Introduction to Finite Volume Methods-II
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Lecture – 02
Linear solvers – II

So, welcome to the lectures of this Finite Volume Method. Now, if you put these things back in the algorithm what will happen?

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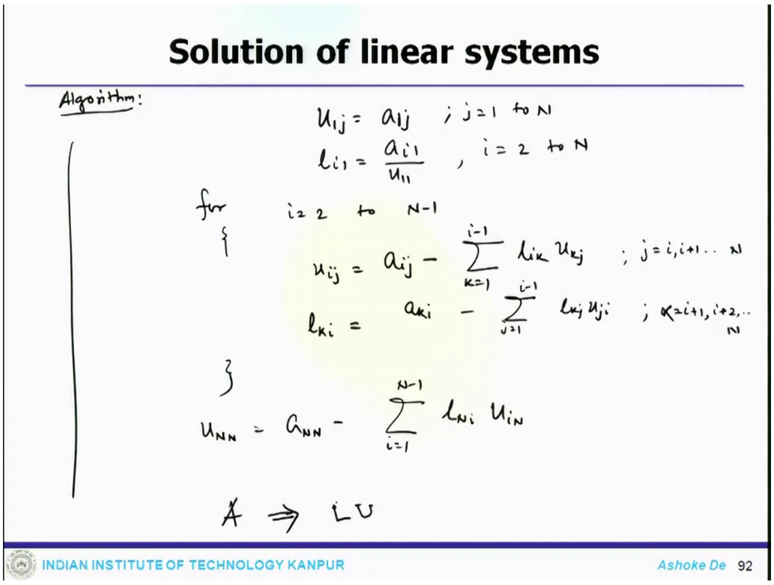

So, first thing that I will have u 1 j equals to a 1 j for which j goes from 1 to N; then I get l i 1 equals to a i 1 by u 1 1 where, i goes from 2 to N. So, these two elements initially are being computed and then you go over loop for i goes 2 to N minus 1. Then u i j equals to a i j minus k equal to 1 to i minus 1 l i k u k j where, j goes from i i plus 1 so on N. And, the lower triangular system is a k i minus summation of j equals to 1 to i minus 1 l k j u j i where k goes from i plus 1 i plus 2 so on N.

So, that is were the loop closes and finally, you get u N N equals to a N; N minus i goes to N minus 1 l N i u i N. So, that is how the whole algorithm works. Then you obtain from A getting decomposed into 2 component lower and upper triangular system and once you get that then you have to get the solution for that. So, now the steps which are required is the substitution step ok.

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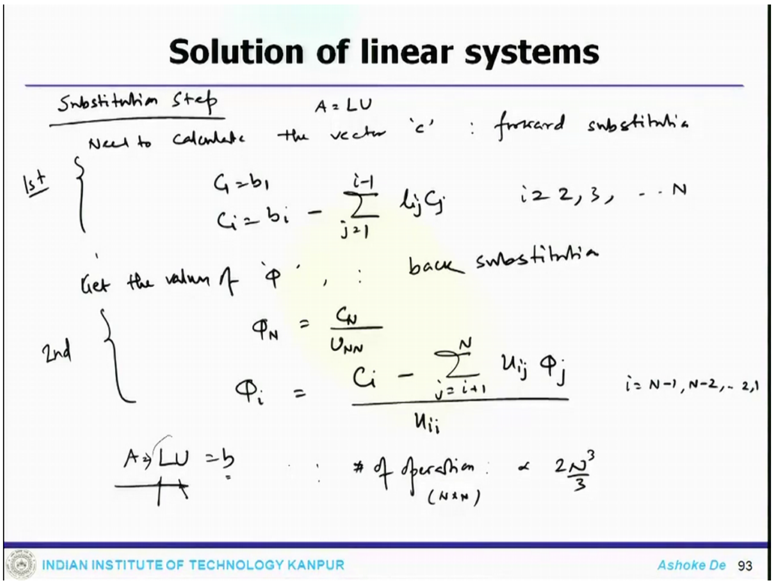

So, now my A got into L and U, one is lower and upper triangular system and first need to calculate the vector c. So, that through forward substitution process where, you get C 1 equals to b 1 and C i equals to b i minus j equals to 1 to i minus 1 l i j c j where, goes to 2 3 to N. So, that is where you get the and the next step you get the values of phi by back substitution process, which will be phi N equals to C N divided by U N N. And, phi i equals to C i minus j equals to i plus 1 to N u i j phi j divided by u i i. So, where i goes N minus 1 N minus 2 to 2 1 like that.

So, initially 1st step this is your first step where, you get the forward substitution to get the vector C and the 2nd step you do the backward substitution to get the. So, essentially you had A equals to LU and where this was solved for b and it was written LU c equals to b and from there you get the solution Now, that important points are few here like the elements of this L and U can be directly stored in the matrix A; if it is no longer needed. This is because that A are only needed when the corresponding elements of either L or U are calculated.

So, the number of operation required to perform here, the number of operation which were required to perform for LU factorization for a system of N by N is 2 something N cube by 3. So, which is double the operation that is required to solve through the gauss elimination process again the advantage of this LU factorization is that when the same matrix A applies to different b, then know need to calculate L and U factorizations again and again, the coefficients just can be directly applied to. So, which makes the solver more efficient.

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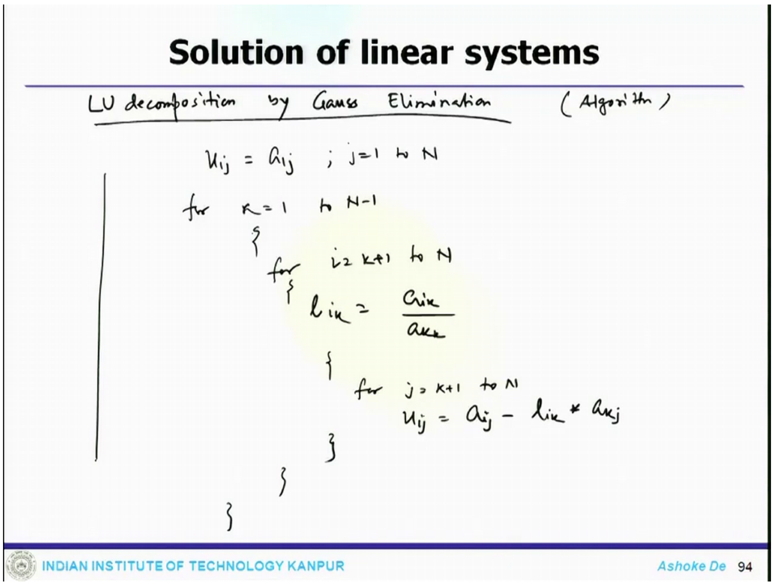

Now, one can obtained this LU factorization through Gauss elimination process. So, LU decomposition by Gauss eliminationthat is another way of obtaining the thing and so, this is sometimes can be also used in some CFD codes. Now, if you look at that algorithm, how you do that you have simple u 1 j equals to a 1 j where, j goes from 1 to N. Now, you go over loop for k equals to 1 to N minus 1 you obtain i equals to k plus 1 to N where, l i k equals to a i k divided by a k k..

And, then here you start then you get for j goes from k plus 1 to N u i j equals to a i j minus l i k multiplied with a k j. Then you closes that loop which will get you this information and then the upper one and the last one. So, that is where the Gauss elimination process can be also used. Now, the important points. So, this is possible all these Gauss elimination or LU factorizations when you have generic.

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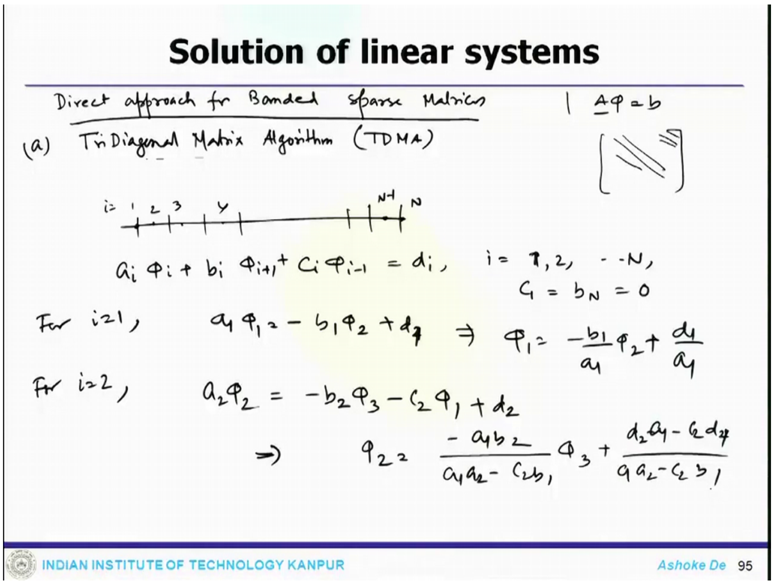

Now, one can look at this method for or other the direct approach for banded sparse matrices. So, this should be very specific case because, what we have discuss so for is in more generic approach because irrespective of what is the property of A; we did the Gauss elimination or we could do the LU factorization. Now, when you say that the banded sparse matrices; that means, the matrix A which is sitting here it could be looking either some sort of a band like that or some sort of a band like that. Or so, the solvers which can be used for this case cannot be generalized for the elimination process.

In this category the first one which we will talk about is the Tri Diagonal Matrix Algorithm which is called TDMA. And, what happens this algorithm is known as the Thomas algorithm and it is a very common and old algorithm which was I mean or other often discussed in any CFD, but it has some specific application for a particular banded system like tridiagonal system; this is one of the efficient solver. How you get it? Let us say you consider this cell like that starts from i you have 2 1 3 4 these are the points and so on you get N N minus 1. So, number of points N and then the cell centers value.

So, the coefficient matrix for this kind of system if one has to write a i phi i plus b i phi i plus 1 c i phi i minus 1 equals to d i, where i goes from 1 2 dot dot on N c 1 0. So, what one can do for i equals to 1 you can directly solve this equation a 1 phi 1 equals to minus b 1 phi 2 plus d d 1 which you can see phi 1 can be solved minus b 1 by a 1 by phi 2 plus d 1 by a 1.

Similarly, for i 2 if one has to write you will get a 2 phi 2 equals to minus b 2 phi 3 c 2 phi 1 d 2 which will get you the equation for phi 2 which is minus a 1 b 2 a 1 a 2 minus c 2 b 1 into phi 3 plus d 2 a 1 minus c 2 d 2 d 1 divided by a 1 a 2 c 2 b 1.

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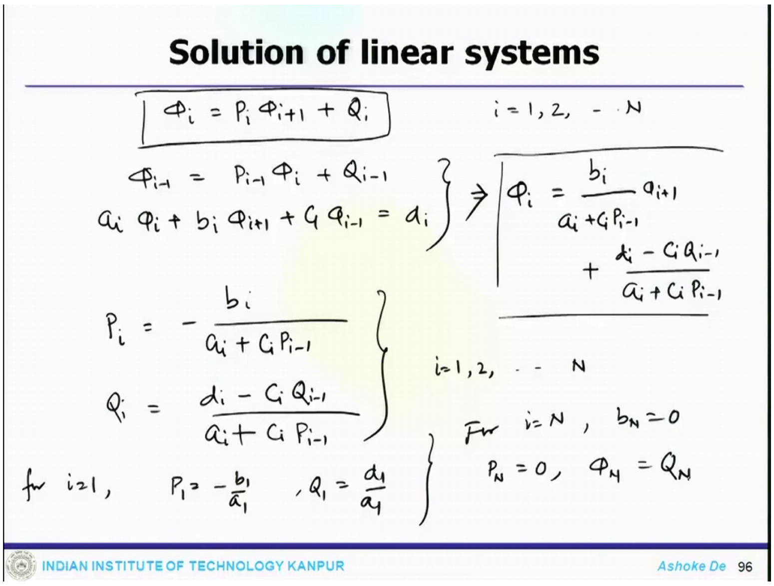

So, if one generalize this then one can write phi i equals to P i phi i plus 1 plus Q i. So, that is the generic formula for any i, where i goes from 1 to 2 N. Now, if you combine with the equation i plus 1 then phi i minus 1 equals to P i minus 1 plus Q i minus 1 a i phi i plus b i phi i plus 1 plus c i phi i minus 1 equals to d i. So, from here you get phi i equals to b i a i plus c i P i minus 1 phi i plus 1 plus d i minus c i Q i minus 1 divided by a i plus. So, that is what you get. Now, if you compare this one and the previous one you can see that P i is nothing, but b i divided by plus a i plus c i P i minus 1.

And, Q i is d i minus c i Q i minus 1 by a i [mi/minus] plus c i P i minus 1. In both these cases i goes from 1 2 to N. Now, for i equals to 1 you can get P 1 equals to minus b 1 by a 1 and Q 1 equals to d 1 by a 1. And, similarly for i equals to N you get since, b N equals to 0 you get P N equals to 0 phi N equals to Q N. So, that is how the algorithm works actually for a system like that you can, if you put in the algorithm then it would be quite handy for this kind of banded system.

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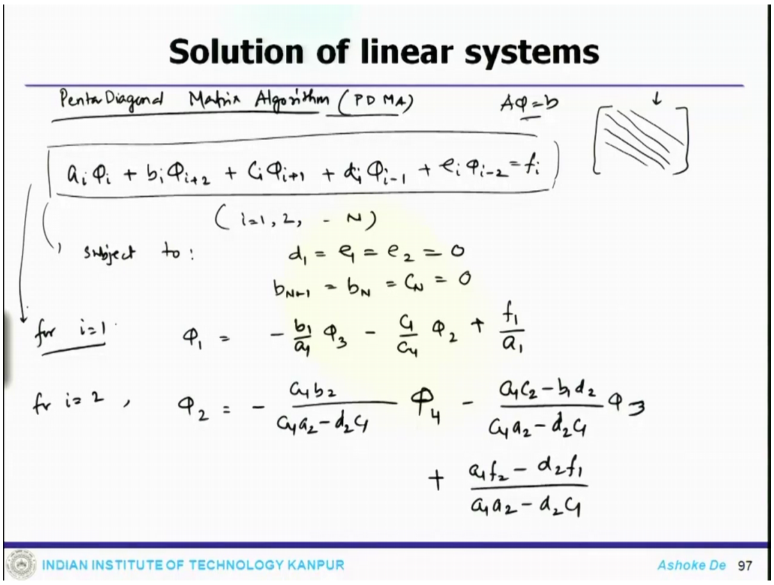

Now, the second one which could be also of interesting is the penta diagonal Penta Diagonal Matrix Algorithm, which actually talks about PDMA. Previous one was the TDMA, this case call the PDMA and again this one you get; see the TDMA one can obtain if somebody discretize the diffusion equation in 1 dimension. Now, when someone discretize the diffusion equation in 2 dimensional system, he gets essentially this kind of a penta diagonal system; that means, not 3 you have 5 banded system. So, this is called the penta diagonal system and in 2 dimensional the diffusion equation discretization can lead to this kind of situation. Now, once you get this kind of PDMA so, you can have a very specific solver which can actually be used for this kind of solution.

And, if one uses the notation very standard notation and like the one which we have used here, the i goes from 1 to N then the equation system for penta diagonal system would be a i phi i plus b i phi i plus 2 plus c i phi i plus 1 plus d i phi i minus 1 plus e i i minus 2 equals to f i, where i goes 1 2 to N. If you see the base variable at ith it is equally distributed between i plus 1 i minus 1 i plus 2 i minus 2 and there are coefficients associated with it.

So, this is the equation which has the 5 points i i plus 1 i plus 2 i minus 1 i minus 2 that leads to a penta diagonal system. This is called the 5 points tensile and which is this particular one is subjected to d 1 equals to e 1 equals to e 2 equals to 0 and b N minus 1 equals to b N equals to c N equals to also 0. So, it has very specific system. Now, if you write for i equals to 1, this particular 1 it gives you back phi 1 equals to minus b 1 by a 1 phi 3 minus c 1 by a 1 phi 2 plus f 1 by a 1.

Now, similarly for i equals to 2 the phi 2 can be obtained minus a 1 b 2 divided by a 1 a 2 minus d 2 c 1 minus. So, this is multiplied by phi 4 minus a 1 c 2 minus b 1 d 2 divided by a 1 a 2 minus d 2 c 1 phi 3 plus a 1 f 2 minus d 2 f 1 by a 1 a 2 minus b 2 c 1. So, this essentially you put the values of i in this equation and then you start obtaining individual once. So, this way one can proceed then you can write an generic expression for phi.

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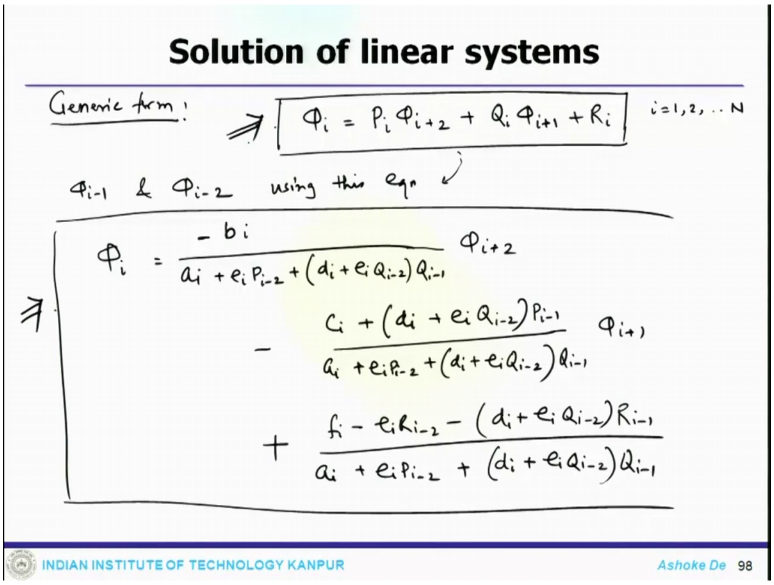

So, in generalized form or rather generic form one can write that phi i equals to P i phi i plus 2 plus Q i phi i plus 1 plus R i. So, which goes for a loop i equals to 1 2 to N. Now, you compute phi i minus 1 and phi i minus 2 using this equation. So, essentially using this you compute that and then from there you obtain phi i i equals to it would be b i a i minus. So, with minus sin sitting there a i plus e i P i minus 2 plus d i plus e i Q i minus 2 Q i minus 1 phi i plus 2. So, that is the first term.

And, the second term would be minus c i plus d i plus e i Q i minus 2 into P i minus 1 divided by a i plus e i P i minus 2 plus d i e i Q i minus 2 Q i minus 1 into i plus 1 plus f i minus e i R i minus 2 minus d i plus e i Q i minus 2 R i minus 1 divided by a i plus e i P i minus 2 plus d i e i Q i minus 2 Q i minus 1. So, you get an generic expression for phi i or the ith element of that component. Now, if you compare this one with this equation.

So, this is one equation for phi i and here is an expression which we have written the generic expression.

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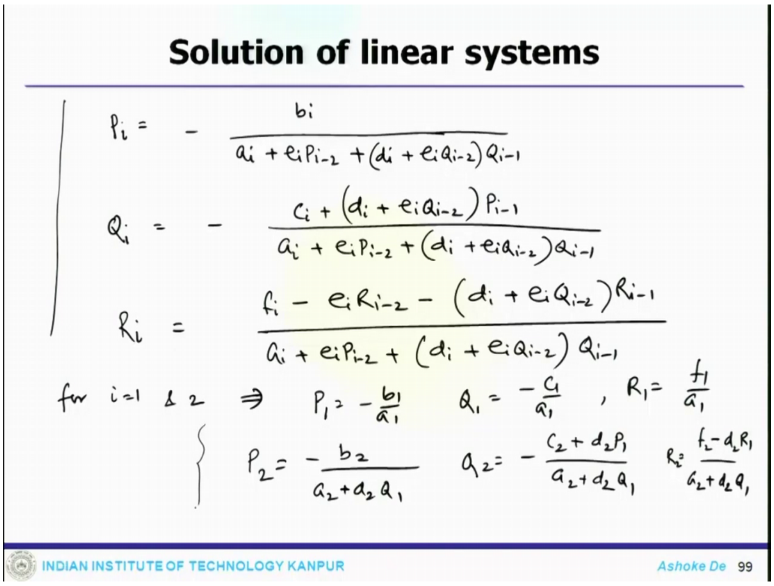

And, if you compare them together you get P i is essentially nothing, but minus b i by a i plus e i P i minus 2 plus d i Q i minus 1. Similarly, one can write Q i equals to minus it is c i plus d i e i Q i minus 2 P i minus 1. And, in the denominator you have the same expression e i P i minus 2 plus d i plus e i Q i minus 2 Q i minus 1. And, the last one is the R i which is f i minus e i R minus 2 d i plus e i Q i minus 2 into R i minus 1. So, and the denominator is the same a i plus e i plus d i Q i minus 1.

Now, for i equals to 1 and 2 which can get you the specific expression like P 1 equals to minus b 1 by a 1, Q 1 equals to minus c 1 by a 1, R 1 equals to f 1 by a 1. Same thing you can get P 2 equals to minus b 2 divided by a 2 plus d 2 Q 1 Q 2 equals to minus c 2 plus d 2 P 1 and R 2 equals to f 2 minus d 2 R 1 divided by a 2 plus d 2 Q 1. So, in this particular expression, if you put i equals to 1 or 2 you can get back this specific numbers.

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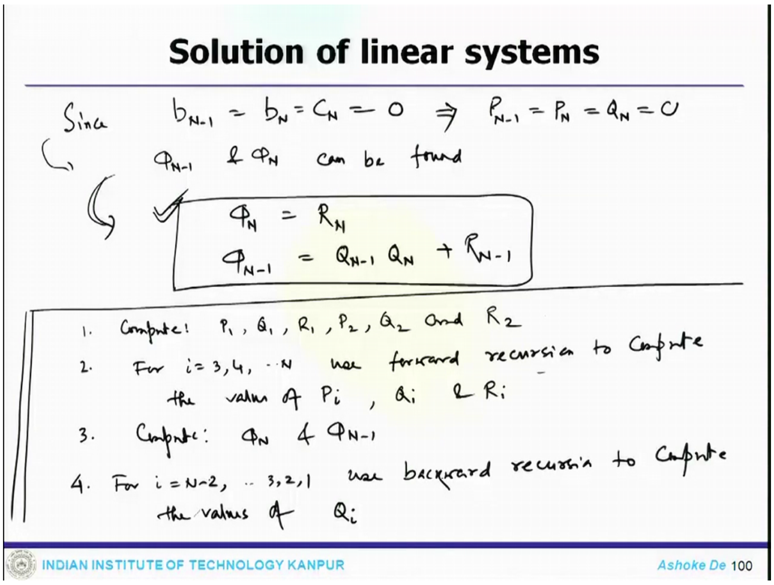

And, since you have b N minus 1 equals to b N equals to c N equals to 0 which will actually lead to P N minus 1 P N equals to Q N equals to 0. So, from here one can find out phi N minus 1 and phi N can be found using this information. And what could be that, once you get this two phi N equals to R N and phi N minus 1 equals to Q N minus 1 Q N plus R N minus 1. So, the specific once one can compute.

So, all these expressions if you put it back in the system then you can get the. So, the compute algorithm if somebody says you first compute P 1 Q 1 R 1 P 2 Q 2 and R 2 and then what you do for i goes from 3 4 to N use forward recursion to compute the values of P i Q i and R i. Then you compute phi N and phi N minus 1 and then finally, for i N minus 2 to 3 to 1 you use backward recursion to compute the values of Q i. So, that is the essentially complete penta diagonal algorithm.

So, what you get first you obtained this specific one P 1 Q 1 R 1 P 2 all these expression and then you go for the forward recursion. And, once you go for the forward recursion as we have obtained here all these P i Q i R i you obtain. And, once you obtain all these you go and calculate the phi N and phi N and phi N minus 1 using this and then finally, you go for the backward recursion to compute the value of Q i. So, this is this completes the algorithm for the penta diagonal. So, these are the discussion on direct approach. Now, in the next lecture we carry on the iterative process.

Thank you.