Computational Science in Engineering
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Lecture - 13
Ordinary Differential Equation

Let us continue the discussion on ordinary differential equation. So, we just started of discussing of ODE and first thing that we have looked at how once when I mean these are all linear first order ODE is where we can do the separation of variables. And then the next situation which may arise is that when the form is not in a given situation where you can do directly the separation of variable. So, but doing some substitution of variable you can finally bring down to the separation of variable.

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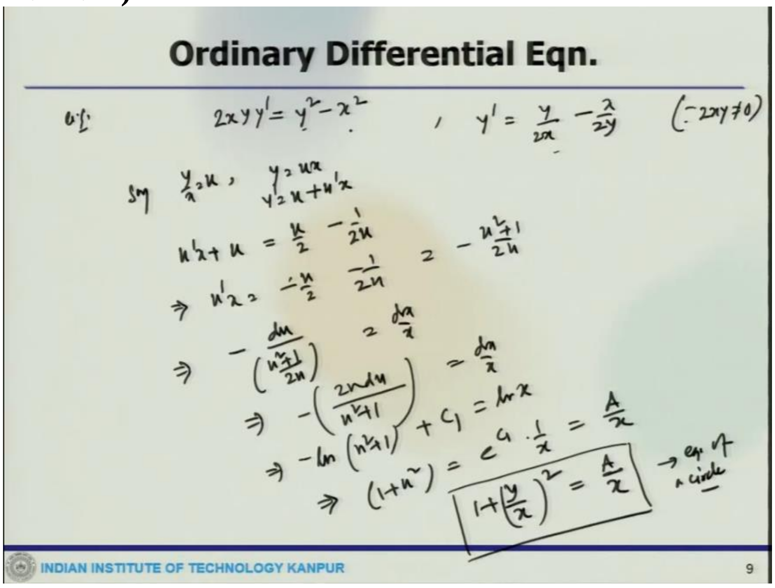

So that you can reduce to the separation of variable this is where we stopped in the last thing where if you look at the ODE directly which we cannot use separation of variable but we reduce to that situation where we can use separation of variable from here and get the solution.

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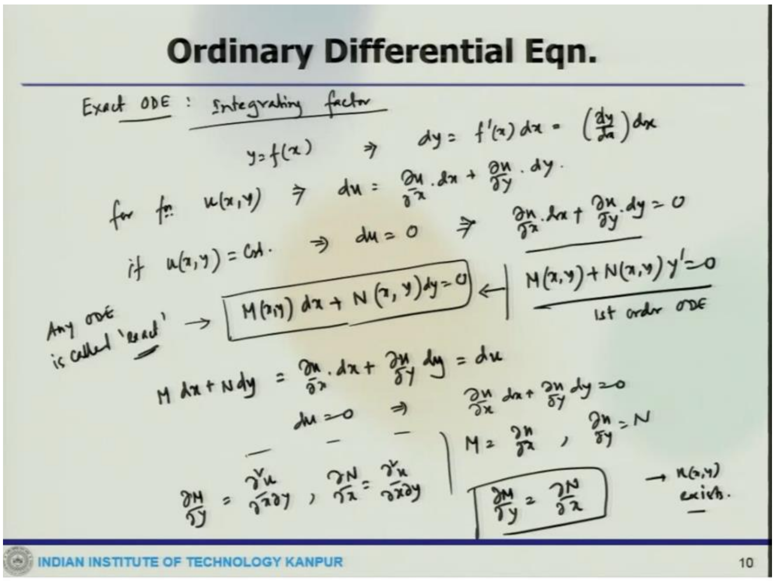

Now we will talk about another situation where there is this is exact ODE which is where we have integrating factor. So let us say any function which is $y = f(x)$ which can be written in another form like $dy = f'(x)dx = \left(\frac{dy}{dx}\right)dx$. So, similarly for function $u(x, y)$ one can write

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Now if this function $u(x, y)$ is constant then this du would be 0.

So, which means the one which you can write is that $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$. So this is in a form something like if we consider the first order ODE of the type where we will write that

$$M(x, y)dx + N(x, y)dy = 0$$

I mean let us say in a form where you have

$$M(x, y) + N(x, y)y' = 0$$

So, this is the first order ODE if this is in form so we can write that

$$Mdx + Ndy = 0$$

So, this is where we can do that now any ODE so essentially what it said that any ODE which can be expressed in this particular form is called exact. So, what we can write

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$$

Now since du = 0 which means this guy is going to be 0. So, my $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$. Now one can see that

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

So, the condition that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

this is an condition that needs to be satisfied to write the exact ODE as such that u that function u exist. Otherwise, system may not be exact but u does not exist. So, one thing is that when any particular first order ODE which can be expressed in this $Mdx + Ndy$ format then also it has to satisfy that $\frac{\partial M}{\partial y}$ would be $\frac{\partial N}{\partial x}$.

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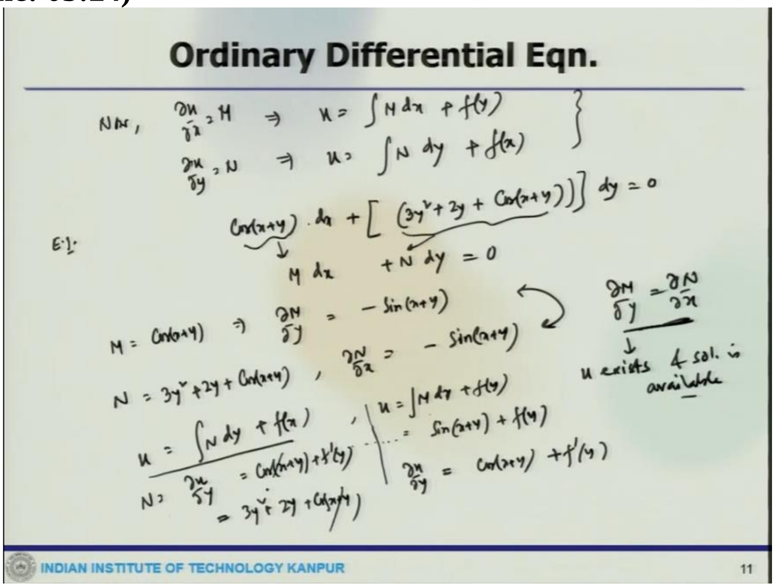

Otherwise, this is not exact now we have $\frac{\partial u}{\partial x} = M$ which means

$$u = \int M dx + f(y)$$

and

$$\frac{\partial u}{\partial y} = N$$

which mean this would be

$$u = \int N dy + f(x)$$

So, this is how you are going to find out the variable or the function. So, we take an example and see how let us say

$$\cos(x+y)\,dx + \,[(3y^2 + 2y + \cos(x+y))]dy = 0$$

So, the equation already given in the form like $Mdx + Ndy = 0$.

So, this guy is supposed to be M and this guy is supposed to be N. Now here M as I said

$$M = \cos(x+y)$$

So,

$$\frac{\partial M}{\partial y} = -\sin(x+y)$$

and

$$N = (3y^2 + 2y + \cos(x+y))$$

So,

$$\frac{\partial N}{\partial x} = -\sin(x+y)$$

So that means they are equal so the condition of

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

so, this is satisfied. So, which means u exist and the solution is available. So, this is one of the conditions that has to be checked before we say anything about it.

Now what do we get

$$u = \int Ndy + f(x)$$

which is now we have so one is this

$$u = \int Mdx + f(y)$$

So, which is like

$$Mdx = \sin(x+y) + f(y)$$

So, this is what we are getting from M dx integration. Now using this we can here if we get

$$\frac{\partial u}{\partial y} = \cos(x+y) + f'(y)$$

So, we will equate that equals to N so which means

$$N = \frac{\partial u}{\partial y} = 3y^2 + 2y + \cos(x+y)$$

that cancels out.

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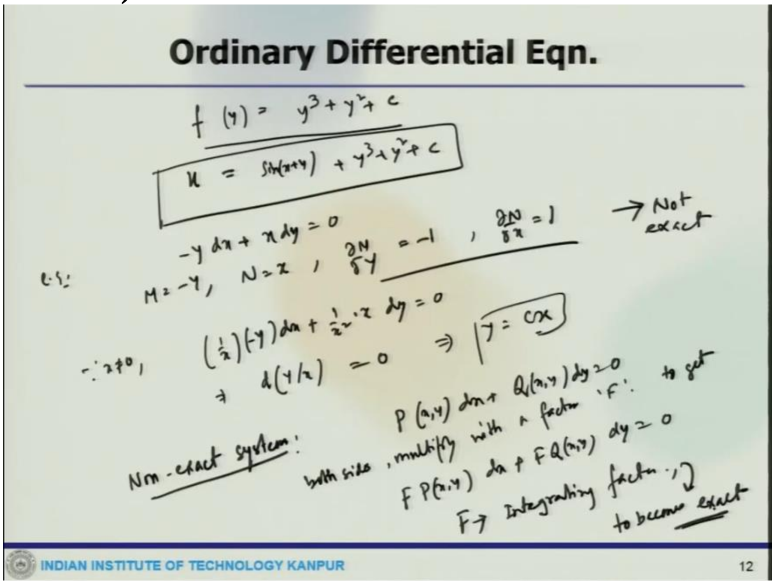

So, what we end up getting is that

$$f(y) = y^3 + y^2 + c$$

So, this is what we get sorry $f(y)$ after getting integration. So, my u would be like if you see that this is what is u

$$u = \sin(x + y) + y^3 + y^2 + c$$

so, this is the solution of that situation. Now we can see like another example of where let us say you have

$$-ydx + xdy = 0$$

So, here $M = -y, N = x, \frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$.

So, this is a situation this is not exact so that system which is not exact cannot be converted to the exact form but what we can do we can multiply the system let us say by since x not equals to 0.

$$\left(\frac{1}{x}\right)(-y)dx + \frac{1}{x^2} \cdot xdy = 0$$

so, what we get

$$d\left(\frac{y}{x}\right) = 0$$

So, here we get

$$y = cx$$

that is the solution but however for a generic non exact system. So, if you have non exact system then how do we deal with it.

So, this is an example where the system is not exact. So, if you have a non exact system and the system is in form of like

$$P(x,y)dx + Q(x,y) = 0$$

So, if the system which is non exact then what we need to do in the both side multiply with a factor ‘F’ to get

$$FP(x,y)dx + FQ(x,y)dy = 0$$

So, this F what is multiplied is called the integrating factor and the system then which makes the system to become exact. So, once it becomes exact then the solution is quite straightforward.

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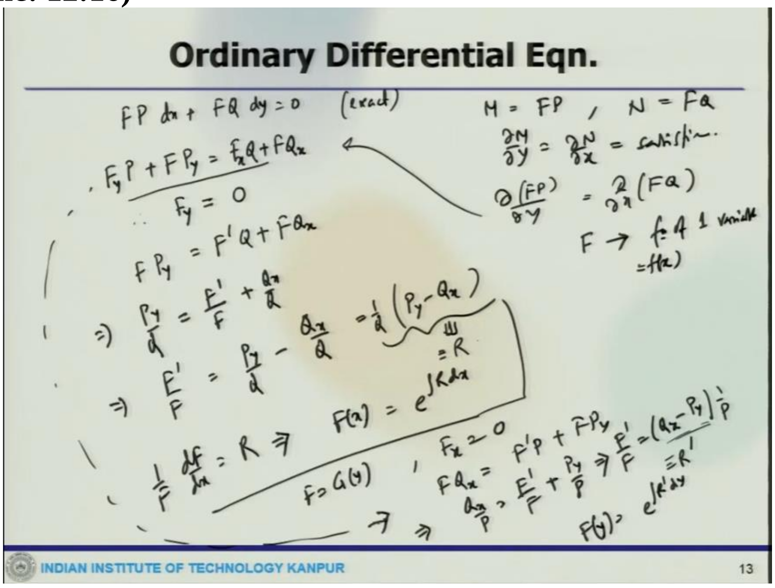

Now when you do that so what do we get

$$FPdx + FQdy = 0$$

So, when this is this becomes exact now in that case the M becomes FP. N becomes FQ and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

is also satisfied. So, now using so this is going to be

$$\frac{\partial (FP)}{\partial y} = \frac{\partial (FQ)}{\partial x}$$

So, we need to find out the function $f(x)$ now so this function would be for f is a function of one variable it is not for like multiple variables.

So, from this condition what we can write

$$F_yP + FP_y = F_xQ + FQ_x$$

so, what do we have that $F_y = 0$ because the function is for the f is a function of one variable and essentially it is in function of $f(x)$. So, what it reduces it to

$$FP_y = F'Q + FQ_x$$

So, what we get?

$$\frac{P_y}{Q} = \frac{F'}{F} + \frac{Q_x}{Q}$$

which is

$$\frac{F'}{F} = \frac{P_y}{Q} - \frac{Q_x}{Q} = \frac{1}{Q}(P_y - Q_x)$$

So, this is whole term let us say equals to R then we can say

$$\frac{1}{F}\frac{dF}{dx} = R$$

So, which get us

$$F(x) = e^{\int Rdx}$$

and where R is $\frac{1}{Q}(P_y - Q_x)$. Now this is when we say now F is a function of x. Now let us say if F is function of y then what will happen? Then $F(x) = 0$ so, this particular expression which come down to

$$FQ_x = F'P + FP_y$$

So, what do we get?

$$\frac{Q_x}{P} = \frac{F'}{F} + \frac{P_y}{P}$$

So, which is

$$\frac{F'}{F} = (Q_x - P_y)\frac{1}{P} = R'$$

So, what do you get?

$$F(y) = e^{\int R'dy}$$

and R prime is given like in this function.

So, this integrating factor could be one variable but it could be either x or y depending on that the system will become or take the form of an exact system and then we can find out that integrating factor.

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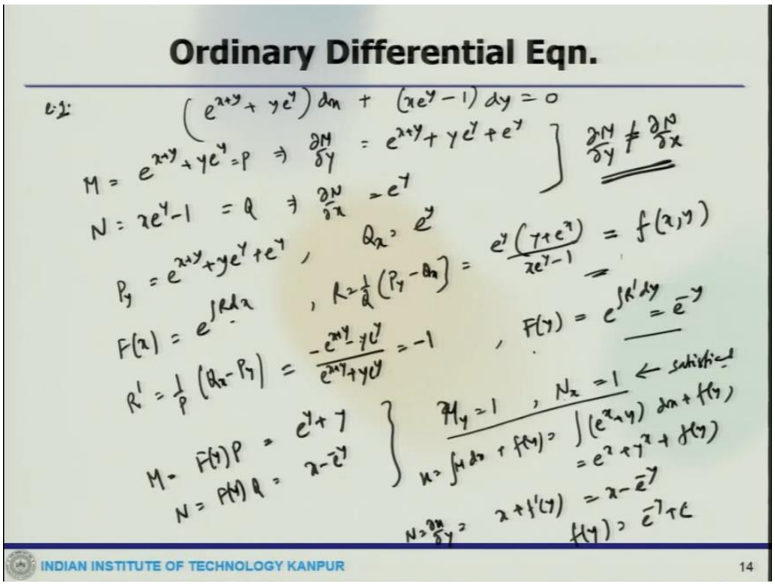

So, let us look at an example like

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$$

here

$$M = e^{x+y} + ye^y = P$$

which gives us

$$\frac{\partial M}{\partial y} = e^{x+y} + ye^y + e^y$$

$$N = \ xe^y - 1 = Q$$

So,

$$\frac{\partial N}{\partial x} = e^y$$

So, from this we can immediately say the form is not in the exact condition so this is not satisfied.

So, once that is not done then we can try to find out the integrating factor. Now what is

$$P_y = e^{x+y} + ye^y + e^y$$

whereas

$$Q_x = e^y$$

So, what you can get so $F(x)$ if we try to find out

$$F(x) = e^{\int Rdx}$$

were

$$R = \frac{1}{Q}\left(P_y - Q_x\right) = \frac{e^y(y + e^x)}{xe^y - 1} = f(x, y)$$

So, here one can see R is a function of both x and y. But this is not possible because it has to be variable.

So, we cannot assuming the integrating factor is a function of x. So, we will look at the second option or like the alternative if it is a function of y. So, we will look at that $\left(Q_x - P_y\right)$. So,

$$R' = \frac{1}{P}\left(Q_x - P_y\right) = \frac{-e^{x+y} - ye^{y}}{e^{x+y} + ye^{y}} = -1$$

So that means we can find out

$$F(y) = e^{\int R' dy} = e^{-y}$$

So, this is going to be the integrating factor. Now once we multiplied with the system my

$$M = F(y)P = e^{y} + y$$

And

$$N = F(y)Q = x - e^{-y}$$

from here now if we check that $\frac{\partial M}{\partial y}$ which is 1, $\frac{\partial N}{\partial x}$ which is N_x which is also 1 so they satisfied. So, this is satisfied so that means the system is after multiplying with the integrating factor this has now become the exact system.

So, we can find out the function

$$u = \int M dx + f(y) = \int (e^{x} + y)dx + f(y) = e^{x} + y^{x} + f(y)$$

And

$$N = \frac{\partial u}{\partial y} = x + F'^{(y)} = x - e^{-y}$$

So, here x cancels out and if we do that f y would be

$$F^{(y)} = e^{-y} + c$$

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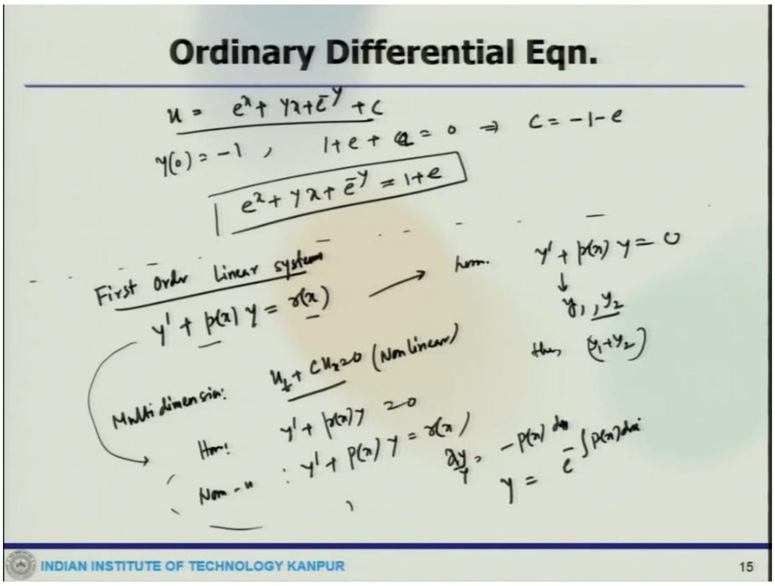

So, my exact solution would be it will be

$$u = e^x + yx + e^{-y} + c$$

so that is the solution. Now if $y(0) = -1$ then one can write. So, this is c,

$$c = -1 - e$$

so, what we get? We get

$$e^x + yx + e^{-y} = 1 + e$$

that is what we get. So, you can see even the system is not exact you can still convert the system to an exact one and that it is quite easy to do that by doing getting some integrating factor.

Now we look at this first order linear system. Now let us say the system looks like

$$y' + p(x)y = r(x)$$

which is also non homogeneous. So, when you have this kind of system then we can write to a homogeneous part can be written at

$$y' + p(x)y = 0$$

So, this is the homogeneous part we will take it out. So, let us say that y_1, y_2 are solutions then $y_1 + y_2$ is also a solution that is a linear combination.

Then $y_1 + y_2$ would be also a solution. Now if you go to multi-dimensional system we are talking about here linear system let us say in multi dimension $u_t + cu_x = 0$ this is an ODE in multi dimension but this is a nonlinear but what we are talking about here is the linear. So, linearity of the first term is the determining factors that whether the system is linear or not. Now coming back to this particular example so, we can get this one into two components one is homogeneous components so this would be the homogeneous part.

So, we will get solution for that and the non-homogeneous part would be $y' + p(x)y = r(x)$. So that homogeneous component the solution of that part is quite straightforward or it can be done easily if you look at from here that one can write

$$\frac{dy}{y} = -p(x)dx$$

So, one get

$$y = e^{-\int p(x)dx}$$

that is the homogeneous part but now when you look at the non homogeneous part.

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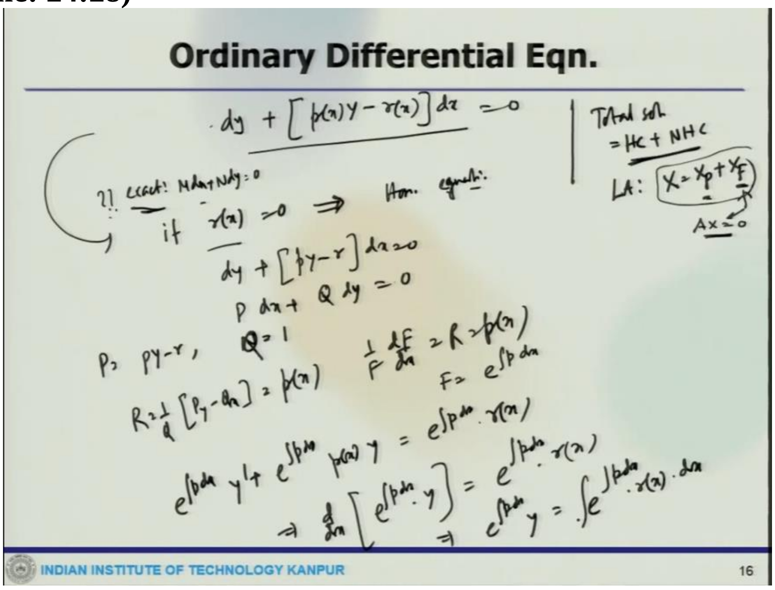

So, non homogeneous component we can write like

$$dy + [p(x)y - r(x)]dx = 0$$

so, we will write in this fashion. So, once we write in this fashion, we can try to see whether the ODE is exact are not. So, this should be in a form of like if it is in exact then it has to be from $Mdx + Ndy = 0$, or rather we can look at show whether exact or not that we can identify if it is exact then the solution is quite straightforward.

If it is not exact then we can use now some integrating factor to bring it to the exact and get the solution. And then finally the total solution would be total solution will have the homogeneous component plus the non homogeneous component. So that will contribute to the total solution. So, this essentially if you look at this and you try to correlate with the concept of the linear system so this linear combination of the homogeneous and non homogeneous is going to be the solution.

So, which is follows the fundamental law of or other fundamental principle of non spill solution and particular solution this is what we have seen in LA linear algebra the total solution is the particular solution and pre solution due to free variable solution due to preferred variable. So, there is also similar kind of situation here homogeneous and non homogeneous. And so, to get the free variable there in linear algebra used to solve Ax = 0.

So that will get us a solution for this free variable and then the other one we will find out from there. Now obviously when you talk about all these what is important here is the domain definition which has to be provided. Now let us see there could be when you talk about this non homogeneous let us say if $r(x) = 0$ that is the simple case if $r(x) = 0$ that means this leads to homogeneous equation.

So that is become a homogeneous equation for the non homogeneous part we will solve this guy

$$dy + [py - r]dx = 0$$

So, which we compare like

$$Pdx + Qdy = 0$$

here $P = py - r$, and $Q = 1$ then we look at

$$R = \frac{1}{Q}\left[P_y - Q_x\right] = p(x)$$

then we can write

$$\frac{1}{F}\frac{dF}{dx} = R = p(x)$$

So,

$$F = e^{\int pdx}$$

Now if we multiply that what do we get

$$e^{\int pdx}y' + e^{\int pdx}p(x)y = e^{\int pdx}r(x)$$

here we can write it is

$$\frac{d}{dx}\left[e^{\int pdx}y\right] = e^{\int pdx}r(x)$$

So that gives us

$$e^{\int pdx}y = e^{\int pdx}r(x)dx$$

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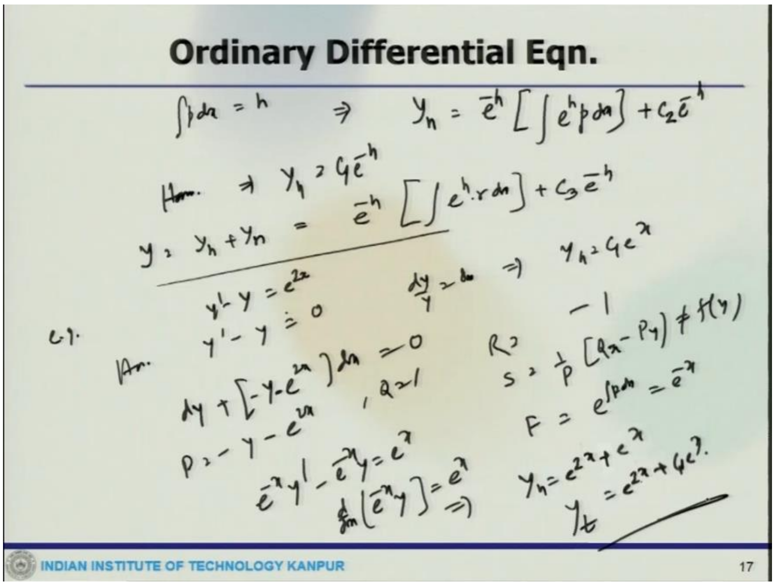

Now let us consider that

$$\int pdx = h$$

then we get the non homogeneous component solution

$$y_h = e^{-h}\left[\int e^h pdx\right] + c_2 e^{-h}$$

and homogeneous solution which is there

$$y_h = c_1 e^{-h}$$

then the total solution would be

$$y = y_h + y_n = e^{-h}\left[\int e^h rdx\right] + c_3 e^{-h}$$

so this is another constant.

So, this is how we can find out both the solution of homogeneous and non homogeneous components we can quickly look at an example like

$$y' - y = e^{2x}$$

So, we get

$$y' - y = 0$$

that is homogenous component. So, from here so

$$\frac{dy}{y} = dx$$

so, we get

$$y_h = c_1 e^{x}$$

and the other part we can write

$$dy + [-y - e^{2x}]dx = 0$$

So,

$$P = -y - e^{2x}$$

and Q is 1.

So, R would be minus 1 so

$$S = \frac{1}{P}\ [Q_x - P_y] \neq f(y)$$

So, integrating factor would be

$$F = e^{\int p dx} = e^{-x}$$

once we get that so this multiplying by this

$$e^{-x}y' - e^{-x}y = e^x$$

So, this will come

$$\frac{d}{dx}[e^{-x}y] = e^x$$

which gives us an solution non homogeneous is so the total solution would be

$$y_h = e^{2x} + e^x$$

$$y_t = e^{2x} + c_1e^x$$

So, you can see that you can have two components and then kind of get the solution for both homogeneous and non homogeneous component. And if it is non homogeneous then split it into two segments and get the solution. So, we will stop the discussion here and continue it from the next session.