Computational Science in Engineering
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Lecture – 11
Linear Algebra

So, let us continue the discussion on this matrix Eigen values and Eigen vectors and we are seeing how the Eigen values and Eigen vector they are important. So, you have looked at the positive definite matrix and also looked at an example where this is not exactly positive definite it is a positive semi definite matrix and there are certain properties that one has to satisfy that. So, now using this nice property of Eigen values and Eigen vectors you can solve some of the linear systems and we have looked at couple of examples like this here.

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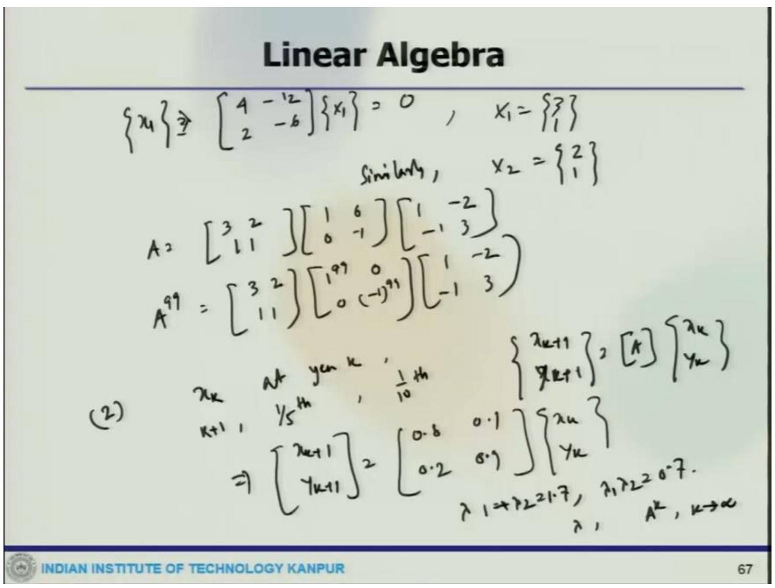

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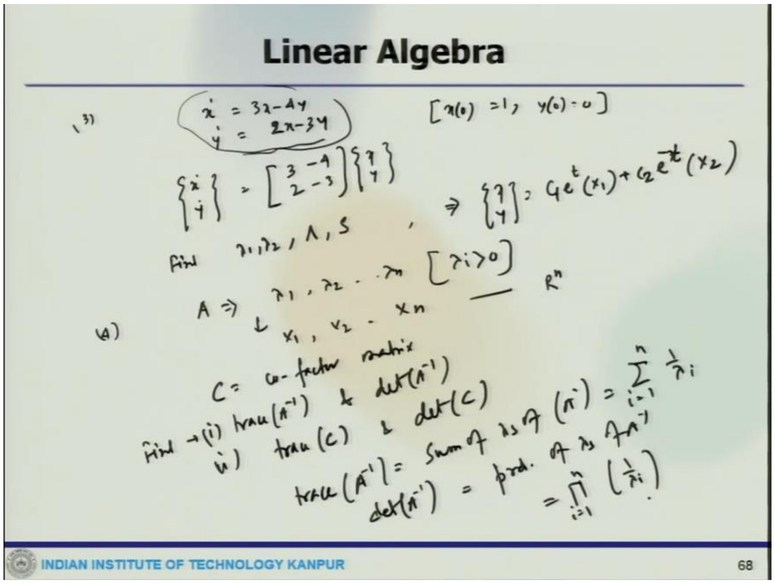

Now we can move to some more example like let us say one kind of you have a system

$$\dot{x} = 3x - 4y$$

and you have

$$\dot{y} = 2x - 3y$$

Where $x(0) = 1, y(0) = 0$. So, you can see these are ordinary differential equation that would be our next point of discussion that so how we can use our concept of matrix Eigen values and all these to solve that. So, this is our system so we can find λ_1, λ_2 then these.

And this Eigen vector matrix Eigen value matrix and finally the solution would be like

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = c_1 e^{t}(x_1) + c_2 e^{-t}(x_2)$$

So, now let us say A has Eigen values of λ_1, λ_2 and λ_n where all the $\lambda_i > 0$ and which is forming the Eigen vectors like x_1, x_2 and x_n. So, which are forming the basis for R^n and C being the co-factor matrix.

So, we can what we can find like

(i) the trace of A inverse and determinant of A inverse.

(ii) trace of C and determinant of C.

So, it would be quite nice to use the property like

$$trace\ (A^{-1}) = sum\ of\ \lambda s\ of\ (A^{-1}) = \sum_{i=1}^{n} \frac{1}{\lambda_i}$$

and

$$det\ (A^{-1}) = product\ of\ \lambda s\ of\ (A^{-1}) = \prod_{i=1}^{n} \frac{1}{\lambda_i}$$

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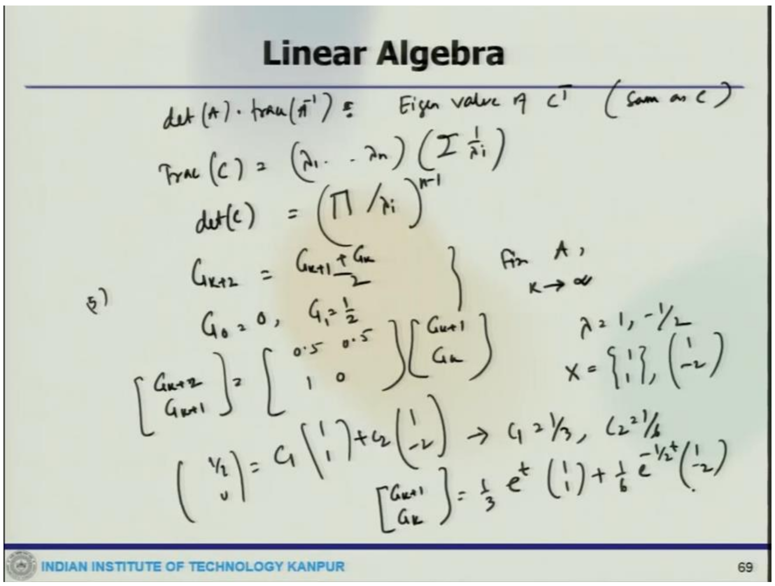

Now

$$det\ (A^{-1}) \cdot trace\ (A^{-1}) = Eigen\ values\ of\ C^T \quad (same\ as\ C)$$

So, what we can say

$$trace\ (C) = (\lambda_1 \dots \lambda_n)\left(\sum \frac{1}{\lambda_i}\right)$$

and

$$det\ (C) = \left(\prod \frac{1}{\lambda_i}\right)^{n-1}$$

So, I mean you can look at multiple examples like this I mean another example one can see like you have a system

$$G_{K+2} = \frac{G_{K+1} + G_K}{2}$$

you have $G_0 = 0$ and $G_0 = \frac{1}{2}$.

So, we can find A and then also the limit for k tends to infinity so what it is

$$\begin{bmatrix} G_{K+2} \\ G_{K+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix}$$

So, λ will be 1, -1/2 so the vector would be $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. So, putting that back we get

$$\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which will get us $C_1 = 1/3$, $C_2 = 1/6$. So, the

$$\begin{bmatrix} G_{K+2} \\ G_{K+1} \end{bmatrix} = \frac{1}{3} e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{6} e^{-\frac{1}{2}t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

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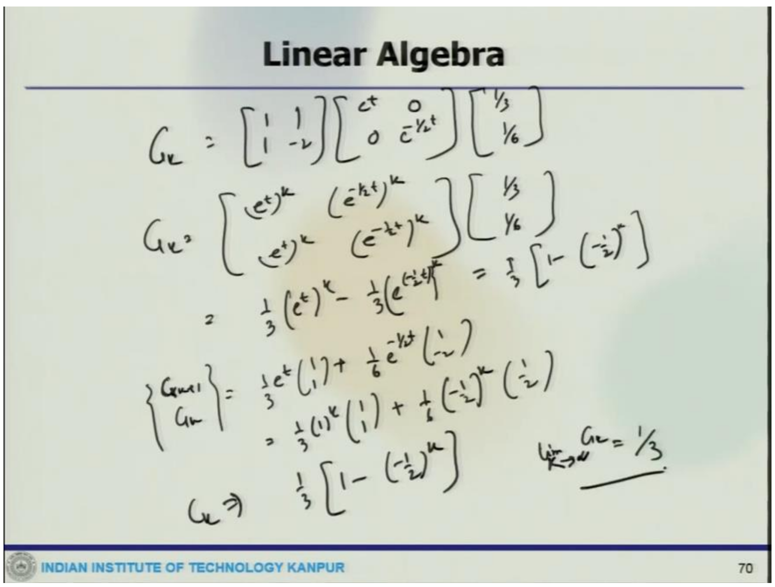

So, which one can write

$$G_K = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix}$$

so, we can write

$$G_K = \begin{bmatrix} (e^t)^K & \left(e^{-\frac{1}{2}t}\right)^K \\ (e^t)^K & \left(e^{-\frac{1}{2}t}\right)^K \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix}$$

$$G_K = \frac{1}{3}(e^t)^K - \frac{1}{3}\left(e^{-\frac{1}{2}t}\right)^K = \frac{1}{3}\left[1 - \left(-\frac{1}{2}\right)^K\right]$$

now if we do that so now expanding this and get this one then we can write again

$$\begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix} = \frac{1}{3} e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{6} e^{-\frac{1}{2}t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So, one can also write

$$\begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix} = \frac{1}{3} (1)^{K} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{6} \left(-\frac{1}{2}\right)^{K} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Where,

$$G_K = \frac{1}{3}\left[1 - \left(-\frac{1}{2}\right)^{K}\right]$$

Now for the limit where k tends to infinity for that $G_K = \frac{1}{3}$. So, I mean the idea here is that we can use this concept of Eigen value and Eigen vector and their decomposition to find out or solve this kind of a linear system.

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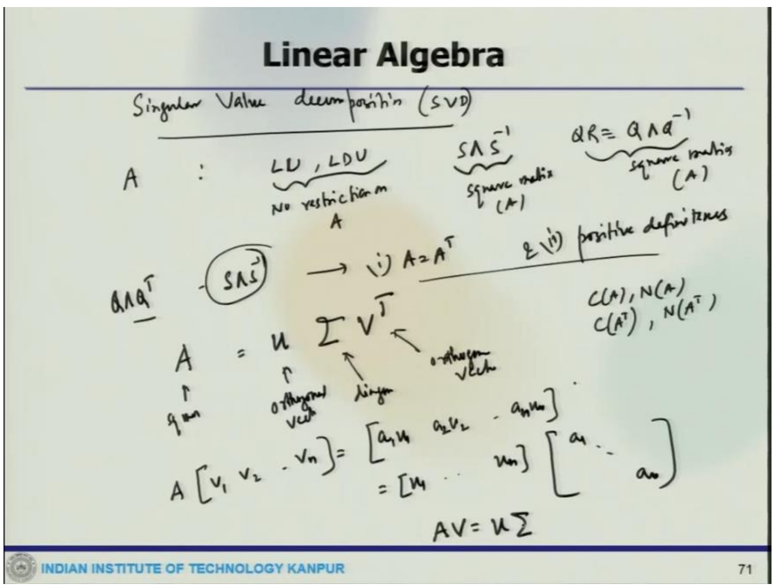

Now another important issue just to discuss the that singular value decomposition which is called SVD. Now let us consider a matrix A so we can do a lot of different kinds of decomposition like LU, or LDU or we can do $S\lambda S^{-1}$ or we can do QR like $Q\lambda Q^{-1}$. So, if you look at this kind of decomposition LU or LDU there is no restriction on A that means he could be anything still we can do this kind of decomposition to have these kinds of decompositions it has to be square matrix.

And also, for this it has to be square matrix. So, there are certain restrictions now $Q\lambda Q^T$ is a special $S\lambda S^{-1}$ decomposition for the conditions given as A = A transpose and positive definiteness for these situations this is a special decomposition of this kind of decomposition $S\lambda S^{-1}$. Now what we can do let us say A is a square matrix we can say

$$A = u \sum V^T$$

this is orthogonal vector this would be diagonal this is again orthogonal vector.

Now this matrix has four fundamental subspaces like column space, null space, column space of A transpose, null space of A transpose. So, these are the what we can write let us say

$$A[v_1, v_2, \dots v_n] = [a_1 u_1, a_2 u_2, \dots a_n u_n]$$

which we can say

$$A[v_1, v_2, \dots v_n] = [u_1, u_2, \dots u_n] \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$

like that. So, what it gives us

$$AV = u \sum$$

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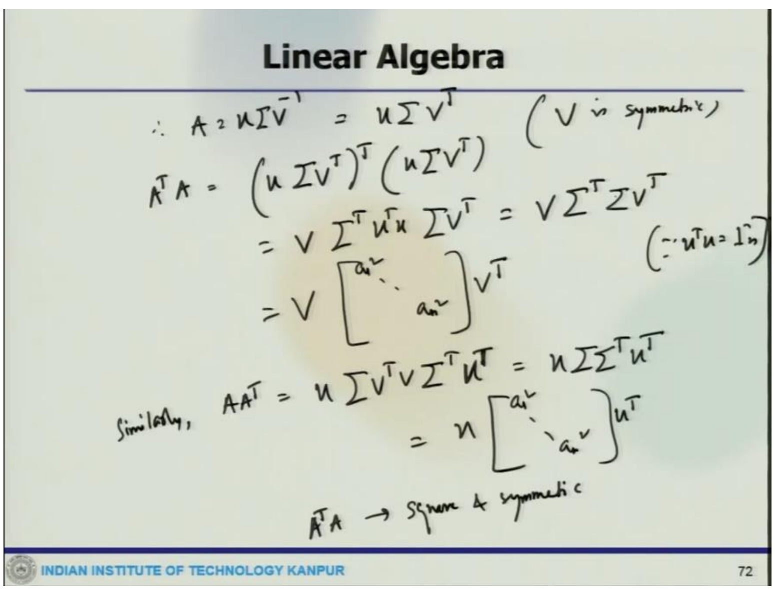

So,

$$A = u \sum V^{-1}$$

or one can say

$$A = u \sum V^T$$

V is symmetric. Now we can write

$$A^T A = \left(u \sum V^T\right)^T \left(u \sum V^T\right)$$

So, you write the

$$A^T A = V \sum{}^T u^T u \sum V^T = V \sum{}^T \sum V^T$$

So, you can write

$$A^T A = V \begin{bmatrix} {a_1}^2 & & \\ & \ddots & \\ & & {a_n}^2 \end{bmatrix} V^T$$

Similarly, we can write

$$AA^T = u \sum V^T V \sum{}^T u^T$$

which is

$$AA^T = u \sum \sum{}^T u^T$$

which is

$$AA^T = u \begin{bmatrix} {a_1}^2 & & \\ & \ddots & \\ & & {a_n}^2 \end{bmatrix} u^T$$

So, what is important here is that $A^T A$ is always square and symmetric. So, we can again see some small examples like 2×2 systems or something like that.

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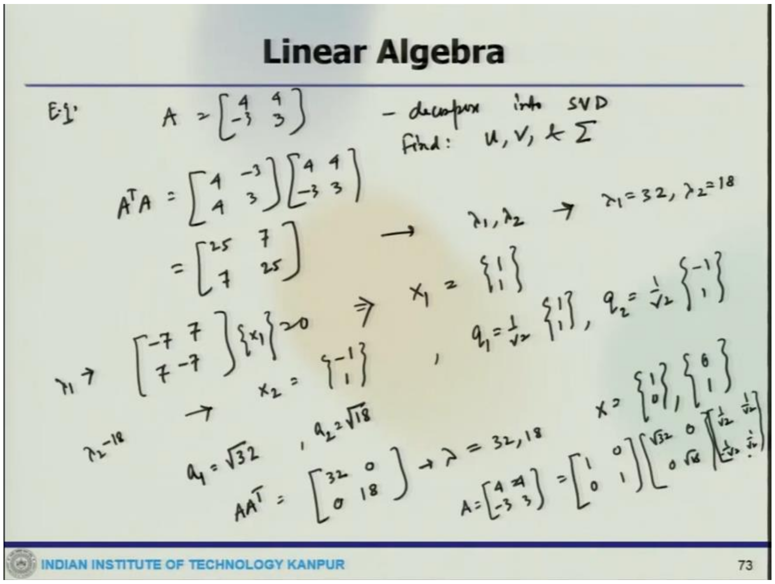

And can see how it works for example let us say we taken system

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

so we can decompose into SVD. So, what we need to find out to do that u, v and sigma these are the matrices that one has to find out. So, first let us see

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

So, A transpose A this would become

$$A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

so which now A transpose A once we get it then for A transpose A.

What do we now we find out the λ_1, λ_2 here. So, once you do that calculation here you get λ_1 = 32, λ_2= 18. So, this is what you get so once you get the λs we can find out the Eigen vectors corresponding to that so like for

$$\lambda_1 = \begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \{X_1\} = 0$$

So, after solving we get

$$X_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Now similarly for $\lambda_2 = 18$ we can get $X_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$.

So, what we get then

$$q_1 = \frac{1}{\sqrt{2}}\begin{Bmatrix}1\\1\end{Bmatrix}$$

And

$$q_2 = \frac{1}{\sqrt{2}}\begin{Bmatrix}-1\\1\end{Bmatrix}$$

which is orthogonal vectors or basis vectors like this. So $a_1 = \sqrt{32}$ and $a_2 = \sqrt{18}$ again we say

$$AA^T = \begin{bmatrix}32 & 0\\0 & 18\end{bmatrix}$$

which is gives λ is 32 and 18. So, for that the vector is $\begin{Bmatrix}1\\0\end{Bmatrix}$ and $\begin{Bmatrix}0\\1\end{Bmatrix}$. So, now when you write the decomposition for that A can be written as which is

$$A = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}$$

which is

$$A = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{32} & 0\\0 & \sqrt{18}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

So, this is how you can actually decompose the system.

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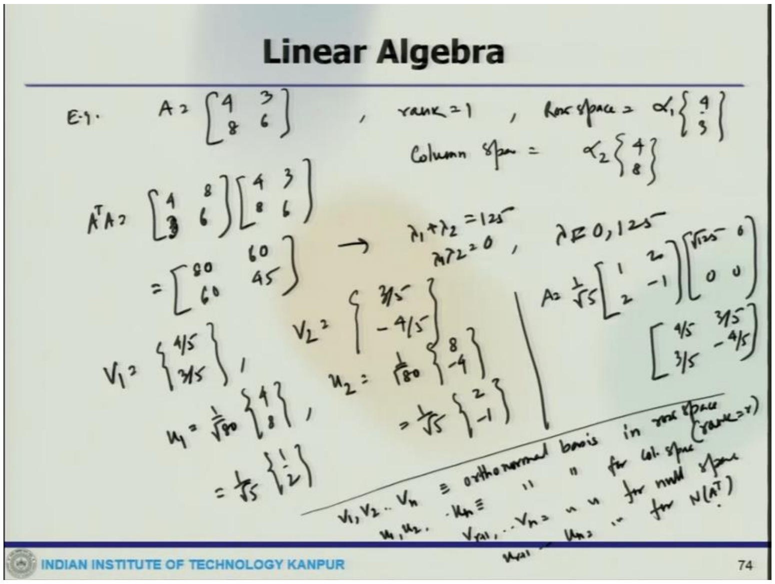

Now we can look at another example of such kind which is let us say slightly

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

here the rank would be 1 and so the row space has $\alpha_1 \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$ vector column space has $\alpha_2 \begin{Bmatrix} 4 \\ 8 \end{Bmatrix}$ that means one column would be independent. So, another column is not so here

$$A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

so, this is

$$A^T A = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

So, this gives us $\lambda_1 + \lambda_2 = 125$ and $\lambda_1 \lambda_2 = 0$.

So, λ would be 0, 125 so we get

$$V_1 = \begin{Bmatrix} 4/5 \\ 3/5 \end{Bmatrix}$$

and

$$V_2 = \begin{Bmatrix} 3/5 \\ -4/5 \end{Bmatrix}$$

So,

$$u_1 = \frac{1}{\sqrt{80}} \begin{Bmatrix} 4 \\ 8 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$u_2 = \frac{1}{\sqrt{80}} \begin{Bmatrix} 8 \\ -4 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

then we have

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$$

So, you can decompose in such fashions so here the important thing to note here is that v_1, v_2 *and* v_n these are all orthonormal basis in a row space.

And rank would be obviously are similarly u, u_2 *and* u_n these are orthonormal basis for column space. That means the vectors that we are finding out these v_1 vectors are the ortho normal basis in row space u_1 vectors are the orthonormal basis in column space and $v_{r+1} \dots v_n$ these are orthonormal basis for null space and similarly $u_{r+1} \dots r_n$ they are orthonormal basis for null space of A transpose.

So, all these vectors that we have sort of found out they are these orthonormal bases they correspond to the orthonormal bases of these fundamental subspaces like row space or column space or null space of A transpose. So, now that is pretty much what I would like to talk on the linear algebra part were you have now so basically, we started with the row picture and column picture then we looked at the linear independence.

And vectors the column vectors and then existence for the solution like when we can have a solution for a linear system like $Ax = b$. Obviously the b has to lie in the column space of A and then from there during elimination process we identified the pivot variables or the independent columns or independent variables and dependent variables and then from there we have identified the independent columns or then the basis vector for the null space and the complete solution.

And then finally Eigen vector determinant and Eigen values and Eigen vectors and finally through the decompositions of singular value decomposition so that we can decompose these matrices and solve a linear system. So, we will stop here and continue the discussion on the other topics in the next class.