Introduction to Finite Volume Methods - II
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Lecture - 4
Linear Solvers – IV

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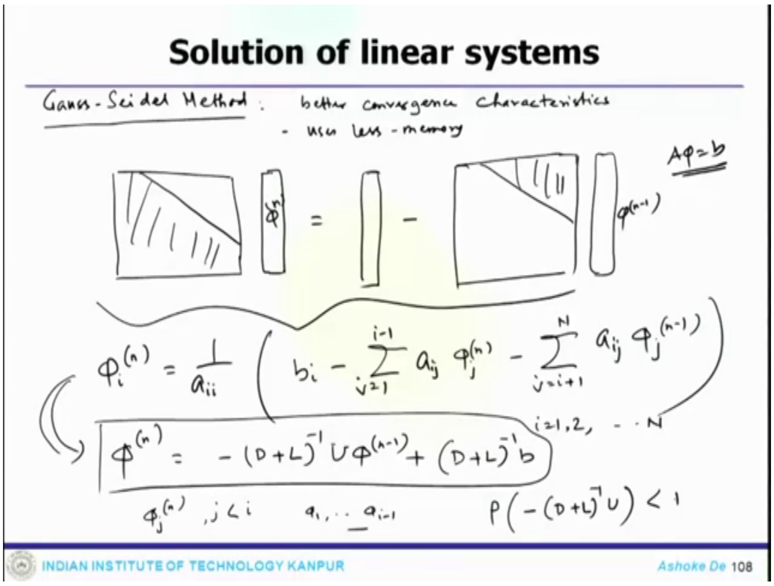

Ok, welcome back. Now, second method that we can talk is the Gauss Seidel method, so that is the second one. And if one can see, so this is another popular method which can be slightly better than Jacobi because it has better convergence criteria or characteristics our Jacobi, and also it uses less memory. So, memory wise also it is less expensive.

And if you see how things done this is the box, you have you are trying to find out at this level, then this minus this. So, this will have some sort of and these things phi n minus some sort of this phi n minus 1, so that is how graphically it can be represented. And but finally, the equation that we are solving for is phi equals to b. So, if you write that mathematically at nth level phi i equal to 1 by a i i b i minus summation of j 1 to i minus 1 a i j phi j minus j equals to i plus 1 to n, a i j. So, i goes from 1, 2 to N.

Now, this expression this is in indicial notation. If you convert that one to a matrix form, one can write this equals to minus D plus L inverse multiplied with upper triangular phi n minus 1 plus D plus L inverse into b. So, this is what one can write. So, the effect of Gauss-Seidel method uses the most recent values in a situation. Specially if you look at phi j at nth level, when the j is less than i, since by the time phi i is to be calculated the value of phi 1 to phi i minus 1, they are used which are the most recently calculated in the ongoing iteration.

So, this approach also saves lot of memory that is why it can be quickly or less computationally expensive, but other criteria like the spectral radius that must be satisfied for this iterative matrix which is this, it is less than 1. So, some cases obviously when we say this still there could be some exceptional cases where Jacobi method converges faster, Gauss-Seidel is the preferred one, so that is also possible.

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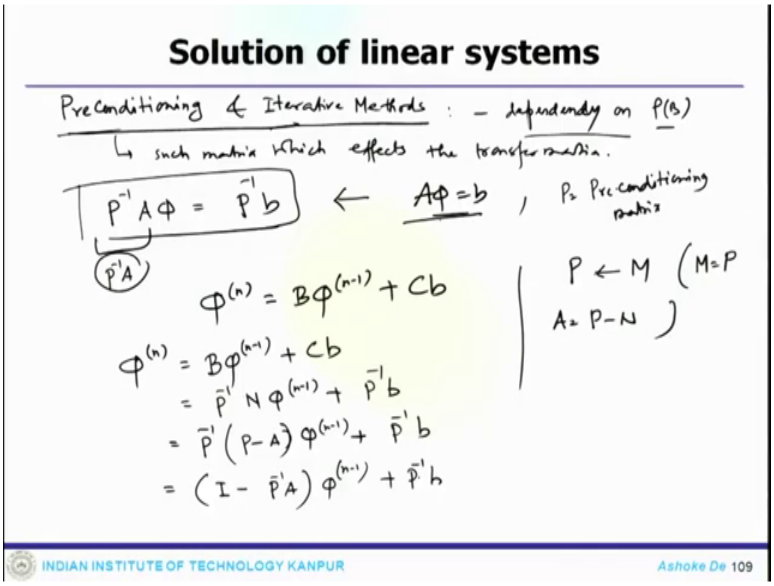

Now, we move to an important topic which one needs to know while talking about the iterative process is the preconditioner or preconditioning, and associated iterative methods, so that is one thing. So, what happens preconditioning why do use preconditioning, because that the rate of convergence of the iterative methods heavily depends on the spectral properties of the radius of the iteration matrix B.

So, the depends or the dependency, dependency on rho B that is a heavily depended. So, what makes it is so that the iterative method can be I mean defined in a better way where you can actually improve its convergence level. So, in that way the it can be transform to a equivalent system. So, essentially the iterative matrix can be transform to equivalent system and then the equivalent system can be solved or the transform system, so that is where the preconditional comments the picture.

So, preconditioner is such matrix which actually effects the transformation of the system. So, P is defined as a preconditional matrix, one can defined as P A inverse phi equals to P inverse B. So, what we are doing we are solving for the equation A phi equals to B, and P is a pre conditioning matrix. So, once you multiplied with that P inverse the both the side of this equation, you get this one.

Now, this guy will have the same solution as the A phi equals to B. But the spectral properties of this iterative matrix that is P inverse A, they are different. And in defining the preconditional P, the difficulty is to find out that approximation approximate A inverse and but it is easy to convert to find P inverse at a reasonable cost. So, we will write the equation, this equation will start with our iterative equation phi n equals to B phi n minus 1 plus c b that is our starting point of the iterative equation.

Now, what you will do, we will replace P or P replacing by M and A equals to P minus N. So, what will do we will now with P replacing M, and A equals to P minus N, so that means, basically what we are saying that M equals to P and A equals to P minus N. So, once we use that for the fixed point iteration equation, this equation look like that B phi to the power n minus 1 plus c b which is P inverse N phi N minus 1 plus P inverse b. Now, this is P inverse P minus A, so this becomes phi n minus 1 plus P inverse b. So, this one can write that identity matrix minus P inverse A phi n minus 1 plus P inverse b.

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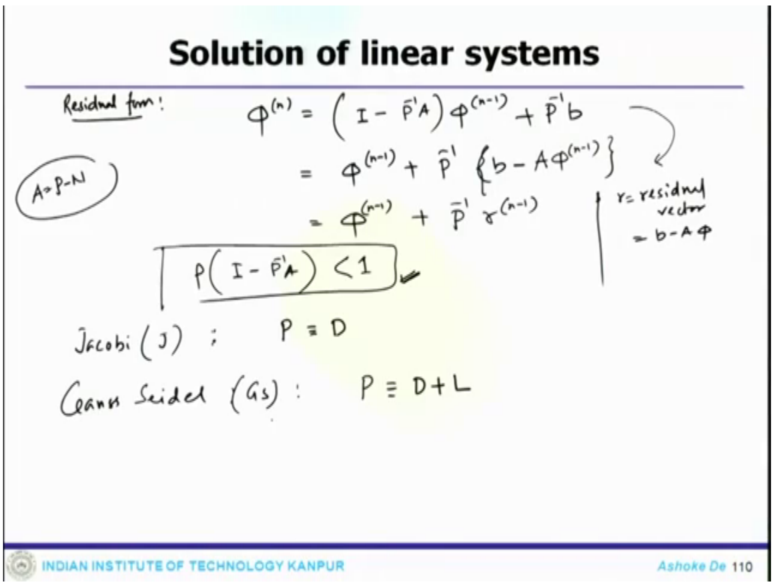

So, so residual form one can write that and that can be written as phi n equals to I minus P inverse A phi n minus 1 plus P inverse b which will be phi n minus 1 plus P inverse b minus it would be a phi n minus 1. So, essentially what we are doing from this step to this step, we just rearranging the stuff. This is identity matrix multiplied with the n minus 1 level of variable, and then you take P inverse in that side. So, this will become like this which one can rewrite phi n minus 1 plus P inverse r to the power n minus 1. Here r is the residual vector which is b minus A phi.

Now, from this equation one can see that the iterative procedure is just a fixed point iteration on a precondition system associate with the decompositions, where A equals to P minus N, where the spectral properties now would be on rho I minus P inverse A which must be less than 1. So, this is the property one has to satisfy.

Now, if you compare this precondition system with two other methods that we have discussed, one is the Jacobi and other case is the Gauss-Seidel. So, if you compare these two with this precondition system, one can immediately see for Jacobi method the P is equivalent to your diagonal matrix and Gauss-Seidel P is nothing but your diagonal plus lower triangular system so that is the equivalence you one can see. That means, when you define a precondition system, the precondition system can be also used for the Jacobi method and other methods.

So, here the D is the diagonal matrix, L is the lower triangular part of the matrix. So, essentially what one can say that preconditioning is a manipulation of the original system to improve its spectral radius as we have seen with the help of this preconditioning matrix P used for the associative iterative process. So, one can develop different kind of preconditioning matrixes in which the coefficients are quite complex one.

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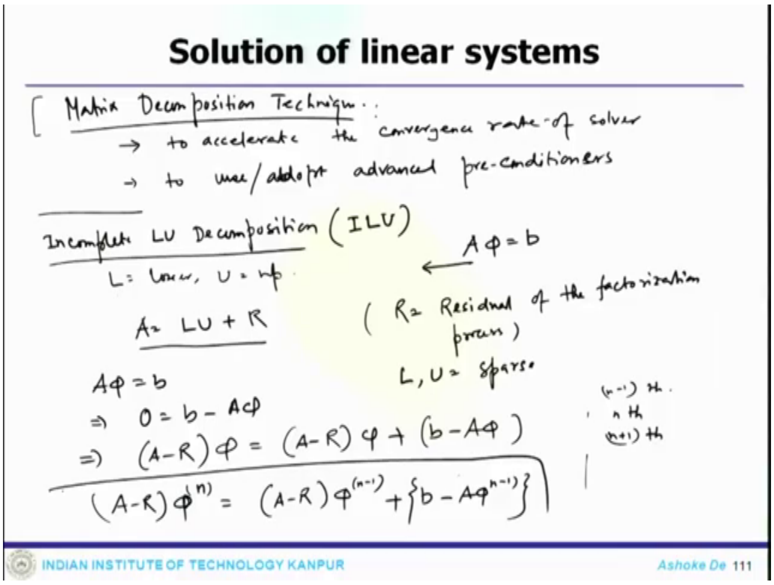

Now, the other think one needs to know that the matrix decomposition, matrix decomposition technique. So, why we need to decompose the matrices, that reason is that one can accelerate so to accelerate the convergence rate of the iterative process. So, the decomposition helps in turn to promote the convergence rate of the solver. Also to some extent use or adopt advanced pre-conditioners. So, we have all ready seen that pre-conditioners helps to get in slightly better convergence because of the change in the spectral properties of the system.

So, simple, but yet an very efficient approach for the purpose is to perform the incomplete factorizations of the original matrix so which call the so there are couple of issues which are very very important or for the iterative process is that one has to improve the convergence rate because it then only you can have faster calculation to be done, and also if you can adopt a advanced level of pre-conditioners. So, and the simple way to that how you decompose the system in incomplete factorizations or the first one we call incomplete L U decomposition.

So, some other way it is known as I L U. So, this is what so L here again stands for lower triangular matrix, U stands for upper triangular matrix, and I is the identity matrix. So, one can incomplete ILU factorization of a and the system that we are solving for is the same A phi equals to b is performed such that the resulting lower and upper matrices of the same nonzero element.

So, the way this incomplete factorization is done from the, from A to this that they have same number of nonzero structure at the lower and upper half. So, the A can be written as L U plus R, where R is the residual of the factorization process. So, the matrices L and U they are also L, U they are also sparse. So, they are being spares are easier to deal with and their obtained from a complete factorizations. But the product being an approximation to A which will also necessitate to use an iterative solution procedure for this system.

And the first step towards that would be writing this original equation in such a form that 0 is b minus A phi which will also lead to A minus R phi equals to A minus R phi plus b minus A phi. So, now if you get this calculation for different iterative level, so that means, nth level n plus 1th level n minus 1th level like that, first, second, third iteration, I can write A minus R phi to the power n equals to A minus R phi to the power n minus 1 plus b minus A phi to the power n minus 1, so that is what I get.

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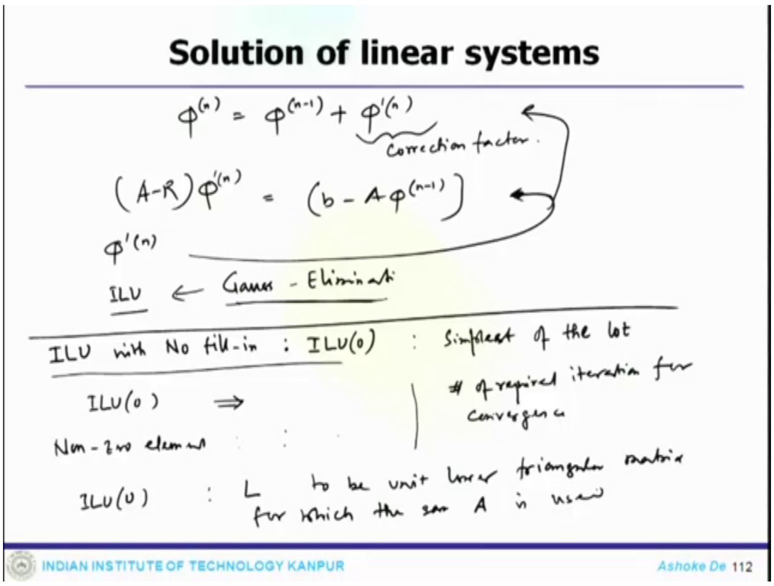

Now, this if you now you can find out at the iteration level n using the value of n minus 1 such that phi n equals to phi n minus 1 plus some sort of n phi prime n which is nothing but some kind of a correction factor which are being used.

Now, the previous equation becomes A minus R phi n equals to or phi prime n equals to b minus A phi n minus 1. Once you get phi prime n, then you can use that value to update the phi n. The ILU factorization can be performed using so this you can perform using

Gauss elimination process while some dropping of diagonal elements at different locations.

So, you see what we have discussed for the direct method like Gauss elimination or the LU factorization, they are now again coming back for the discussion or under the discussion of the iterative process, and they play an I mean critical role here. Because once you try to do ILU factorization, you can use the Gauss elimination process.

Now, one also can do the incomplete factorizations like incomplete ILU with no fill in, that means, which is ILU 0. This is a typical terminology which is used to demonstrate the incomplete LU factorizations with 0 fill or no fill. So, there are different variations of the ILU factorizations which exist in the literature. So, again this is the simplest of the lot.

So, what happens in ILU 0, ILU 0, the pattern of zero elements in the combined L and U matrices is taken to be precisely the pattern of zero elements in the original LU matrices in the original A. So, Gaussian elimination or can be performed in the case of full LU factorizations. But in any nonzero elements exist; there you cannot use that kind of Gaussian elimination. Hence, the combined L U matrices combined LU matrices have together the same number of nonzero at the original matrix A.

So, with this particular approach, what one can do with that the filling problem that usually arises when factorizations of the sparse matrices takes place, so that in those non zero elements are location with the original matrices has zeros, they can be avoided. So, in the process the accuracy is reduced, thereby increasing the number of required, number of required iteration for the convergence number of required iteration for convergence also high to remit to this what coming for more accurate ILU factorization methods which are often more efficient and more reliable. So, these are the different blames of ILU factorizations which are devised.

And there are what you do in that ILU0, factorizations assume L to be unit lower triangular matrix for which the same A used, A is used to store the elements of their unit lower and upper triangular matrices. And L and U can be written in that kind of approach.

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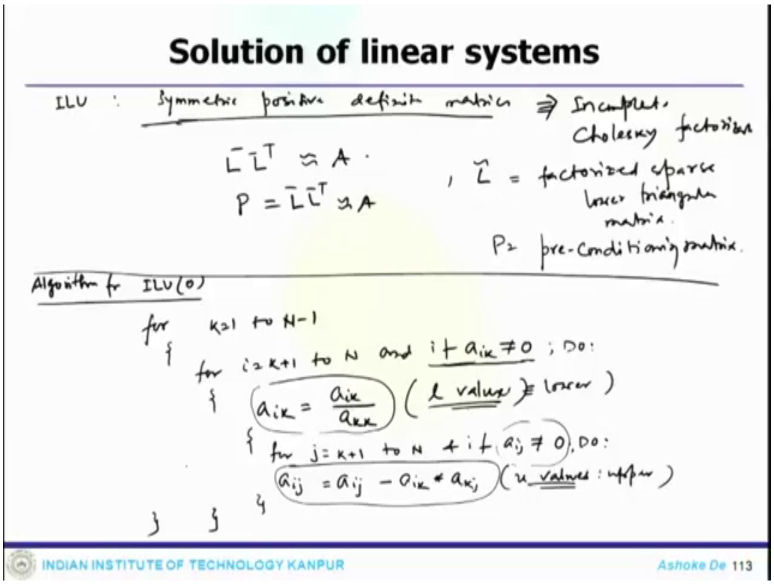

Also in this I L U factorization of the symmetric, so if you do this for symmetric positive definite matrices, so if it is symmetric positive it should be mentioned that if you are that the ILU decomposition of symmetric positive definite matrices is denoted by incomplete Cholesky factorization which is called Cholesky factorisation. In this case, the factorizations is done such that L bar L bar transpose equals to A where, L bar is the factorized sparse lower triangular matrix ok.

And pre-conditioner which is given as P equals to L bar L bar transpose equivalent to A, where P is the pre conditioning matrix. Now, that is for a very specific case if you have a symmetric positive definite matrices. Now, if you put the algorithm for ILU 0, what one has to do? This is for ILU 0. You go for a loop where k equals to 1 to n minus 1, first loop starts, then the second loop you go I equals to k plus 1 to n. And you check if a i k not equals to 0, then you do the following. You calculate a i k equals to a i k divided by a k k which is the lower values.

And then you go for another loop for j goes from k plus 1 to n, and if a i j not equals to 0, you do the following calculation for a i j which is essentially a i j minus a i k multiplied with a k j which is the upper values, upper this is essentially the lower. And that brings back to the closer of first this loop, and then you close back this loop, and finally, you close down the third one, so that is the essentially the ILU 0 decomposition algorithm which you can use.

And so what essentially do you go from this loop k equals to 1 to n minus 1, then the second loop you check this criteria. This is a very important criteria if this is not satisfied even then you go what this so you are going over the loop, but checking this criteria if this particular criteria is satisfied, then only you do this calculation to obtain the lower values of the system.

And then in the second level of iteration you check these criteria. If that is satisfied, then you finally calculate the upper values. Here U stands for the upper value; L stands for the lower values. And then you close this loop. So, this is how you get the ILU 0 factorizations. So, we will stop here today. And in the subsequent lecture, we will discuss the other variant of the iterative process.

Thank you.