Introduction to Finite Volume Methods-II

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Lecture – 31
Fluid Flow Computation: Incompressible Flows -I

So welcome back to the lecture series of Finite Volume and today we are going to discuss an important system that is the Navier stokes system. And what we have done if you just quickly recall. We have looked at diffusions term individually it is discretization process linear then solution of the linear solver whether its a direct approach and iterative approach. Then we have looked at the convection diffusion system and when talking about the convection a diffusion system we looked at higher order scheme or high resolution scheme. And then after that we have done the discussion on unsteady diffusion system.

Then after doing all these we have touched upon or had some discussion on few important other and or rather relevant topics like the source term discretizations or linearization of the source term and how that impacts the convergence of the linear solver. Then we looked at the under relaxation factor and how to implement the under relaxation for the linear system. And while the requirement of the under relaxation is there it can be also I mean implemented implicitly or explicitly. And both the approaches have their pros and cons, but it is nevertheless one may need or most of the time you need the under relaxation, because your realistic problems that you are dealing with their of having a stiffness in the linear solver.

Then we looked at the criteria for how to define the rate of convergence and in that case we looked at the residual approach where you can compute your absolute residual, you could sometime calculate the maximum residual, maybe the RMS of residual or more importantly if you have a multivariable system like you have a pressure velocity, then you have scalar transport or multispecies multiphase system that kind of situation it is very difficult to look at a particular variable and defining your convergence criteria.

So, in that case what is important that one has to look at individual variable separately or which may not be a computational efficient instead of doing that a good idea to approach towards that kind of system is to define some kind of a scale residual and that also we have looked at it. And that is very much pertinent to a system when you have multiple variables. And now today we are going to in this particular lecture discuss on the fluid flow system which is essentially governed by your Navier stoke solver.

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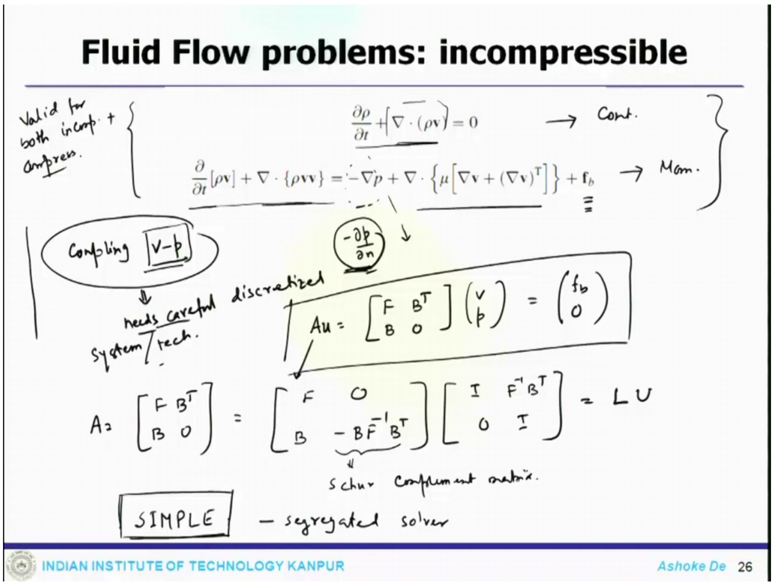

Now, if you look at your equations, these are your set of equations and one is that. So, first thing that we will start doing the discussion on the incompressible solver. And then once we are done with the incompressible system and this the definition comes, because incompressible or compressible dependent on the mach number which is a very standard notation of the fluid flow problems where one can define whether the flow is compressible or incompressible. But other way you can think that in a incompressible system density does not vary too much in the domain whether it is specially or temporally.

So, that remains pretty much constant and so, this guy goes away and you end up getting a only this much for a continuity equation, but these equation is valid for both incompressible and compressible. So, these has nothing to do with, but when you comes down to the incompressible system, in that case this boils down to this much the continuity equation. And the momentum equation this is your continuity equation and this is your momentum equation and there is a source term which is associated with your momentum equation.

And now, if you look at this particular system. So, what we have done the discussion or finished our discussion looking at a unsteady term, looking at a convection system. This is a diffusion term or diffusion system with source term, but what we have not talked about this pressure gradient. So, there is a pressure gradient term. So, del p by del n, one can think about depending on the phase this will lead to either del p del x del p del y or del p del z these term we have not looked at it.

So, this is a gradient term, now in addition to that not only the discretization of that particular term, the thing which is associated with this kind of fluid flow problem is the coupling, coupling between pressure and velocity. So, these brings the completely new paradigm shift in solving the Navier stokes solve problem, I means as long as you did not have or if you do not have the pressure velocity sitting in a system you do not have that much of problem which we have already solved.

I mean that is a if you drop out this particular term then you get only a unsteady convection diffusion system where you have a convection term which dominates the flow field or the diffusion term depending on the problem statement, but as soon as you bring in this pressure gradient term in place that adds to the complexity of the whole system.

And that is where one has to be extremely careful number 1, number 2 this coupling between pressure and velocity needs to be handled. So, these needs very careful needs careful discretizations of the numerical technique to be handle discretize needs careful discretize system or technique. So, what will be doing primarily, we will be looking at this whole system connected with continuity and momentum and try to see how one can device and get a system where the pressure velocity coupling is handle and you get an discretized equation.

As we move along with the discussion, it will get more and more complicated, but as we have done throughout our lecture series, we will do that step by step show that one can grab the idea, but again let me put a note here this is the important part of the whole thing that if one can easily handle this then he can have a good shape decode which can handle fluid flow problem. And then later on I can add to any other scalar transport equation or anything. Now having said that if you look at these two equations and review it while the velocity fill is computed using the momentum equation the pressure field is appearing here in the momentum equation cannot be directly computed, but from the continuity and these.

So, which actually yield a sort of a implicit coupling between these two and one can write a system like this let say A u equals to F B transpose B 0 v p equals to f b 0. And this particular format or form, it shows a 0 diagonal block in the system which is a characteristics of the saddle point problems which can say that it cannot sustain the solution of the pressure and velocity fields by any iterative means. So, one approach to simply reformulate the system of momentum and continuity equation by decomposing A into lower and upper triangular system.

So, my A is here F B transpose B 0 which can be decompose like F 0 B minus B F inverse B transpose which is a lower triangular system. And then other side one can write I F inverse B transpose 0, I which is a upper triangular system. So, it is a L U where this one is known as Schur complement matrix. So, this is known as complement matrix. Now this is the expense of this approach that needs to be followed to iteratively solve the Navier stokes equation.

So, this kind of technique is embedded in classical segregated solver segregated solvent means where which is known as a very famous solver called simple and again it was proposed by Patankar and Spalding.

So, this is a very- very famous algorithm in finite volume context which stands for semi implicit method for pressure linked equation. So, where it is known as it is a segregated solver or in the sense segregated solver where you decouple the pressure from the velocity. And then you solve momentum equation with certain initial guess values and correct it and then look at the pressure corrections from the continuity equation through the guess values or correct values to conserve the mass conservation. So, what happens that a algorithm which is define this kind of iterative procedure where the momentum equation is solved.

And then without the pressure, then pressure is corrected through the continuity equation, get pressure corrections equation and follow the mass conservation system.

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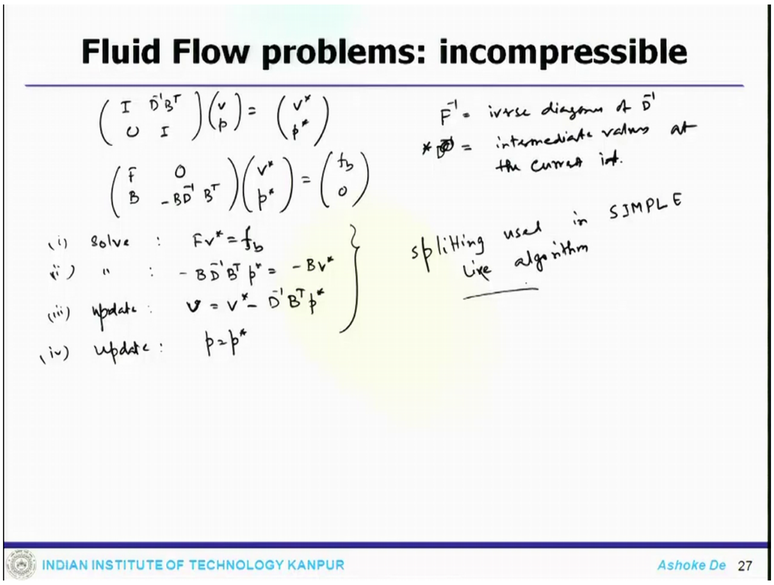

So, one can describe that in a matrix form what happens that I D inverse B transpose 0 I v p is some sort of a v star and p star. Now it can be followed by an updation by F B 0 minus B D inverse B transpose v star p star f b 0. So, F inverse is the inverse diagonal d inverse refers to the intermediate. So, star intermediate values at the current iteration. So, this is inverse diagonal of D inverse. So, the steps which is required is that one can solve F v star equals to f b.

Then one can solve minus B D inverse p transpose p star equals to minus B v star then one can solve or rather update, it is V equals to V star minus D inverse B transpose P star. And then finally, also update p equals to p star. So, this kind of splitting is essentially used this is called some sort of a splitting which is used in simple like algorithm. And we will see that in details as we move ahead with our discussion. Now what we can do? We can look at a preliminary derivations, how we do that.

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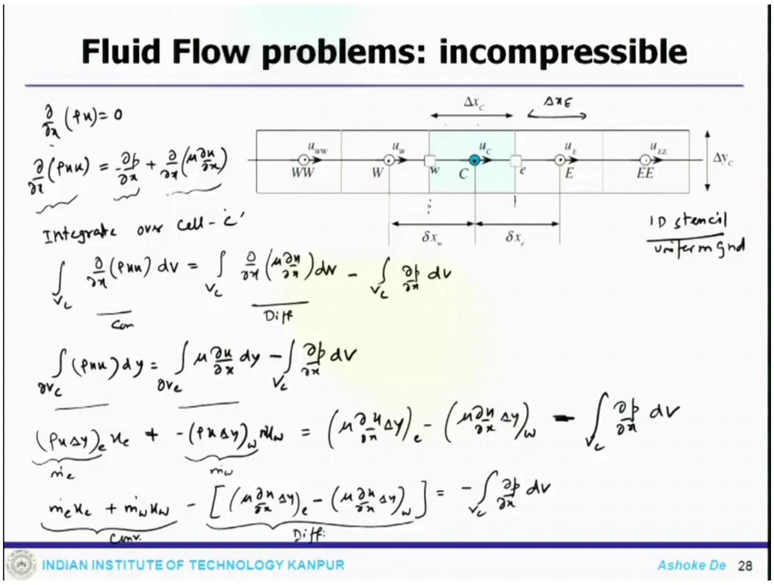

And we can start with a one dimensional system where you define a stencil like that and this is your 1 D stencil and it is the discretize indexing. And again let me reiterate that the notations which are taken here it is consistent throughout the lecture series.

When we take a cell C ahead that E behind that W, two cell behind W W two still upstream East East. Similarly, in the other direction north south, north north north south south like that again its a 1 dimensional stencil with uniform grid. So, which means the distance between another cell with of this cell C and E they are equal. When we concern about cell C, this is the east face, E this is the west face W and we can start with our continuity equation rho u equals to 0 it say 1 D so, only we take the derivative. And our momentum equation gives rho u u equals to minus del p by del x plus del del x of mu del u del x that is our diffusion term pressure gradient term and convection term.

So, how do you discritize the system. So, again you integrate over cell C, if you integrate over cell C then what you get, you write del del x of rho u u d v equals to del del x of mu del u del x d v minus del p by del x d v. So, that is how you get. Now the volume integral of convection this is convection and this is diffusion these are then transform into surface integral by using the divergence theorem. And that becomes like del v c rho u u d y equals to del v c del u by del x d y minus v c del p by del x d v.

So, now the surface integral, we can represent the fluxes or the sum of the fluxes over faces and using the single Gaussian point of the integration rule then the semi discretized equation looks like rho u delta y u e.

Then plus with a minus sign of rho u delta y w u w equals to mu del u by del x delta y east face minus mu del u by del x delta y west face equals to or minus v c del p by del x d v. Now if you look at this is nothing, but your m dot e this is nothing, but your m dot w. So, one can write m dot e u e plus m dot w u w minus mu del u by del x, delta y east minus mu del u by del x delta y west equals to minus del p by del x d v. So, that is so, this is your so called convection term, this is your diffusion term and one can think about the pressure gradient term as a source term.

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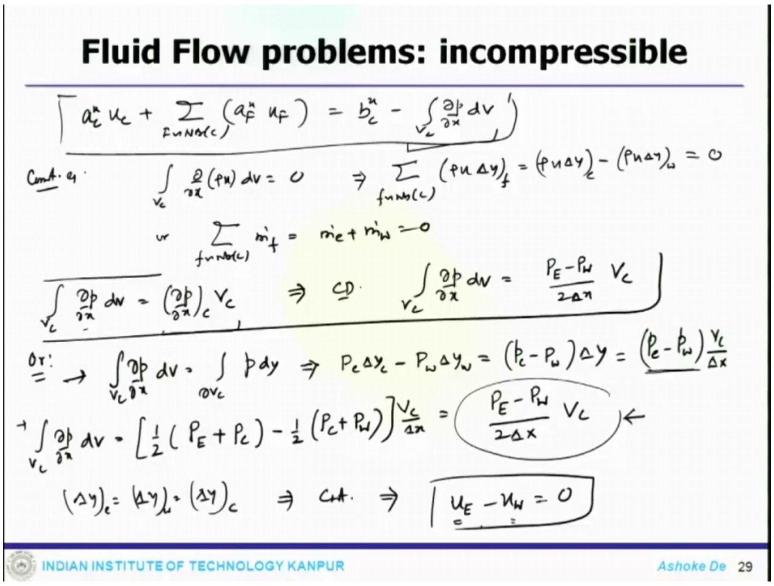

So, if you use your standard discretization technique then one can write a c u, here we will use a superscript u which will stand for the velocity component in the s direction, because when we start discussing the two dimensional problem also the discretized equation for v velocity; that means, the second directional velocity would be looking similar. So, its better to use and superscript to substantiate or differentiate between the two velocity component. So, this would be a F u u F which is b c u minus v c del p by del x d v; so, that is what one get.

So, this is a discretization of the pressure term is differed till the discretization of the continuity is discuss; we can now will not touch this till you do that. First if you look at the continuity equation. So, the continuity equation, if we integrate this would give you del del x rho u d v equals to 0. Again the system the volume integral if you use the divergence theorem and transform to a surface integral this will get you back that F n b c rho u delta y f which is rho u delta y e minus rho u delta y w which is 0 or one can think about summation of f m dot f equals to m dot e plus m dot w which is 0. So, that is satisfy the continuity

Now the problem which comes one can think about which is a term we are is the pressure gradient term del p by del x d v equals to del p by del x c V c. So, one can think about. So, the volume integral you can using the single Gaussian integration point you can integrate like that. And if you do like that and now use some sort of an central different scheme, then this integration will become del p by del x d v which will become P E minus P W by 2 delta x V c.

So, we are referring to the same 1 dimensional stencil that we have taken here. So, we still written in that 1 dimensional system and trying to or alternatively what one can do the volume integral of the pressure gradient term can be transformed to a surface integral like I used del p by del x d v to p d y. Now if you rewrite the surface integral term, this will become P e delta Y e minus P w delta Y w which is nothing, but P e minus P w delta Y which is P e minus P w you can use P like that V c divided by delta x assuming uniform system.

Now one can a linear interpolation profile can be assumed. Now the pressure gradient term can be written and then if you rewrite this term which is one can write half of P E plus P c minus half of P c plus P w which is multiplied by V c by delta x that is also P E minus P W by 2 delta x V c.

Now either of these approach whether this or this leads to the same expression involving the pressure difference term, but one case you have the pressure at the surface and the other case. So, where you can write like this or alternatively you obtain like this. Now both the cases if you see whether you convert them to surface integral and then find out the face value of the pressure and then if you convert that to the cell centre value you end up getting this. Now either of the cases, now the similar way a linear interpolation profile if your density is constant then one can say this guy is going to be your delta y of c.

Now, the continuity equation can be expressed as from continuity equation one can write U E minus U Q equals to 0 that is one thing one can write. So, it is a relative velocity in alternative grid point. Now what is happening here if you look at this equation. So, the pressure gradient term in element c depends on the values which is the upstream and the downstream of c rather two alternative points in between which the c is sitting.

So, which is a non consecutive and grid points around that element, the same is true for the continuity also, you can see the relative velocity is dependent on upstream grid and the downstream grid. So, this implies that some sort of a non physical zigzag pressure velocity field which one can be sensed through the numerical scheme.

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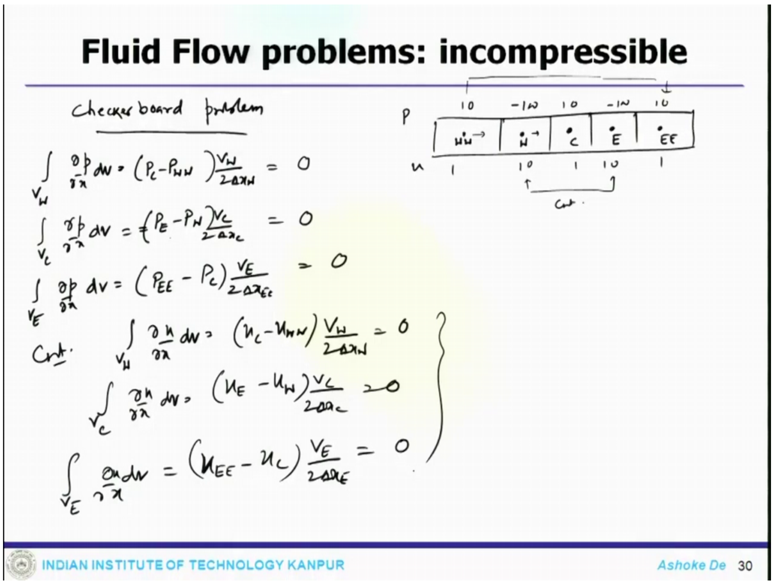

If you think what can possibly happen you look at this stencil where you have C then E E, so, this is C E E E W W W. Now you think about the pressure which can have 10 minus 100, 10 minus 100, 10 the gradient is 0 and u you can think about 1 10, 1, 10, 1 something like that. So, that also satisfied the continuity; so, this is also satisfying continuity.

These kind of situation is known as checker board problem. And what happens that your gradient if you look at it its a V W del p by del x d v, it is P C minus P WW V W by 2 delta x w which is if you look at the values it is 0. Similarly, V C del p by del x d v this is also P E minus P W V c by 2 delta x c which is also going to be 0 minus 100 plus 100.

Similarly V E del p by del x d v P E E minus P C V E divided by 2 delta x E E which is also 10 minus 10 0. Now similarly if we look at the continuity equation.

The continuity equation you can see V W del u by del x d v U C minus U WW V W by 2 delta x W is 0. Similarly, V C del u by del x d v U E minus U W VC divided by 2 delta x e 0 del del x d v which is U E E minus U C V E by 2 delta x E which is also 1 minus 1 0. So, you see this. So, in the multidimensional system, if the similar non physical behaviour can arise even it is harder to visualize. So, these sets the ground for want to think about how to dissolve this kind of problem because one.

So, you see immediately when you couple pressure and velocity together this checker board or zigzag kind of problem. So, which allows or effectively what it does there is no gradient. So, it makes the system to change to certain situation that it actually get you back a 0 gradient condition. And if you look at your equation that is not true you do not have the, you have certain source term as a gradient term. So, we will stop here and we will continue the discussion in the next lecture.

Thank you.