Introduction to Airbreathing Propulsion

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Lecture – 08
Review of Compressible Flows (Contd.,)

Okay so we are talking about the normal shock and we just started deriving the normal shock relation and what we obtain finally in the last lecture is that the Prandtl relations which is which provides the relationship between a star and the velocities between upstream and the downstream.

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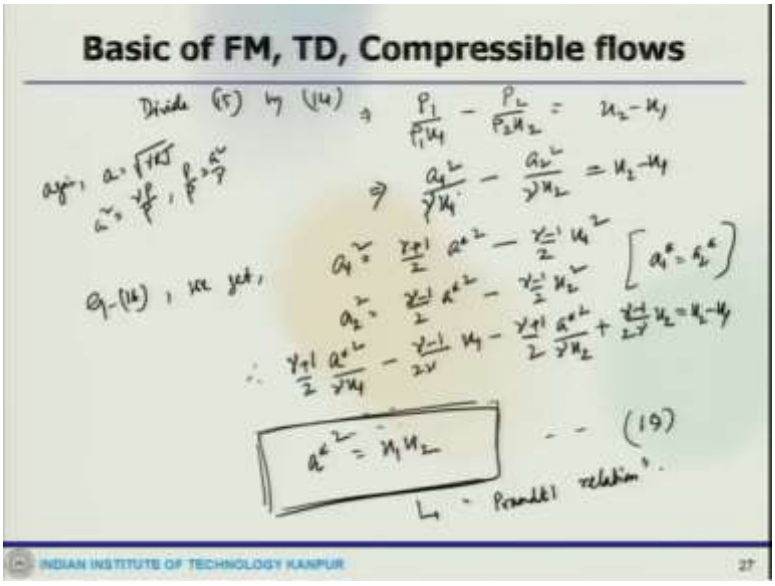

So this Prandtl relations that we obtain this is what we can see the proof of it.

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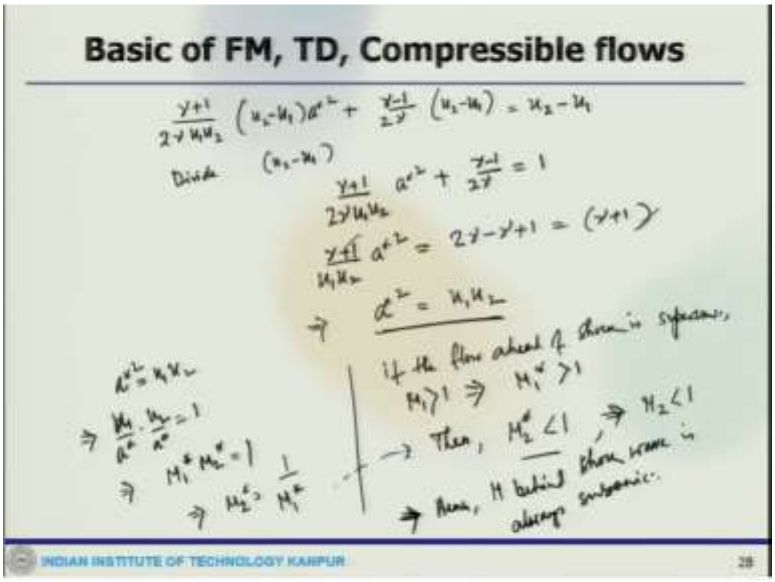

So that is now let us say what we have is that from here what we have is that

$$\frac{\gamma+1}{2\gamma u_1 u_2}(u_2-u_1)a^{*2}+\frac{\gamma-1}{2\gamma}(u_2-u_1)=u_2-u_1$$

So now we divide (u_2-u_1) and what we get is that

$$\frac{\gamma+1}{2\gamma u_1 u_2}a^{*2}+\frac{\gamma-1}{2\gamma}=1$$

So even doing be top little bit more maths what we get

$$\frac{\gamma+1}{u_1 u_2}a^{*2}=2\gamma-\gamma+1=\gamma+1$$

So once that cancels we get

$$a^{*2}=u_1 u_2$$

So this is what you get now if you solve for the a^* then that can gives us like

$$a^{*2}=u_1 u_2$$

So what we can do

$$\frac{u_1}{a^*}\frac{u_1}{a^*}=1$$

and that provides our

$$M_1^* M_2^*=1$$

$$M_2^*=\frac{1}{M_1^*}$$

So if the flow ahead of shock is supersonic that means $M_2^*>1$ which means $M_2^*<1$.

Then from this M_2^* and M_1^* relationship from this one we can say thus we will have shock behind the flow field behind the shockwave M_1^* would be less than 1. So if M_2^* is less than 1 that means the M_2 would be less than 1. So which proves that M behind shock wave is always subsonic. Okay so this is behind normal shock because this relationship is for normal shock relationship.

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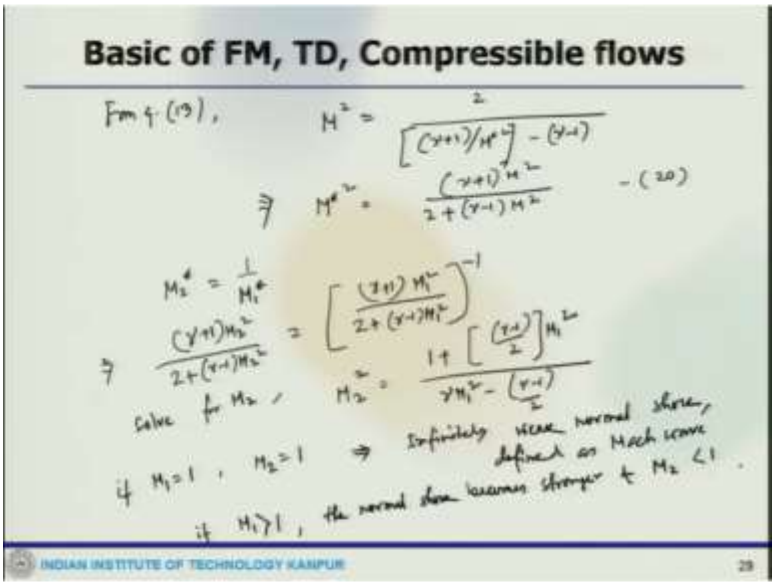

Now further one can extend this what we got from let us say equation (13) what we do or can write

$$M^2 = \frac{2}{\left[\frac{\gamma+1}{M^{*2}}\right] - (\gamma - 1)}$$

So, from here what we get

$$M^{*2} = \frac{(\gamma+1)M^2}{2 + (\gamma-1)M^2}$$

okay. So, we can again see this is equation (20) so what we have is

$$M_2^* = \frac{1}{M_1^*}$$

now we put equation (20) here what we get

$$\frac{(\gamma+1)M_2^2}{2+(\gamma-1)M_2^2} = \left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{-1}$$

Now if we solve for M_2 that yields

$$M_2^2 = \frac{1 + \left[\frac{(\gamma-1)}{2}\right]M_1^2}{\gamma M_1^2 - \frac{(\gamma-1)}{2}}$$

So this is what we get so let us see if M_1 is 1 that means at sonic then M_2 would be 1. So this is something you can say this is infinitely weak normal shock which is sort of defined as Mach wave if M_1> 1 then the normal shock becomes stronger and M_2 becomes less than 1.

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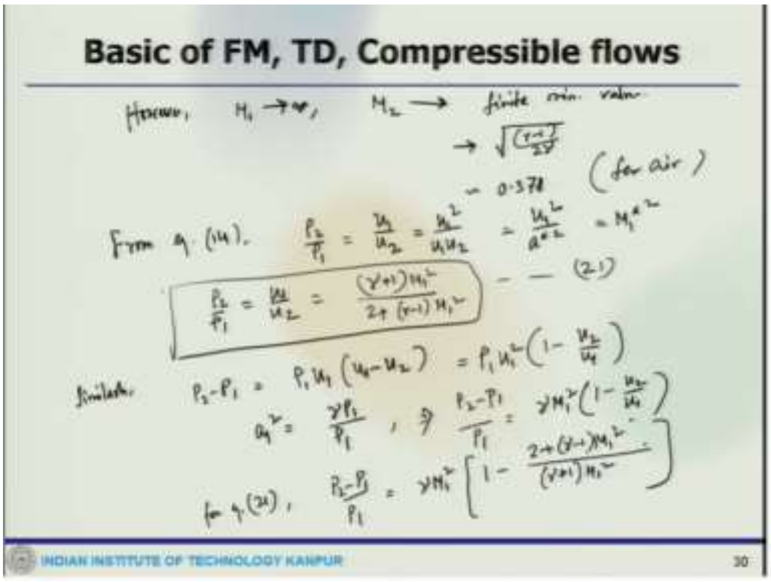

However, M_1 if it tends to infinity M_2 tends to some finite minimum value which is in

$$M_2 \rightarrow \sqrt{\frac{(\gamma - 1)}{2\gamma}}$$

So that comes around 0.378 per air. So, these are the different relationship that one can obtain. Now again from equation (14) we get

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} = M_1^{*2}$$

So that provides the relationship like

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$

So we get in that relationship between the density across the normal shock. So same thing one can obtain for the pressure let us say

$$p_2 - p_1 = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$a_1^2 = \frac{\gamma p_1}{\rho_1}$$

Now since this has a ratio of the velocity from equation (21) if we use value of $\frac{u_1}{u_2}$ we get

$$p_2 - p_1 = \gamma M_1^2 \left(1 - \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right)$$

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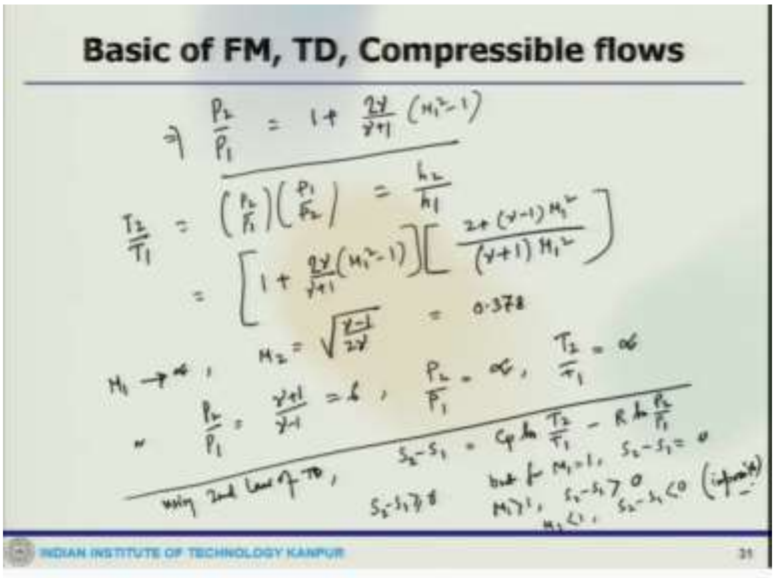

So

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

okay so that is the relationship between the pressure ratio across the normal shock. The other way the

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} = \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}$$

$$= \left(1 + \frac{2\gamma(M_1^2 - 1)}{(\gamma + 1)}\right)\left(\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right)$$

when M tends to infinity

$$M_2 \rightarrow \sqrt{\frac{(\gamma - 1)}{2\gamma}}$$

which is 0.378 or

$$\frac{p_2}{p_1} = \frac{(\gamma + 1)}{(\gamma - 1)} = 6$$

$$\frac{\rho_2}{\rho_1} = \infty$$

$$\frac{T_2}{T_1} = \infty$$

So now using the second law of thermodynamics what we can write using second law of thermodynamics what we can write

$$S_2 - S_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

so this is what we have already derived now $S_2 - S_1$ > 0. Now so what we can do let us say if T2 = T1 and P2 = P1 then this becomes 0 but for M = 1 $S_2 - S_1$ = 0 but for M1 > 1 $S_2 - S_1$ > 0 M1 < 1 also $S_2 - S_1$ < 0 so this is an impossible situation why?

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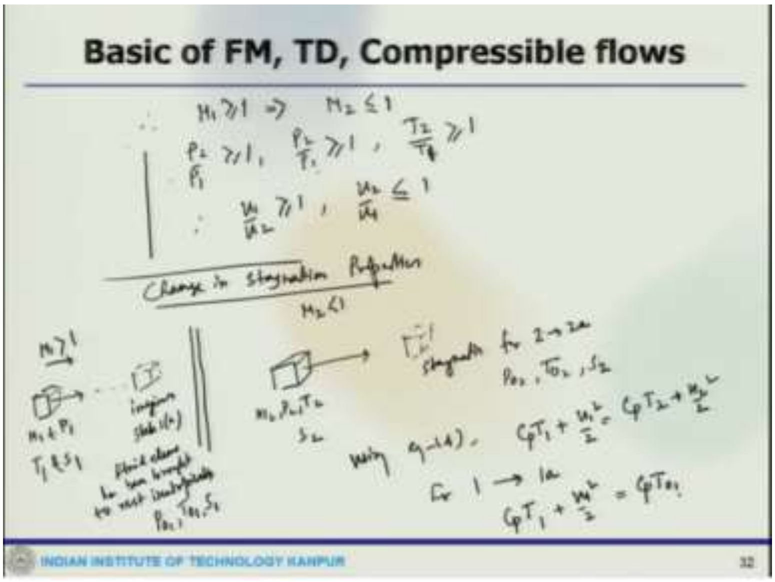

Now another situations since $M_1 \geq 1$ which means $M_1 \leq 1$ so

$$\frac{\rho_2}{\rho_1} \geq 1$$

$$\frac{p_2}{p_1} \geq 1$$

$$\frac{T_2}{T_1} \geq 1$$

$$\frac{u_1}{u_2} \geq 1$$

$$\frac{u_2}{u_1} \leq 1$$

So, which clearly means pressure, density and temperature they increases across the shock wave and velocity actually and Mach number decreases across the shock wave okay. So that is what happens when you have a normal shock now, we look at the change in stagnation properties.

So we have a normal shock here so let us say we have a fluid particle like this so that has been M1 > 1. So M1 and P1 and T1 and S1 this is an imaginary place it is a state 1a where the fluid element has been brought to rest isentropically. Now M2 < 1 here where the fluid particles so this is M2, P2, T2, S2 stagnation for 2 to 2a which is P02. So this would be P01, T01, S1, Po2, T02, S2.

Now from energy equation what we can write that is equation 4 using equation 4 we write

$$C_pT_1 + \frac{u_1^2}{2} = C_pT_2 + \frac{u_2^2}{2}$$

that means the free particle from here this was brought to rest which is a sort of an imaginary point isentropically the then this would be

$$C_pT_1 + \frac{u_1^2}{2} = C_pT_{01}$$

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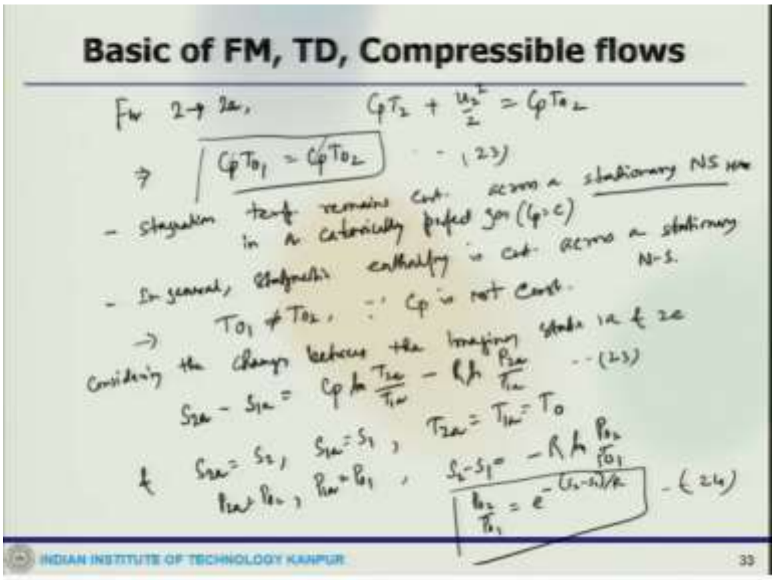

Similarly, for 2 to 2a

$$C_pT_2 + \frac{u_2^2}{2} = C_pT_{02}$$

okay now

$$C_pT_{01} = C_pT_{02}$$

so that means what we get the stagnation temperature remains constant across a stationary normal shock wave in a calorically perfect gas it is $C_p = C$ that means now this is very important that we say it is a stationary normal shock wave. Because if the normal shock wave is moving then this condition may not be valid.

So when the shock wave is stationary and it is a normal shock wave then the stagnation temperature across the normal shock wave remains constant. In general, the stagnation enthalpy is constant across a stationary normal shock. Okay now for chemically reacting gases if there is a chemical reaction then T01 would not be T02 since Cp is not constant anymore.

Now consider the changes between the imaginary state so considering the changes between the imaginary states 1a and 2a

$$S_{2a} - S_{1a} = C_p \ln\frac{T_{2a}}{T_{1a}} - R\ln\frac{p_{2a}}{p_{1a}}$$

and what we have is $S_{2a} = S_2$, $S_{1a} = S_1$, $T_{2a} = T_{1a} = T_0$ and $p_{2a} = p_{02}$, $p_{1a} = p_{01}$ so what we get

$$S_2 - S_1 = R\ln\frac{p_{02}}{p_{01}}$$

so what we could write

$$\frac{p_{02}}{p_{01}} = e^{-(S_2 - S_1)/R}$$

So that is what you get for stagnation pressure ratio relationship.

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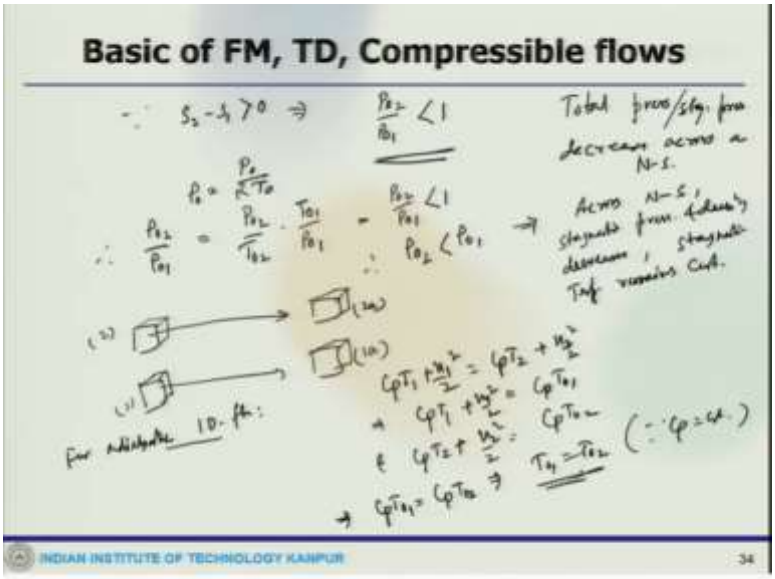

Since $(S_2 - S_1) > 0$, $\frac{p_{02}}{p_{01}}$ will be less than 1 which clearly means that the total pressure or stagnation pressure decreases across a normal shock. Now we can get the density too

$$\rho_0 = \frac{p_0}{RT_0}$$

so what we can write

$$\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{T_{02}} \cdot \frac{T_{01}}{p_{01}}$$

which is since the stagnation temperature is same this will become P02/P01 which is also less than 1 that means the stagnation density is also so is rho 0 1 that means across normal shock stagnation pressure and density decreases whereas stagnation temperature remains constant okay.

So what we had is that fluid particles from here so this has gone to state 2a and this has gone to 1a. So this is 1 this is 2 for adiabatic 1D flow we can again write

$$C_pT_1 + \frac{u_1^2}{2} = C_pT_2 + \frac{u_2^2}{2}$$

$$C_pT_1 + \frac{u_1^2}{2} = C_pT_{01}$$

$$C_pT_2 + \frac{u_2^2}{2} = C_pT_{02}$$

$$T_{01} = T_{02}$$

and Cp is constant.

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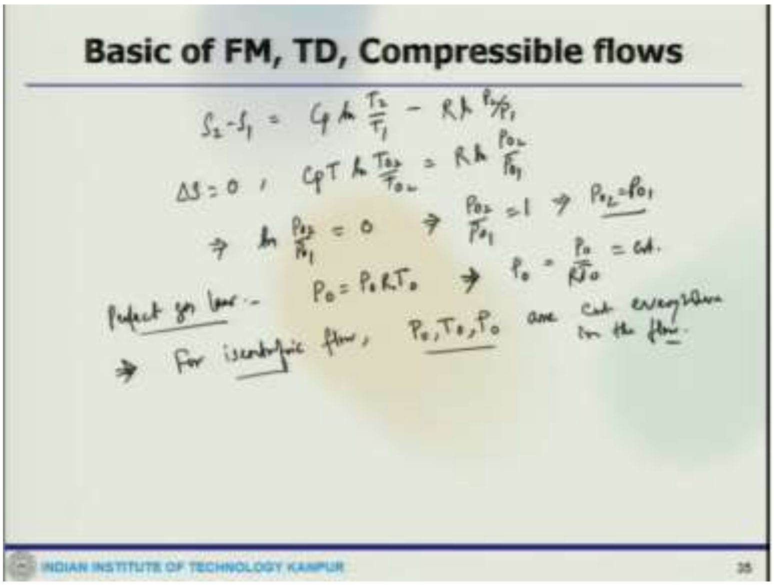

And again, we can write the change in entropy which is delta S let us say

$$S_2 - S_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$S_2 - S_1 = 0$$

$$0 = C_p \ln\frac{T_{02}}{T_{02}} - R\ln\frac{p_{02}}{p_{01}}$$

$$\ln\frac{p_{02}}{p_{01}} = 0$$

$$\frac{p_{02}}{p_{01}} = 1$$

$$p_{01} = p_{02}$$

So, from perfect gas law what it gets

$$p_0 = \rho_0 R T_0$$

$$\rho_0 = \frac{p_0}{R T_0}$$

which is constant.

So, this brings to a very important conclusion that for isentropic flow $p_0, \rho_0, and\ T_0$ are constant everywhere in the flow. So, which means whenever you have an isentropic flow your stagnation properties remain constant everywhere but whenever you have stationary normal shock just again, I am reiterating that it is a stationary normal shock then your stagnation density stagnation pressure that decreases but stagnation temperature only remains constant.

And across a normal shock density increases I mean static density increases, static pressure increases, static temperature increases whereas the flow velocity decreases in the downstream of the shock. So that is pretty much what we wanted to talk about in normal shock relations. So, what we will do we will just talk about the oblique shock and other stuff in the next lecture we will stop about this here today.