensional nature of the assumed profile and can be reduced either by interpreting in the direction of the flow.

So, the cross stream diffusion one can reduce by interpolating the profile along the direction of flow or by using or using higher order profile or scheme, higher order interpolation profile or scheme. So, we will stop here today and will take from here in the follow up lectures.

Thank you.