Computational Science in Engineering

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Lecture - 10
Linear Algebra

So, let us continue the discussion on Eigen values. So, what we started off with the talking about the different Eigen values of the system.

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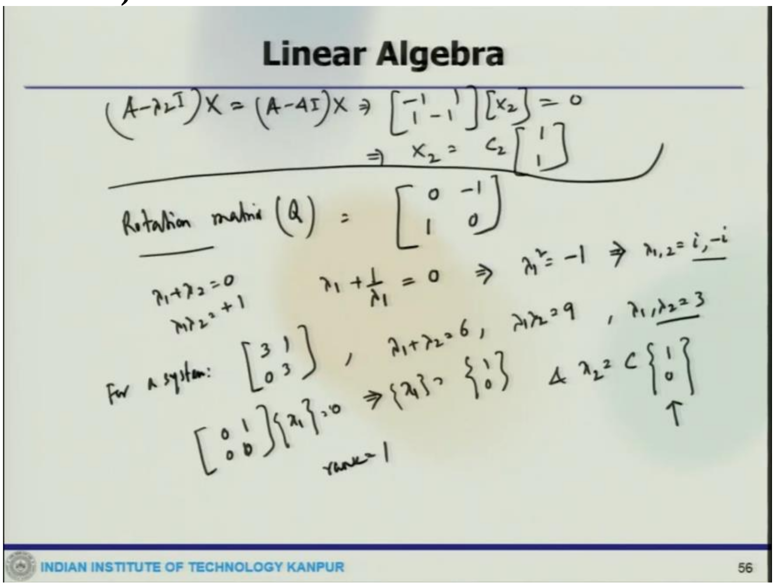

And we have looked at these examples like where you have real Eigen values and which are distant and then you have equal Eigen values which are real and then imaginary Eigen values also.

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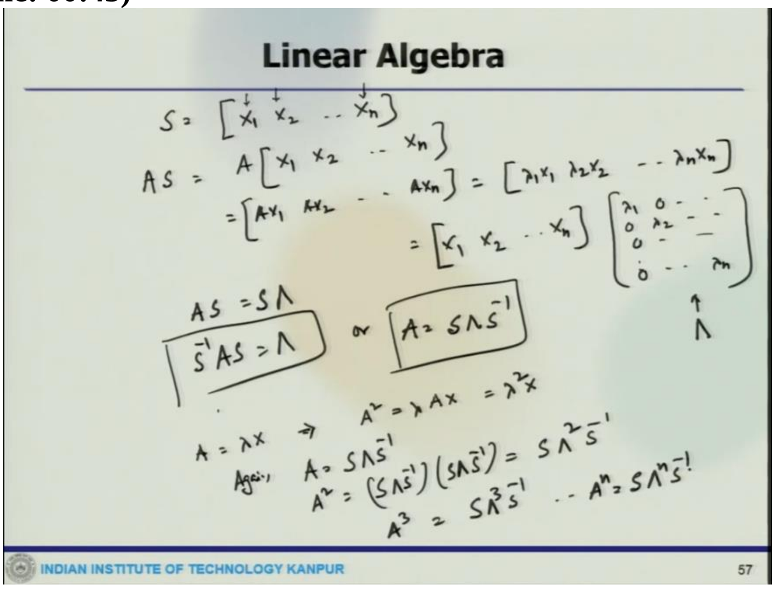

And then we stopped here where now, let us say A=λx then we can write $A^2 = \lambda$Ax which is $\lambda^2 x$. Again, we have A= $S\lambda S^{-1}$ so, $A^2 = (S\lambda S^{-1})(S\lambda S^{-1}) = S\lambda^2 S^{-1}$ similarly, we can says that $A^3 = S\lambda^2 S^{-1}$ and so $A^n = S\lambda^n S^{-1}$.

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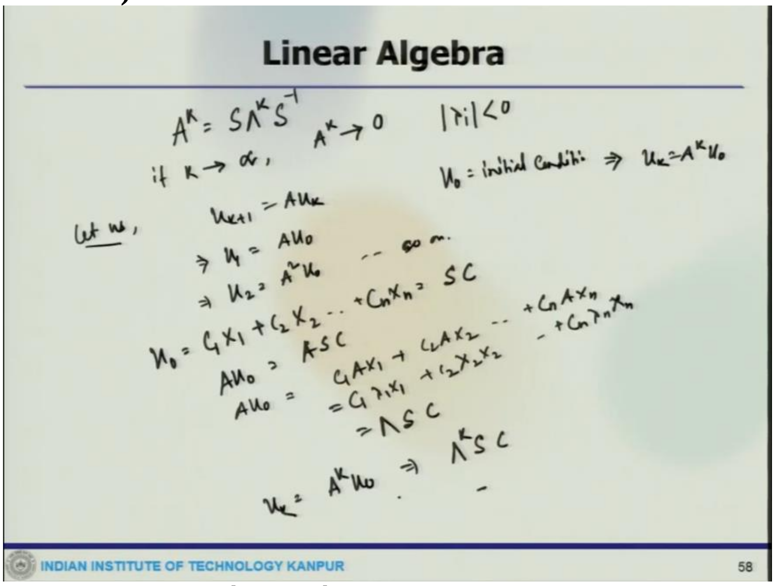

Now, we can write similarly, for $A^k = S\lambda^k S^{-1}$ now, if k tends to infinity or k tends to 0 or if k tends to infinity then A^k tends to 0 and that is only possible when we have this λ_i less than 0 and we can see that let us take a system let us say we have $u_{k+1} = Au_k$ the system like this where u_0 is the initial condition and we have $u_k = A^k u_0$.

you have so on like so, now, we write

$$u_0 = C_1 X_1 + C_2 X_2 + \cdots + C_n X_n = SC$$

So,

$$Au_0 = C_1 A X_1 + C_2 A X_2 + \cdots + C_n A X_n$$

which is

$$Au_0 = C_1 \lambda_1 X_1 + C_2 \lambda_2 X_2 + \cdots + C_n \lambda_n X_n = S\lambda C$$

Now, the same system one can see that can be written in different ways. So, I can write even

$$u_k = A^k u_0 = \lambda^k SC$$

so, that a way also one can write.

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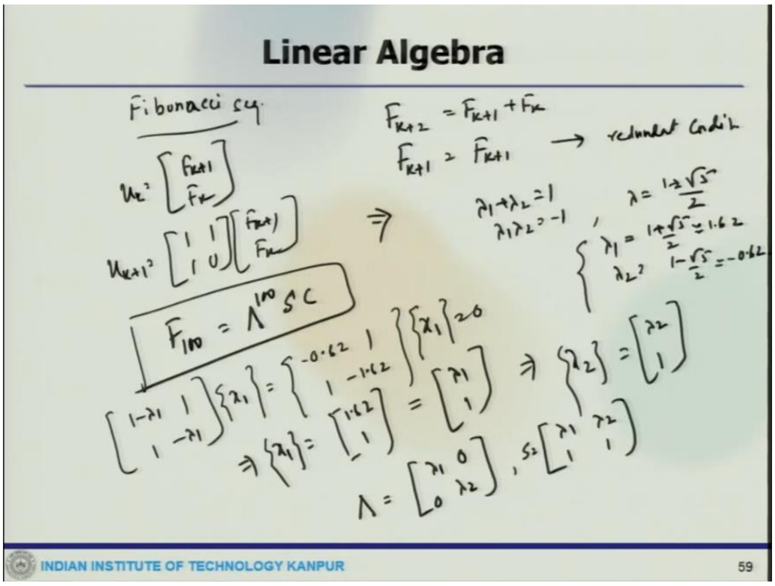

Let us look at another example like which is Fibonacci series, Fibonacci sequence. So, we write $F_{k+2} = F_{k+1} + F_k$ and $F_{k+1} = F_{k+1}$ this is some sort of an one can think about this is redundant condition. So,

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

and it give so,

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

So, from here what do we get $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \lambda_2 = -1$.

So, $\lambda = \frac{1 \pm \sqrt{5}}{2}$. So, $\lambda_1 = \frac{1+\sqrt{5}}{2}$ roughly 1.62 and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ roughly – 0.62. So, one can find let us say what would be the 100 digit or the number in that sequence. So, this one can write is S C now, how do you get that choose that these two lambdas so, let us get

$$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \{x_1\} = \begin{Bmatrix} -0.62 & 1 \\ 1 & -1.62 \end{Bmatrix} \{x_1\} = 0$$

$$\{x_1\} = \begin{bmatrix} 1.62 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

So, similarly, one can say $\{x_1\} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$. so, our Eigen value matrix would be $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and S would be $\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ eigenvector matrix.

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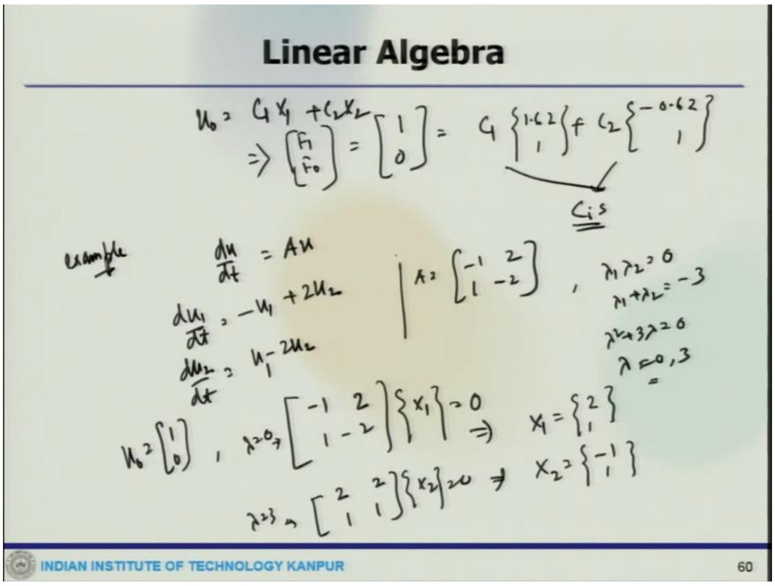

So, what we can write

$$u_0 = C_1 X_1 + C_2 X_2$$

which is

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1.62 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -0.62 \\ 1 \end{bmatrix}$$

now, we can find out what the C_1 and C_2 from here. So, from here we can find out C_i s and once you find out then you can put it back here, because this is known this is known and that is known. So, one can estimate the now, we can look at another example of that, that $\frac{du}{dt} = Au$.

So,

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

so, which gives us an system like A is $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$. So, our $\lambda_1 \lambda_2 = 0$,$\lambda_1 + \lambda_2 = -3$ so, we get $\lambda^2 + 3\lambda = 0$ or three our initial conditions here $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, what do you get $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \{x_1\} = 0$. So, which gives us so, this is corresponding to lambda 0, so, you get $\{x_1\} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$. this is λ = 3. What do we get is $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \{x_2\} = 0$ which gives us $\{x_2\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$.

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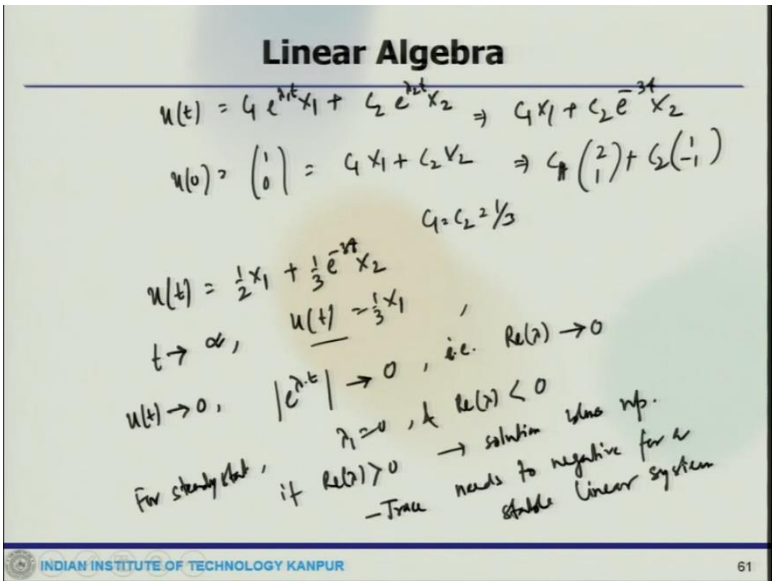

So, our solution

$$u(t) = C_1 e^{\lambda_1 t} X_1 + C_2 e^{\lambda_2 t} X_2 = C_1 X_1 + C_2 e^{-3t} X_2$$

now,

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = C_1 X_1 + C_2 X_2 = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which gives us $C_1 = C_2 = \frac{1}{3}$ so, the solution would be $u(t) = \frac{1}{2} X_1 + \frac{1}{3} e^{-3t} X_2$ when t tends to infinity the ut tends to $\frac{1}{3}X_1$ now, one can see from stability point of view what can happen is that.

Now, as ut tends to 0 it will be necessary for $e^{\lambda t}$ also tends to 0 which means that our Re(λ) or the real part of the lambda also tends to 0 now, for steady state λ_1 would be 0 and the Re(λ) would be less than 0 now, if Re(λ) greater than 0 then the solution closeup that means it diverges so, that trace leads to negative for stable linear system.

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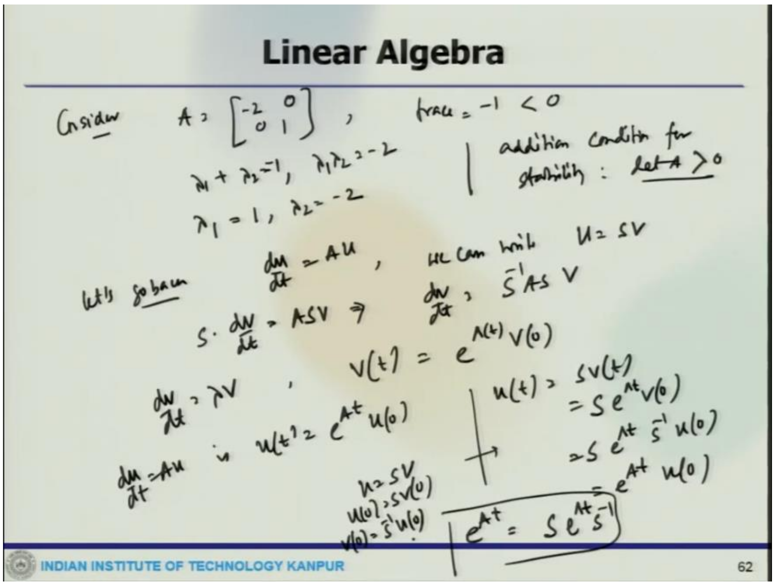

You can just again see they can consider an example of

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

where the trace is - 1 less than 0 here $\lambda_1 + \lambda_2 = -1$, $\lambda_1\lambda_2 = -2$ and we get $\lambda_1 = 1$ and $\lambda_2 = -2$. So, the additional condition for stability is that determinant of A has to be greater than 0 and let us go back to this example of let us $\frac{du}{dt} = Au$ we can write u = SV. So, means S$\cdot \frac{dV}{dt} = ASV$ and$\frac{dV}{dt} = S^{-1}ASV$. So, what do we get $\frac{dV}{dt} = \lambda V$. So, $V(t) = e^{\lambda(t)}V(0)$.

Now the solution of $\frac{du}{dt} = Au$ is $u(t) = e^{At}u(0)$ and what we write here $u(t) = SV(t)$ which is $Se^{At}V(0)$= $Se^{At}S^{-1}u(0) = e^{At}u(0)$. so you can say $e^{At} = Se^{At}S^{-1}$. so, this is you are doing that because you have u = SV and u(0)= SV(0). So you have V(0) = $S^{-1}u(0)$.

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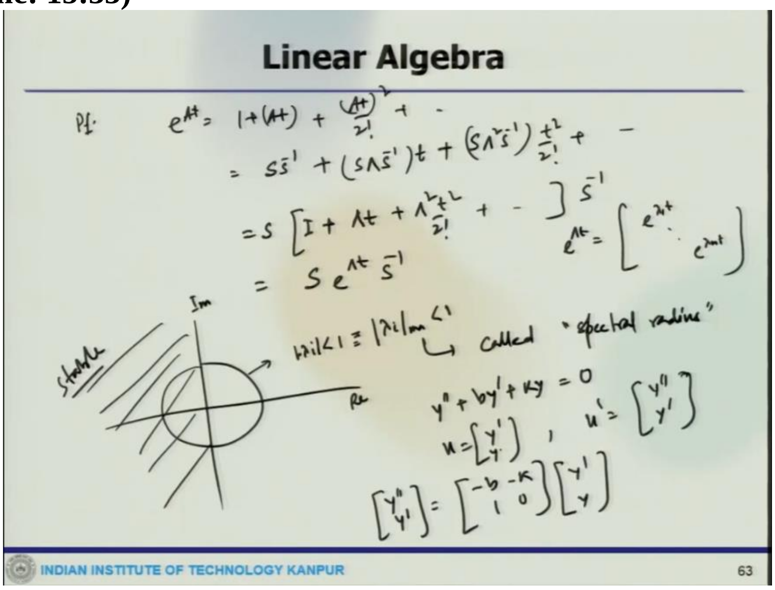

So how do we show that $e^{At} = 1 + (At) + \frac{(At)^2}{2!} + \cdots$ just we are expanding the exponential series and so on. Now, this we can write

$$e^{At} = SS^{-1} + (S\lambda S^{-1})t + (S\lambda^2 S^{-1})\frac{\lambda^2}{2!} + \cdots$$

So, this we can take

$$e^{At} = S\left[I + \lambda t + \lambda^2 \frac{t^2}{2!} + \cdots\right]S^{-1}$$

where you have this is nothing but

$$e^{At} = Se^{\lambda t}S^{-1}$$

So, if you look at the conditions here this is the imaginary part this is the real part. So, what has to happen is that and this is the circle which say magnitude of λ_i less than or equal to 1 which is rather λ_i max less than 1. So, this compasses so, this side is the stable so, for stability this has to be satisfied that λ_{max} has to be less than 1 and this maximum Eigen value is called the spectral radius of the system.

So, for let us say you have $y'' + by' + ky = 0$ where you have $u = \begin{bmatrix} y' \\ y \end{bmatrix}$, $u' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$. So, we can write $\begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix}$. So, here when A is a matrix for x = b.

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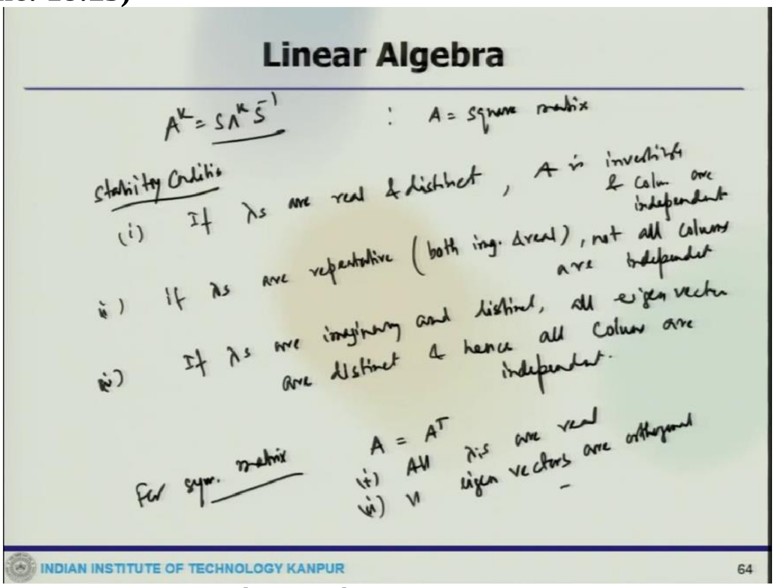

Now, for a differential equation $A^k = S\lambda^k S^{-1}$ symbols all calculations are true for any A provided it is a square matrix. So, A is has to be the square matrix. Now, what are the stability conditions

i) if lambdas are real and distinct then A is invertible, and columns are independent

ii) if lambdas are repeat did repetitive both imaginary and real case. Not all columns are independent.

iii) if lambdas are imaginary and distinct all Eigen vectors are distinct and hence all columns are independent now, for a symmetric matrix A is A transpose. So, what happens that here all lambdas are real to all Eigen vectors are orthogonal we can show that I mean that is a quick thing one can show.

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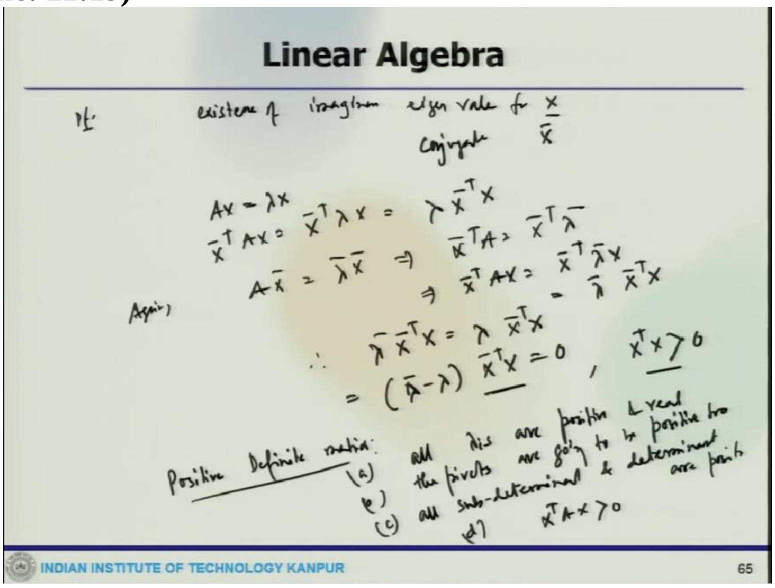

So, let us take in contradictory situation the like there exists an imaginary Eigen value, so, existence of imaginary Eigen value so, Eigen value for x so, obviously, if there is an imaginary Eigen value there will be a conjugate value and the conjugate vector is $\bar{x}$ which is the conjugate. So, the equation is Ax = λx now, $\bar{x}^T Ax = \bar{x}^T \lambda x = \lambda \bar{x}^T x$ again the $\bar{x}$ is a conjugate value so, we can write again we can write $A\bar{x}$=$\bar{\lambda}\bar{x}$. so, it $\bar{x}^T Ax = \bar{x}^T \lambda x$ which is $\bar{\lambda}\bar{x}^T x$.

So, what do we get $\bar{\lambda}\bar{x}^T x = \lambda \bar{x}^T x$ so, $(\bar{\lambda} - \lambda)\bar{x}^T x$ =0. So, which means $x^T x$ > 0. So, determinant a would be the product of the pivots or product of the lambdas and sign of the pivots would be sign of the lambdas. So, that means, what do we have assumed that is not correct. So, we have the real system.

Now, there is another kind of matrix which is called positive definite matrix for this kind of matrix all lambdas are positive and real the pivots are going to be positive too then all sub determinant and determinant are positive last $x^T Ax$ > 0. So, when you have a positive definite matrix, this is what happens.

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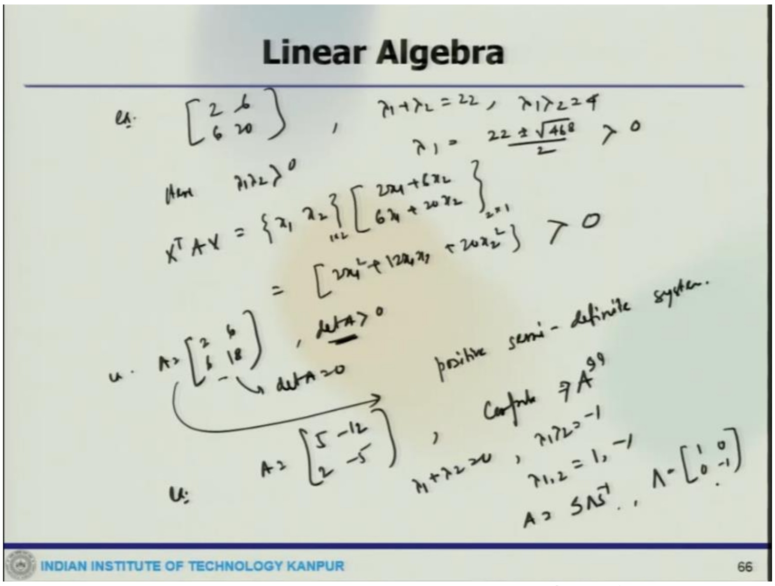

Now, we can look at an example for let us say you take $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$. Then here $\lambda_1 + \lambda_2 = 22$, $\lambda_1 \lambda_2 = 4$. So, if you put it back we get $\lambda_1 = \frac{22 \pm \sqrt{468}}{2}$. So, which is greater than 0. Here $\lambda_1 \lambda_2$ also greater than 0 all pivots are positive sub determinant is also greater than 0 and what do we get $x^T A x = \{x_1 \quad x_2\} \begin{Bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 20x_2 \end{Bmatrix}$ this is 1×2 system this is 2×1 system.

So, this is also to some extent positive. Now, another example, where A is like $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ except determinant A which is greater than 0, all properties are satisfied. So, this is again a specific case as here the determinant of A is 0. So, this is called so, all these properties like all lambdas positive and real pivots are going to be positive to sub determinant and determinant to positive $x^T x$ greater than 0.

Now, since the determinant is A= 0, this is called positive semi definite system. So, they are also useful while deriving linear system of equations. So, the square matrix is the most ideal system. Now, we can just quickly look at a few examples, like let us say we have a matrix $\begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix}$, we need to compute A^{99}. So, here, we get $\lambda_1 + \lambda_2 = 20$, $\lambda_1 \lambda_2 = -1$. So, $\lambda_1, \lambda_2 = 1, -1$. So, A, so, this is how we can write A, and this guy would be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

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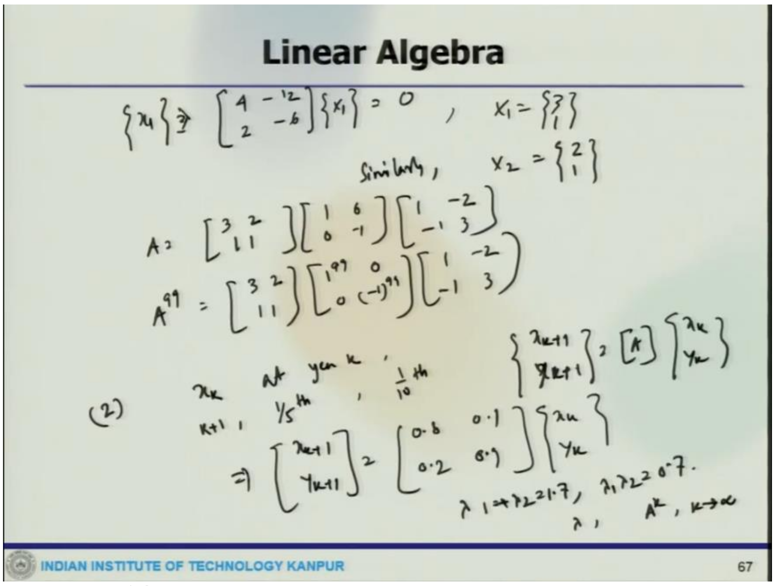

And we have $\{x_1\}, \begin{bmatrix} 4 & -12 \\ 2 & -6 \end{bmatrix} \{x_1\} = 0$, so, we get $\{x_1\} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. And, we get similarly, we get $\{x_2\} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, A would be $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. So, we can compute A^{99} would be $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{99} & 6 \\ 0 & (-1)^{99} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. So, this is how one can compute. So, like another example one can take similar way, let us say suppose, x_k is the fraction of students preferring calculus over linear algebra at here k.

The remaining fraction prefers linear algebra. Now, at here k+1 1/5th of those preferring calculus changes their mind and the same here 1/10th of linear algebra changes to calculus. So, we can find the limit for k and when it reaches a steady state so, we can formulate the problem like

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{Bmatrix} x_k \\ y_k \end{Bmatrix}$$

So, now here after doing the calculation, we get $\lambda_1 + \lambda_2 = 1.7$, $\lambda_1 \lambda_2 = 0.7$. So, then you can find out all the lambdas and then we find out the A^k and then we can see when k tends to infinity, what would be the limit for this. So, we can look at some more examples before moving to the next discussion, so, that you will get an idea how the system of the linear system can be used for this kind of solving of this kind of linear system. So, we will stop here and continue the discussion in the next session.