Computational Science in Engineering
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Lecture - 12
Ordinary Differential Equation

So, you finish the discussion on linear algebra part now we will move to the discussion on the differential equation. And in differential equation there are two different kinds of differential equations. One is the ordinary differential equation and the other would be partial differential equation. So, first we will continue the discussion on ordinary differential equation or rather we will talk about ordinary differential equation. And then we will touch upon partial differential equation before we move to the other topic. So, for how we define an ordinary differential equation.

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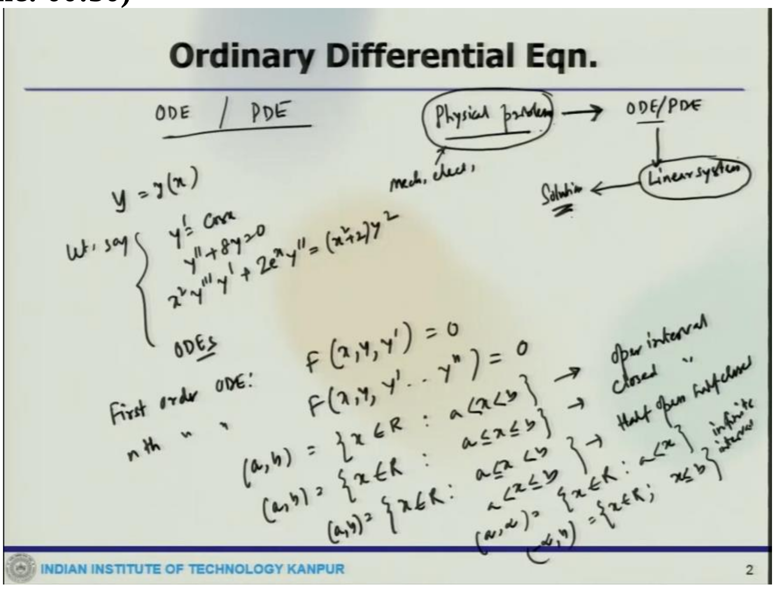

Essentially so as I said there are two different kinds of differential equation one could be ODE other could be PDE. So, these are called ordinary differential equation is ODE and other would be partial differential equation. Now where do we get these differential equations so basically, we are trying to solve some physical problem and when you are trying to solve some physical problem they lead to this ODE or PDE.

So, to represent this and this physical problem they are kind of represented through these ODE or PDE or system of ODE or PDE and then through some approximation they finally come down to the linear system. So, and this solution of the linear system or after solving the linear system we get the solution. So, this is where you see where it is important and that is what we have done so much of discussion on the linear system.

Because finally our goal or interest is to sort of solving some of this realistic physical problem and now, they are through physical problem through some approximations they finally come down to linear system and the solution of that only give us the solution to that particular problem. So, this is why we have talked about so much about linear. Now the physical problems they could be of any time like mechanical, electrical or any such system an ordinary differential equation is essentially in equation.

Which containing one or more derivations of the variables or like function of like this let us say you have

$$y' = \cos x$$

$$y'' + 8y = 0$$

Or

$$x^2 y''' y' + 2e^x y'' = (x^2 + 2)y^2$$

So, these are all different sorts of ordinary differential equations they are of different kinds and how they are defined and all these that as we go on with the discussion you can see that.

And here the highest derivative will govern the order of the differential equation. So, the first order ODE can be written as

$$F(x, y, y') = 0$$

So, if it is an nth order ODE then in general one can write

$$F(x, y, y' \dots y^n) = 0$$

Now when you define something like this ODE is the one important things is the definition of the intervals like for example if you see (a, b) for x belongs to R where x is between a and b this is called open interval.

Now we can have (a, b) which is for x belongs to R we have x this is closed interval that means this is bounded between a and b. Now one can have (a, b) where x belongs to R and x less than b or we can have x less than b greater than R. So, this is called half open half-closed interval or we can have an infinity where again x belongs to R and x is greater than a or minus infinity b where x belongs to R x less than b this is infinite interval.

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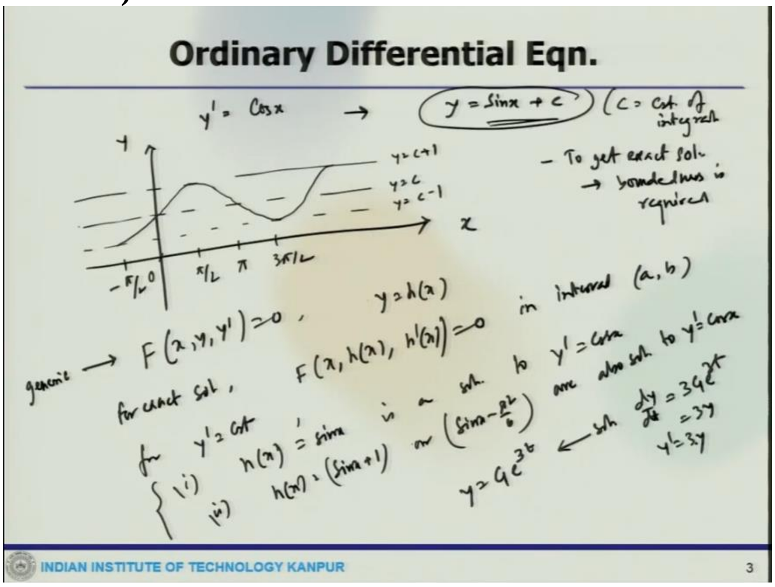

So, let us see a simple example of $y' = \cos x$, so the solution of that would be $y = \sin x + c$ here c is the constant of integration. So, if you look at visually or graphically this is x and this is how y is varying so this is 0, $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$. So, this is the function how it is going to vary like then it come down like that. So, this could be $y = c,\ y = c - 1,\ y = c + 1$ so that is how the function would vary.

Now as I said c is the constant of the integration if c is the unknown then a family of curves is formed. So, this is the generic solution that $y = \sin x + c$ representing a family of curves to obtain the exact solution. Hence it is important that the boundedness so to find exact solution the boundedness or other intervals should be provided. So, to get exact solution boundedness is required then only one can find otherwise this is going to represent a family of curves.

Let us consider a first order system like $F(x, y, y') = 0$, where $y = h(x)$ is a situation or solution to the system. So, this statement is complete only if generic solution is needed for exact solution, it should be stated in a different way as $F(x, h(x), h'(x)) = 0$ in interval of a and b. So, this is for exact solution and this is more like a generic representation. Now for $y' = \cos x$ what one can write $h(x) = \sin x$ is a solution to $y' = \cos x$.

Or one can say $h(x) = \sin x + 1$ or $\left(\sin x - \frac{\pi^2}{6}\right)$ are also solution to $y' = \cos x$. So, this is how things can change depending on how you define. So, without interval this could give to a provide with a generic solution. Similarly, if you have a let us see if you have $y = c_1 e^{3t}$ this is a solution to the problem $\frac{dy}{dt} = 3c_1e^{3t}$ or $3y$ so one can say $y' = 3y$. So, this equation is representing an exponential growth or decay. So, now it is quite important to note that these things play a role how define the system.

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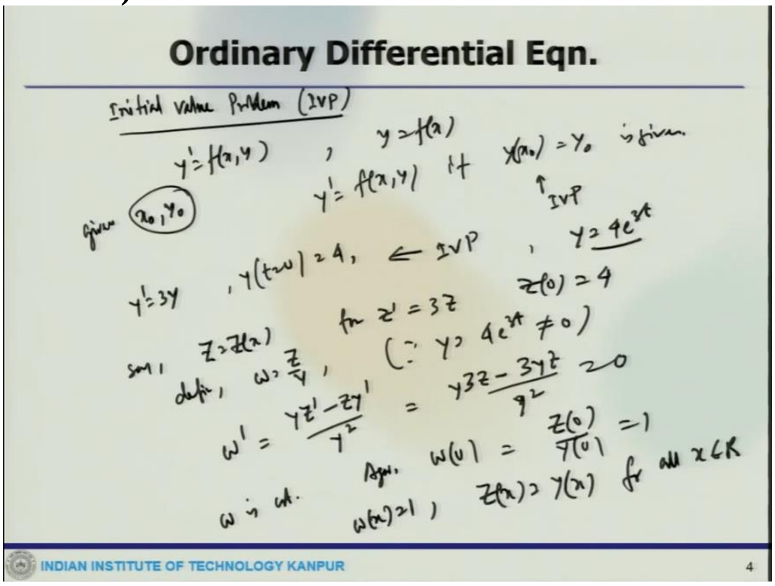

Now let us start with initial value problem which is IVP say we have a problem $y' = f(x, y)$ which solution $y = f(x)$. Now for a particular value of x_0 and y_0 if that is provided let us say this is given this solution is going to be the exact solution. So thus, for $y' = f(x, y)$ if $y(x_0)$= y_0, is given this is called initial value problem. So, here we are starting or rather dealing with the first order ODE at the beginning.

And as I said the highest derivative would determine the order of that ODE. So, we are starting with that now for example $y' = 3y$ is only first order ODE say $y(t_0) = 4$ then this value will make the problem and initial value problem. And where we can get an exact solution that $y = 4e^{3t}$ is the exact solution. And we can show that the solution for functional curve equals to 3y.

Now let us say Z is function of $Z(x)$ or so, one can show that this is a solution like that so $Z(x)$ equals to solution other solution for $Z' = 3Z$ given that $Z_0 = 4$. Let us define

$$w = \frac{Z}{y}$$

since $y = 4e^{3t}$ which is not equal to 0 so we can write this is

$$w' = \frac{yZ' - Zy'}{y^2} = \frac{y3Z - 3yZ}{y^2} = 0$$

this is a constant again $w(0) = \frac{Z(0)}{y(0)} = 1$. So, what do we get? This is 1 so $Z(x) = y(x)$ for all x belongs to R.

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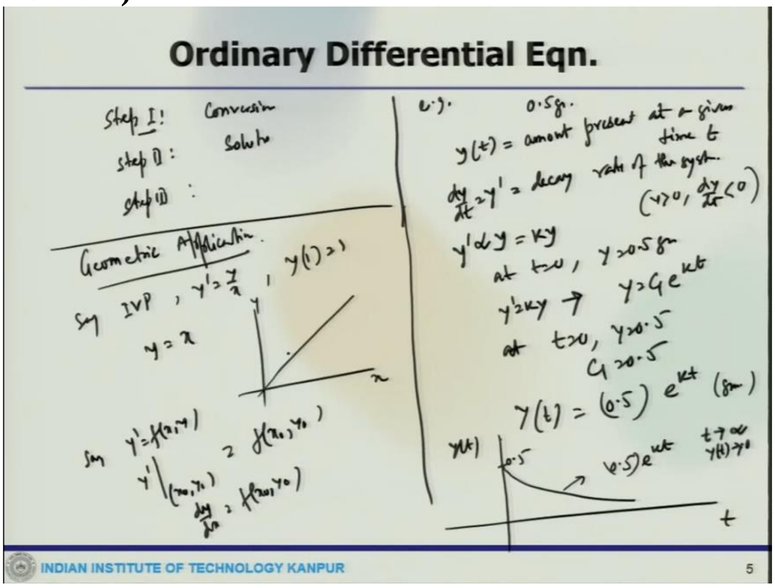

Now how we define? There are different steps first step 1 we have to convert the physical problem to a mathematical formulation. In step 2 we have to solve the mathematical problem so this is conversion solution. Step 3 we can provide the physical interpretation of the mathematical solution. Now like if we say let us say consider a radioactive substance with initial quantity of 0.5 gram and the mass at a given point of time is to be found.

Now as per the physics law, law of physics says that the substance decays at a rate proportional to the amount present at a certain time. So that means $y(t)$ is amount present at a given time t so the decay rate of the system like $\frac{dy}{dt} = y'$ this is the decay rate of the system where $y > 0$, $\frac{dy}{dt} < 0$. So, $y' \propto y$, so we can say this is $y' \propto y = ky$, and $\frac{dy}{dt}$ is the decay rate.

Now if we know k has to be negative and initial condition is given at $t = 0$, $y = 0.5$ gram. So, the solution of this $y' = ky$ would be $y = c_1 e^{kt}$. Now at $t = 0$, $y = 0.5$. So, c_1 is 0.5 so our exact solution is

$$y(t) = 0.5e^{kt}$$

that much of gram. So, if you look at that this is our $y(t)$ this is with time this is how the decay would be and this would be 0.5 and this curve is $0.5e^{kt}$.

So, when t tends to infinity $y(t)$ also tends to 0 that means for a larger timeframe the remaining amount is not enough for significant decay. Now we can also have a geometric application where let us say the initial value problem is $y' = \frac{y}{x}$ with $y(1) = 1$. So, the solution would be $y = x$ so this would be a straight line if I draw it this would be a straight line like that which is passing through the origin.

So, it helps to identify the geometric configuration. Now let us say $y' = f(x, y)$ and for any given point of time x_0, y_0 this is $\frac{dy}{dx} = f(x_0, y_0)$. So, these $\frac{dy}{dx} = f(x_0, y_0)$ that indicates this at a given point of time at x_0, y_0 indicates the slope of the solution at $x_0, and\ y_0$. So, this can also help us to draw small lines of different slopes which when assembled gives the entire solution like this.

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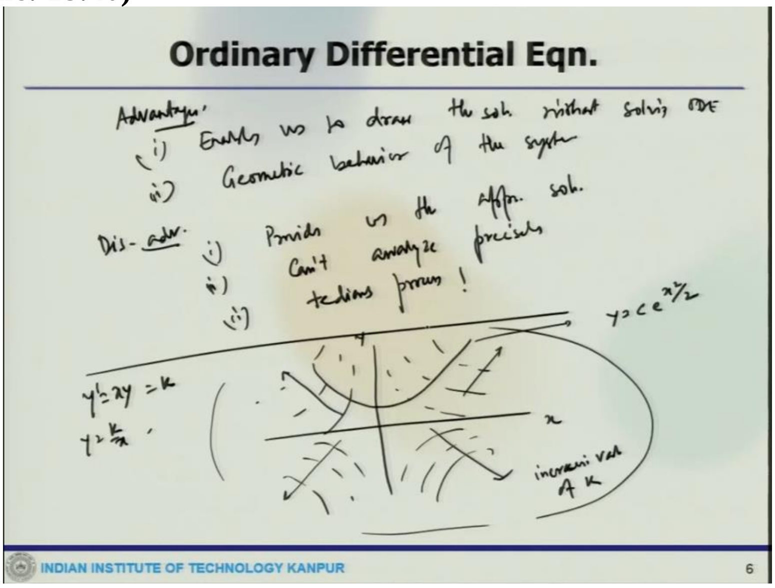

So, there are some advantages like it enables us to draw the solution without solving ODE they are also geometric behavior of the system at the same time there are certain disadvantages like it provides us the approximate solution cannot analyze precisely also this is obviously tedious process. So, these are the things which are associated with that kind of situation. So, let us see $y' = xy = k$ and $y = \frac{k}{x}$.

So, without solving this guy $y' = xy$ we can get a curve for x and y with the different values of k. For example, let us say for a draw this is x this is y. So, we get this and this is how so these are the values of k then this is the $y = ce^{\frac{x^2}{2}}$. So, these are all increasing value of k so these are all family of curves of hyperbolic nature. So, the solution would be hyperbolic in nature but this is if you look at this complete picture, they provide you instead of a particular one they provide you a family of curve for this.

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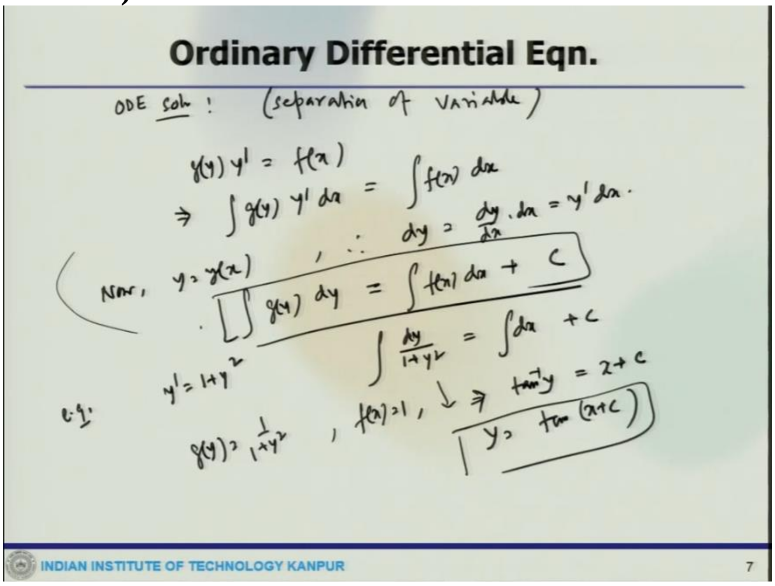

Now we try to find out the ODE solution so the first thing that we will look at separation of variable. So, most of the ODE's can be transformed into like

$$g(y)y' = f(x)$$

after doing some algebraic manipulations. So, which gives us that when we do the integration

$$\int g(y)y'\,dx = \int f(x)\,dx$$

now $y = y(x)$. So, what we can write $dy = \frac{dy}{dx}dx = y'dx$ so this guy turns into

$$\int g(y)\,dy = \int f(x)\,dx + c$$

So, this is how you can get a solution out we can look at an example then it would be quite easy to understand what is going on there. So, here how do we separate

$$\int \frac{dy}{1+y^2} = \int dx + c$$

if you correlate with this kind of system then $g(y) = \frac{1}{1+y^2}$ and $f(x)$ here is 1. So, these guy after integration gives

$$tan^{-1}y = x + c$$

or

$$y = \tan(x + c)$$

So that is the solution to this particular system.

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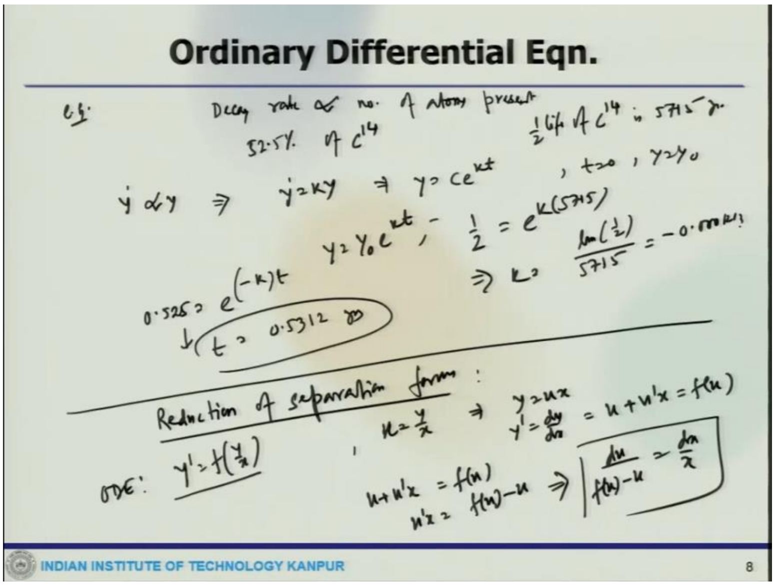

Similarly, we can look at another example of radioactive material like the decay rate is proportional to the number of atoms present. So, the sample of original material is found to have 52.5% of C^{14} of living tissue. So, where you can assume that amount of C^{14} remains constant and half life of C^{14} is 5715 years. So, we can find the age of the sample so here $y' \propto y$.

So, it turns out to be equation of $y' = ky$ and the solution is $y = ce^{kt}$. Now at $t = 0$, $y = y_0$. So, we get

$$y = y_0 e^{kt}$$

Now

$$\frac{1}{2} = e^{k(5715)}$$

so, we get k is something so if you get that this would be

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5715} = -0.0001213$$

some think like that. So, from there now if you put it

$$0.525 = e^{(-k)t}$$

it gives us t is somewhere 0.5312 years or something like that.

So, now you get an idea how this system is actually converted to the linear system. Now the other way one can solve the reduction of separation form. So, let us say ODE is $y' = f\left(\frac{y}{x}\right)$. So, in this kind of situation when you have an ODE of this kind let us assume that we define a variable $u = \frac{y}{x}$ so what happens that which means $y = ux$. So, y' which is $\frac{dy}{dx} = u + u'x = f(u)$.

So, now y' here is expressed as a function of u so, what we can write $u + u'x = f(u)$. So, $u'x = f(u) - u$, and now we do the separation of variable like

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

So that means we use the separation of variable approach. But before that we have to reduce that form.

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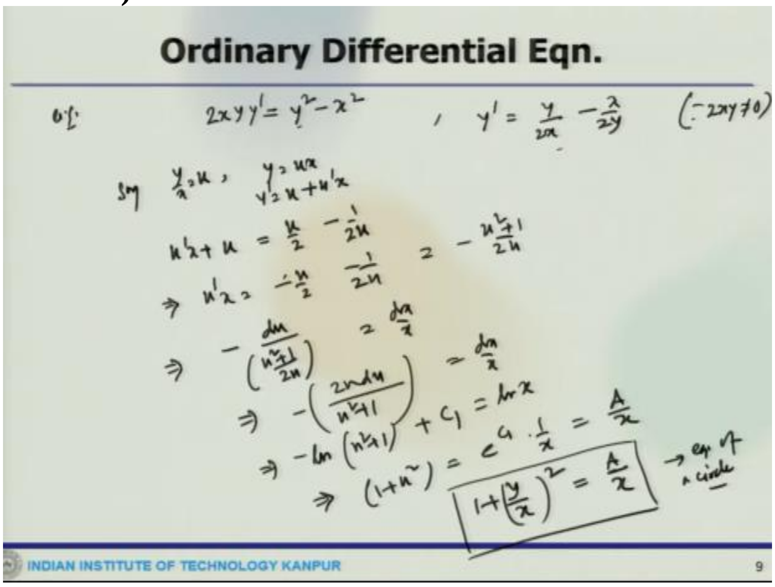

So, let us see an example how it looks like. Let us say we are equation

$$2xyy' = y^2 - x^2$$

So, here

$$y' = \frac{y}{2x} - \frac{x}{2y}$$

So, we divide by 2xy assuming that 2xy is not 0. So, let us say

$$\frac{y}{x} = u$$

so

$$y = ux$$

$$y' = u + u'x$$

So,

$$u + u'x = \frac{u}{2} - \frac{1}{2u}$$

so, we can write

$$u'x = -\frac{u}{2} - \frac{1}{2u} = -\frac{u^2+1}{2u}$$

So, now we will do the separations of variables

$$-\frac{du}{\left(\frac{u^2+1}{2u}\right)} = \frac{dx}{x}$$

So, we get

$$-\left(\frac{2udu}{u^2+1}\right) = \frac{dx}{x}$$

which is

$$-\ln(u^2+1) + c_1 = \ln x$$

So, we write

$$(1+u^2) = e^{c_1}\frac{1}{x} = \frac{A}{x}$$

So,

$$1 + \left(\frac{y}{x}\right)^2 = \frac{A}{x}$$

So, if you see this, this is an equation of a circle actually this is an equation of a circle.

So, I mean you can see how we reduce the form and reduce to the form where we can do the separation of variable and then finally, we can get the solution. So, there are so we have to kind of deal with ODEs in a different way and we will see how different system I mean as we go along with the discussion you can see how things are complicated or getting complicated and then how you reduce the form further getting the solution we will stop here and continue the discussion in the next session.