Introduction to Finite Volume Methods - II
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Lecture -19
High Resolution Schemes-II

So, welcome back to the lecture series of Finite Volume, and what we will continue our discussion where we left in the last lecture on the High Resolution Scheme.

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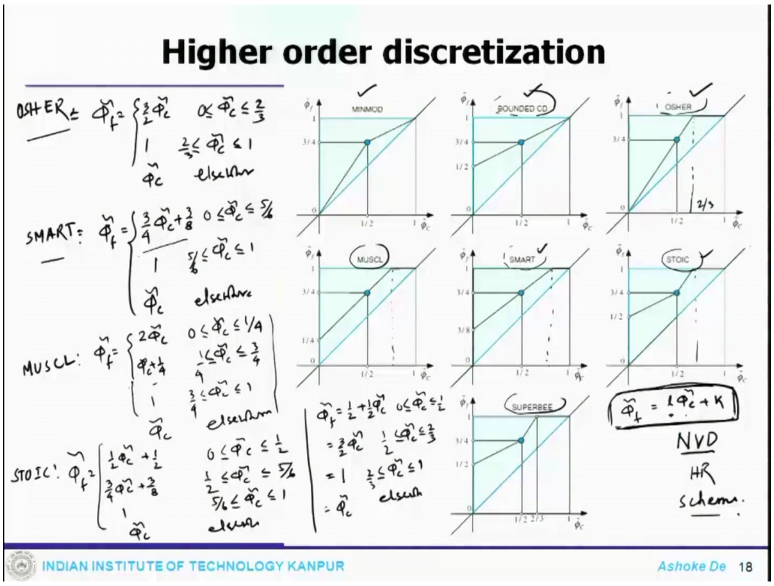

So, if you just quickly recall on the last lecture this is the invity diagram which we call it the normalized variable diagram. And we are looking at it the functional representation of the phase value using this kind of representation.

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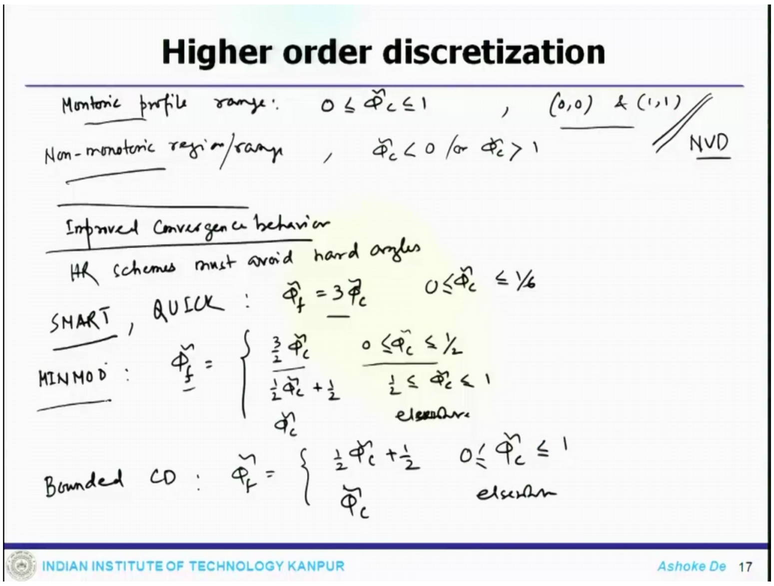

And we have already seen how the MINMOD scheme can be represented, how the bounded CD can be represented, then the OSHER scheme and SMART scheme. So, this is the representation of the MINMOD, this is bounded CD, this is OSHER, this is SMART.

Now, we can see how one can look at the MUSCL scheme. So, this MUSCL scheme says this phase value could be 2 phi C when phi C tilde lies between one-fourth and 0; that means, this will be between here and here, this is 2 phi C. So, this is the 2 phi C slope, and then it is phi C tilde plus 1 by 4 when phi C tilde lies between 3 by 4 and 1 by 4. So, between that it goes phi C plus 1 by 4, till 3 by 4 which could be this is the point which is 3 by 4, and then it is become 1 between 1 and 3 by 4 which is the constant value and other than that, that is elsewhere it is phi C. So, other than that it is phi C. So, the MUSCL scheme is kind of splitted on this normalised variable in 4 different segment. So, that is how you represent the MUSCL.

Now, you look at the STOIC S T O I C scheme. So, what it does? The STOIC says you use phi f when it is half phi tilde plus half when phi C tilde lies between half and 0; that means, 0 and half it is half plus phi C tilde, so this is the line. Then it is 3 by 4 phi by C by 3 by 8 if it is lies between 5 by 6 and half. So, half and 5 by 6 this is the line and after that it is 1 which is between 5 by 6 and 1 and elsewhere it is phi C. So, elsewhere this is 1. So, that is how the STOIC profile.

And SUPERBEE profile, which is this one, this as phi f equals to half plus half phi C tilde if phi C tilde lies between half and 0; that means, 0 to half it follows this profile then it follows 3 by 2 phi C tilde when phi C tilde is 2 by 3 and half 2 by 3 is this point. So, this follows that then it is one when phi C tilde is between 1 and 2 by 3 and it is phi C elsewhere it goes like that. So, these are the different schemes which are represented with this kind of. Now, there are few more like this SMART or STOIC or SUPERBEE, they are modified version.

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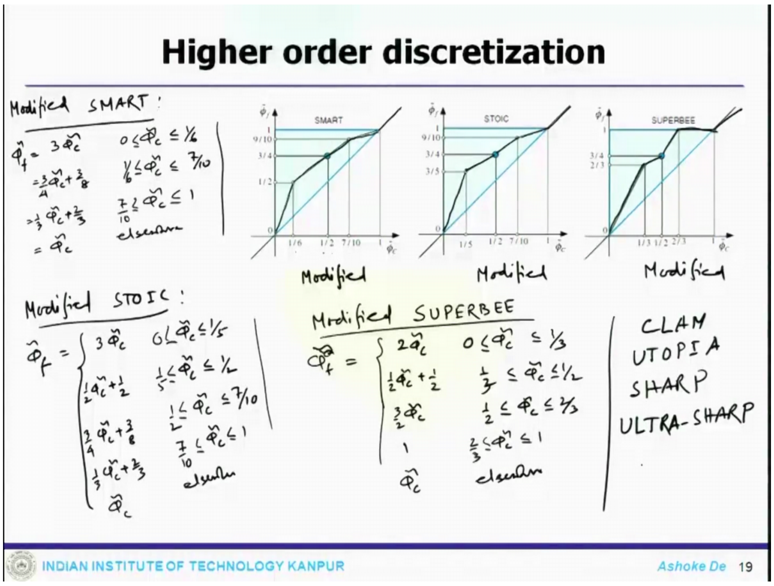

So, this is modified SMART, this is modified STOIC and this is modified SUPERBEE. And once they are modified, they are kind of having slightly involved profiles. So, first look at modified SMART profile what it says.

So, one can easily see from this figure that there is a profile which goes from 1 to 0 to 1 by 6, then there is a segment between 1 by 6 to half then half to 7 by 10 and then 7 by 10 to 1. So, exactly this is how the phi f is going to be. It is 3 phi C when phi C tilde is between 1 by 6 and 0, it is 3 by 4 phi C plus 3 by 8 when phi C tilde lies between 7 by 10 and 1 by 6, that is 1 by 6 and 7 by 10 and then 1 by 3 phi C plus 2 by 3 when this is lies between 1 and 7 by 10 and finally, phi C tilde elsewhere.

So, in between 7 by 10; so, one profile is this, another profile is this, another is this and finally, this that is what the modified SMART does. So, you split in four different segment. Similarly, how modified STOIC is splitted over. We said phi f equals to 3 phi

C, when phi C tilde lies between 1 by 5 and 0. So, 0 to 1 by 5 this is the profile. Now, when phi C tilde lies between half and 1 by 5, it is half phi C tilde plus half. So, it is 1 by 5 and half this is another profile, and then it is between 7 by 10 and half where it is 3 by 4 phi C plus 3 by 8. So, half and 7 by 10 this is the profile. And then finally, it between 1 and 7 by 10 which is one-third phi C plus two-third, so 7 by 10 to 1, is like that and then finally, it is phi C elsewhere. So, it goes like that. So, these are the different segment where you divide the profile.

And then, the one which remains is the modified SUPERBEE, so modified SUPERBEE which is phi f equals to I mean phi f tilde equals to 2 phi C if phi C tilde lies between one-third and 0. So that means, this goes with this profile 0 to one-third. And then half phi C tilde plus half which goes from one-third phi C tilde less than half. So, one-third to half this is the second profile and then it follows 3 by 2 phi C which is half phi C tilde less than 2 by 3 half to 2 by 3 and then one where it goes between 1 to 2 by 3 it stays there and then elsewhere like that, elsewhere phi C. So, that is how.

Now, there are some other methodology which are also exist in the literature like CLAM C L A M, UTOPIA, SHARP, ULTRA-SHARP. So, these are also different different schemes one can have a different profile based on this MBD and MBF.

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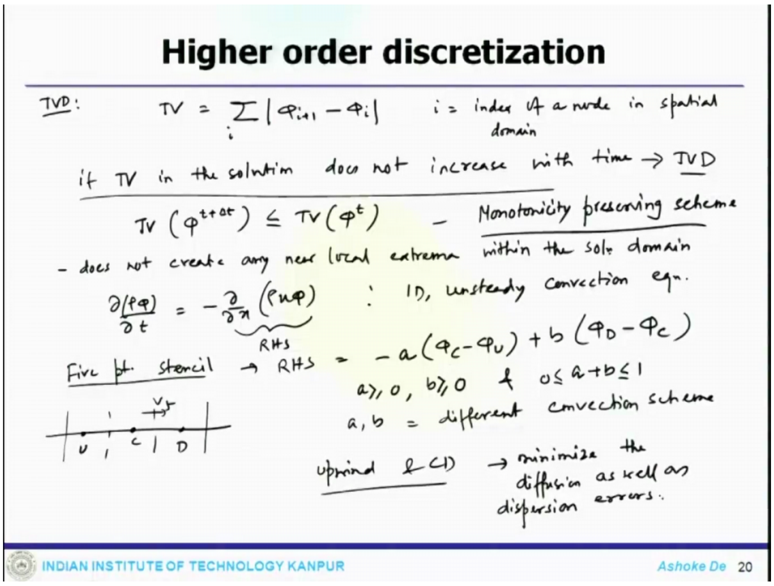

So, that gives you the platform or provide us the platform to start developing the TVD kind of framework. So, once I will come to the TVD framework, which means the total variational diminishing framework. So, what it does that in solving? Numerically, the advection partial differential equation for a variable phi if the total variation of a variable phi should vanish between phi i plus 1 minus phi I, where i represent the index of a node in spatial domain.

So, the total in numerical method can be set as a TVD scheme, if TV total variation in the solution does not increase with time. So, that is the then the scheme can be said total variational diminishing scheme, TVD scheme. So, mathematically if one has to represent that. So, it means the total variation of phi t plus delta t should be less than TV phi at t. So, the total variation at t-th time step should be higher than the next time level. So, this is a monotonically or monotonicity, preserving mono tonicity preserving scheme. TVD schemes are all monotonicity preserving scheme, unless the monotonicity is preserved this actually lose out the property of the higher order resolution with stability.

So that means, monotonicity preserving scheme which says that it does not create any new local extrema within the solution domain, within the solution domain that means, in other way one can say that the value of a local minimum is non-decreasing, and the value of local maximum is non-increasing. So, one now we can actually discuss the methodology for this kind of or construction of that TVD scheme; so, to start with we can start the unsteady one dimensional convection equation which is rho U phi, this is on the so this is 1 D unsteady conviction equation.

So, we start with that. So, once we start with that then general discritized form for the right-hand side you use some sort of an 5 point stencil. So, that you can derive an higher order scheme for the right hand side and that discritized equation for the right hand side one can write minus a phi C minus phi U plus b phi D minus phi C, where U C, D C is the cell which is concerned U is the upstream cell, D is the downstream node, ok. Now, one can see this if this is C and the velocity direction is this is upstream, this is downstream. So, that is how it is represented.

Now, for this condition for this TVD scheme or monotonies for coefficient for unit mass polarate to satisfy couple of inequalities, one is that a greater than 0 and b greater than 0, and a plus b must be less than 1 and greater than 0, where a and b these are adopted from different convection scheme. So, these are adopted from different convection scheme. Now, you can refer back to the convection scheme that we have just discussed. So, one can find that the first order upwind scheme is very diffusive while the second order central different scheme is highly dispersive.

So, the need for this is that we want to lie between somewhere where both the flavour of upwind and central differencing scheme they are preserved. So, that we can minimize, minimize the diffusion as well as dispersion errors.

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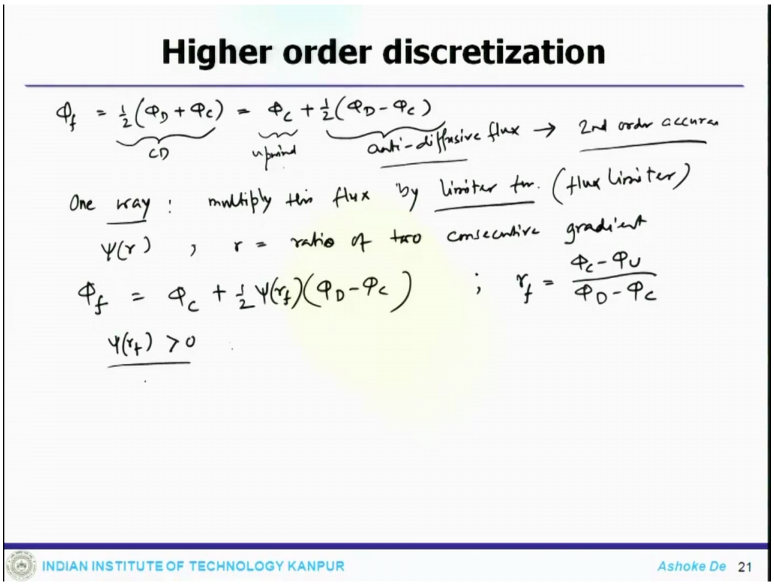

So, what can construct scheme such that the phase value can be constructed like half of phi D plus phi C equals to phi C plus half of phi D minus phi C. So, this is your CD, this is your upwind and this is your sort of anti diffusive flux. So, one can like that; so, CD and all these are the standard notation of the upstream and downstream point. Now, the central different scheme can be written in sum of this upwind and this anti diffusive flux.

Now, the flux is desirable as it makes the scheme second order; so, this makes it second order accurate. Now, the side effect is the unphysical oscillation it prevents due to the decrease in numerical diffusion. So, one has to have a better approach where the portion of this anti diffusive flux is added to this upwind scheme in such a way that second order accuracy is maintained, but at the same time you get less dispersion and less diffusion error.

So, that is why one way to do that, one way to do that is that to multiply this flux or this anti diffusion flux by a limiter function. So, also called the flux limiter or something like that, also called flux limiter in this. So, once you do that that will prevent excessive use in the region where oscillation might occur, while maximizing its contribution in the smooth areas.

So, one can denote such limiter like psi r, r is usually taken the ratio of ratio of 2 consecutive gradients, and phi f can be calculated as phi C plus half of psi r f phi D minus phi C, where r f is phi C minus phi U and phi D minus phi C. So, U is the upstream node D is the downstream node and C is the node which we are concerned.

Now, in order to preserve the sign of anti diffusive flux this psi r f is always greater than 0. So, now, one can develop a TVD scheme which reduces to finding limiters that will make the numerical scheme activity based scheme or monotonicity preserving scheme. So, the condition that this limiters have to satisfy in order to have a conviction scheme to be monotonicity preserving is that the how we can get to get in different different scheme.

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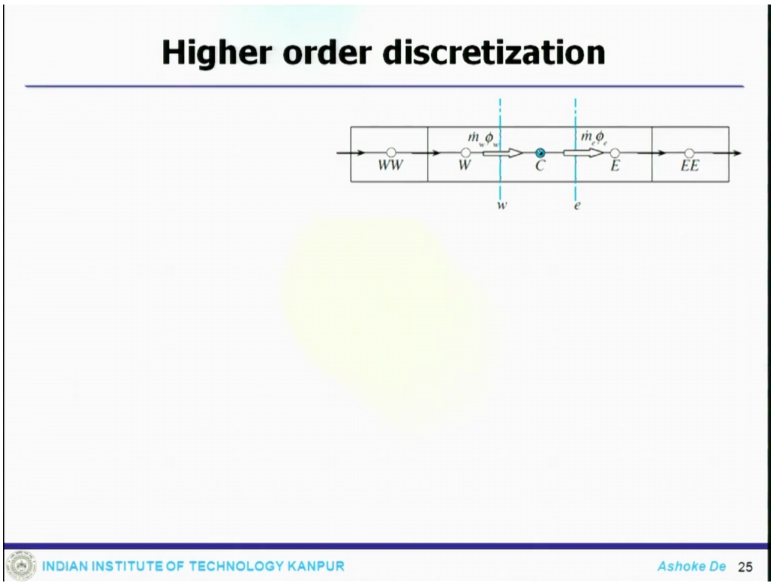

Now, we can write like you consider this one-dimensional, consider this one-dimensional stencil, where this is the cell which are concerned.

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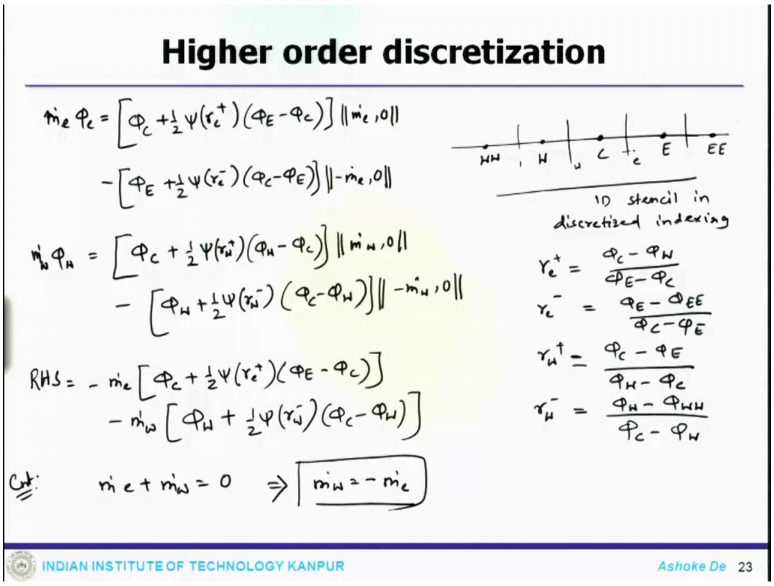

And then you have a West East, so that is typically a stencil of one dimension where you have a node C, this is East, this is essentially you are up downstream this is West that is standard discritized system. And now if one has to write this phase is the east phase, this is the W phase m dot e phi e, then one has to write phi C plus half of psi that is in one-dimensional 1 D stencil in discritized indexing. So, r e plus phi E minus phi C multiplied with m dot e 0 minus phi E plus half psi r e minus phi C minus phi E multiplied with m dot e 0. And similarly, m dot w phi W equals to phi C plus half of psi r w plus phi W minus phi C which is m w 0 minus phi W plus half of psi r w minus phi C minus phi W minus m dot w 0 like that, where r e plus is the ratio of the gradients; at the east phase the positive sides.

So, it should be phi C minus phi w divided by phi e minus phi C; r e minus is phi E minus this is EE and this is WW minus phi EE divided by phi C minus phi r w plus is phi C minus phi E divided by phi W minus phi C and r w minus equals to that will take care of this and this node. So, it will become phi W minus phi WW, phi C minus phi W. Now, the simply the derivation which one can actually rearrange this and the right-hand side will become that minus m dot e which phi C plus half psi r e plus phi E minus phi C minus m dot w phi W plus half psi r w minus phi C minus phi W. While the continuity equation gives m dot e plus m dot w equals to 0; that means, m dot w equals to minus m dot e. So, that is from the continuity one can get from the mass conservation.

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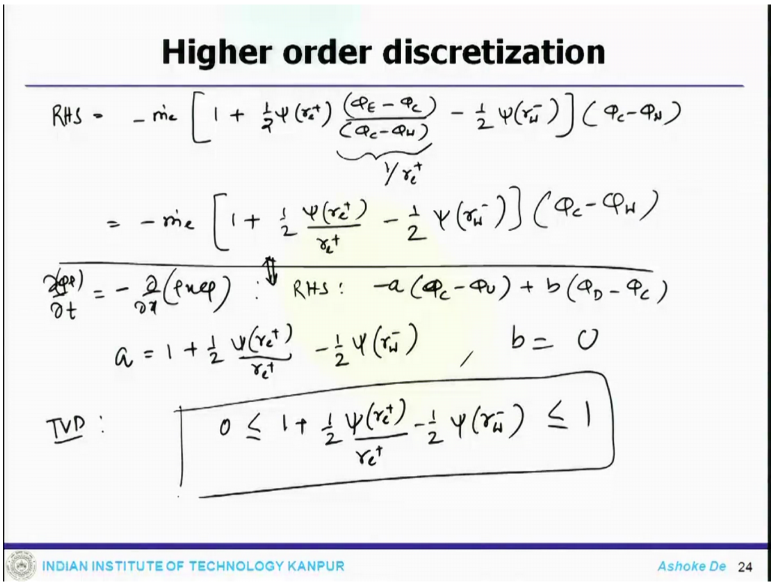

Now, once we apply this what we can write this right-hand side term is that minus m dot e, 1 plus half psi r e plus phi E minus phi C divided by phi C minus phi W minus half psi r w minus multiplied with phi C minus phi W. This one actually 1 by r e plus. So, if you write m dot e 1 plus half psi r e plus by r e plus minus half psi r e w minus multiplied with phi C minus phi W.

So, once you comparing this particular equation with the; so this is the equation and the we started with a equation where del phi by del rho phi by del t equals to minus del del x of rho u phi. So, when you compare these two equation what one can say, that which was the written as from there, the right hand side which was written at minus a phi C minus phi U plus b phi D minus phi C.

So, once this two right hand side, these two are compared one can say a is half plus half psi r e plus by r e plus minus half psi r e w minus and b is 0. So, for this scheme to be a TVD based scheme, what it has to have 0 1 plus half psi r e plus by r e plus minus half psi r w minus less than 1. So, to have the scheme to be a TVD based scheme like this which if you expand; so, this was that stencil we are talking about.

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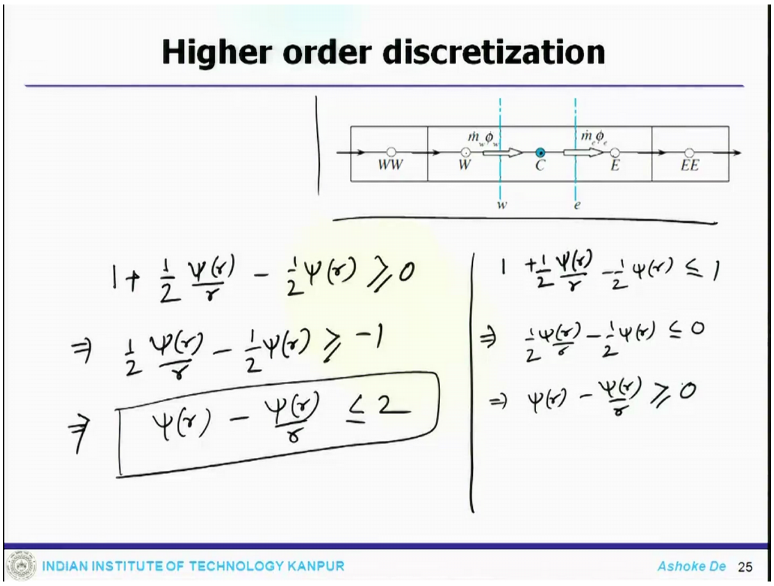

Now, if we expand that one what it shows is that 1; so, what it shows one plus half psi r by r minus half of psi r greater than equals to 0 which means half of psi r by r minus half of psi r greater than equals to minus 1 that leads to psi r minus psi r by r less than equals to 2. So, that is one system that you get.

And the other one that one can get is that one plus half psi r by r minus half psi r less than equals to 1; that means, half psi r by r minus r less than 0 psi r minus psi r by r greater than 0. Now, if you combine these two, it simplifies to psi r minus psi r by r, they 2, and this side it is 0.

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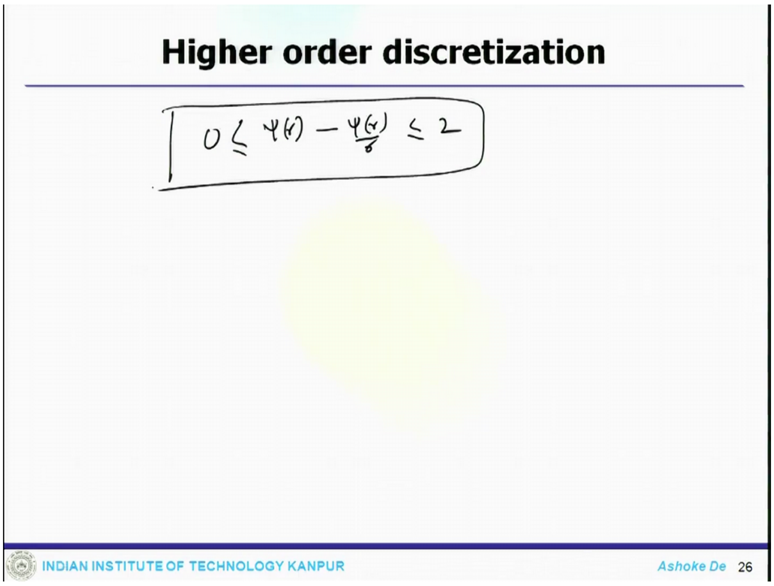

So, we will stop here, and we will continue the discussion in the next lecture.

Thank you.