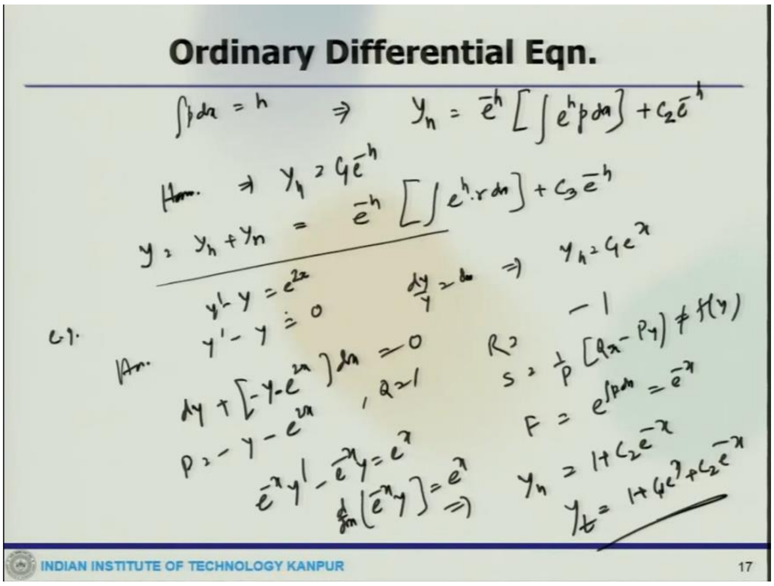
Computational Science in Engineering
Prof. Ashoke De
Department of Aerospace Engineering
Indian Institute of Technology – Kanpur

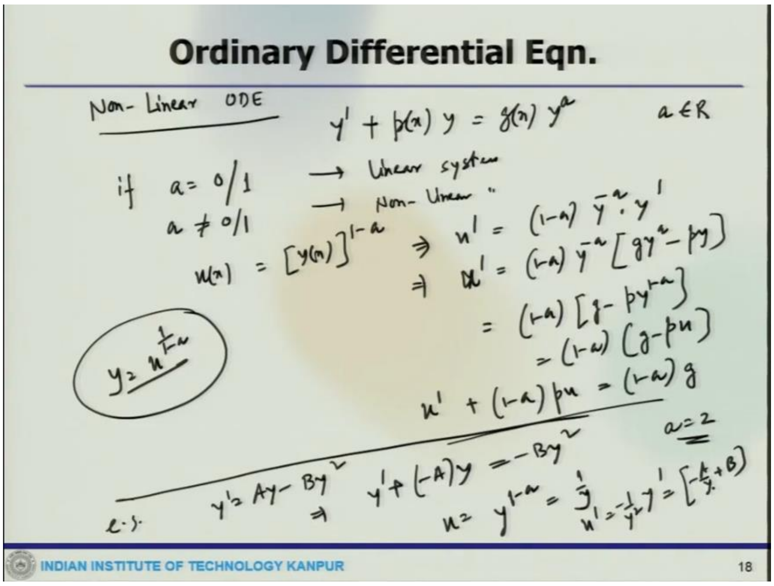
Lecture – 14
Ordinary Differential Equation

So, let us start with now, the other form of the ODEs like nonlinear ODEs. So, before that we have talked about how we can solve the linear system. If it is in homogeneous equation then that is quite straightforward and we can look at whether it is in exact form or not or we can bring down to the exact form. And if not then if it is non homogeneous equation then we basically split it into 2 components homogeneous and non-homogeneous. And then try to find out the solution for both. And then the linear combination of the homogeneous and non-homogeneous is going to give the total solution.

(Refer Slide Time: 00:56)



(Refer Slide Time: 01:02)



So, this is what we looked at the example of such a system. Now we are going to talk about nonlinear system. So, nonlinear ODE which is like, you have a system which is in the form like

$$y' + p(x)y = g(x)y^a$$

where a belongs to R. Now if $a = 0$ or 1 then the system becomes linear then it goes to the linear system. So, to have a nonlinear system this has to be 0 or 1. So that goes to nonlinear system. Now let us say you have a function

$$u(x) = [y(x)]^{1-a}$$

That what we can write

$$u' = (1-a)\, y^{-a}y'$$

So, what we get

$$u' = (1-a)y^{-a}\,[gy^a - py]$$

So, this is what we can write

$$u' = (1-a)\,[g - py^{1-a}]$$

$$u' = (1-a)\,[g - pu]$$

So, we get

$$u' + (1-a)pu = (1-a)g$$

So that is what we get. Now you satisfy the linear ODE condition. So, solving the system we can get

$$y = u^{\frac{1}{1-a}}$$

So that means it has to be some sort of a kind of an variable which we substitute back and then finally bring down the system where we can get this solution done.

Now here also we will look at and system where this is

$$y' = Ay - By^2$$

So, we have

$$y' + (-A)y = -By^2$$

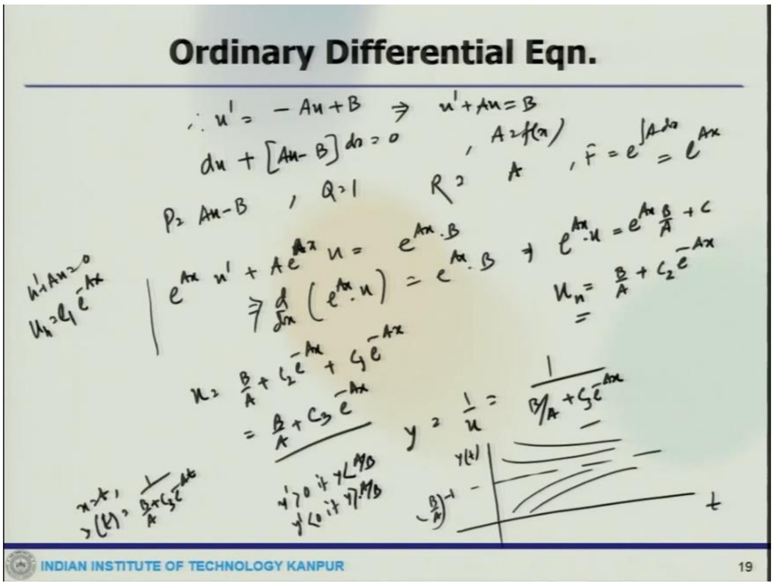
So, $a = 2$ which is definitely not 0 or 1 that means it is nonlinear. So, we can do

$$u = y^{1-a} = \frac{1}{y}$$

So,

$$u' = -\frac{1}{y^2}y' = \left[-\frac{A}{y} + B\right]$$

(Refer Slide Time: 04:15)



So,

$$u' = [-Au + B]$$

So, which in turn on can write this is $Au + B$. So, we get

$$du + [Au - B]dx = 0$$

given that A is function of x which is given, so $P = Au - B$, Q is 1. So, we can have R which is R going to be A. And then we can get the integrating factor like

$$F = e^{\int Adx} = e^{Ax}$$

So, if we multiply that e^{Ax}

$$e^{Ax}u' + Ae^{Ax}u = e^{Ax}B$$

We will have

$$\frac{d}{dx}(e^{Ax}u) = e^{Ax}B$$

So, what that gives us back

$$e^{Ax}u = e^{Ax}\frac{B}{A} + c$$

Another; this is the non-homogeneous part and homogeneous part one can write. So, this would be

$$u_h = \frac{B}{A} + c_2e^{-Ax}$$

and this is the non-homogeneous solution. So, the total solution would be

$$u = \frac{B}{A} + c_2e^{-Ax} + c_1e^{-Ax}$$

So, which one can club that and write some sort of

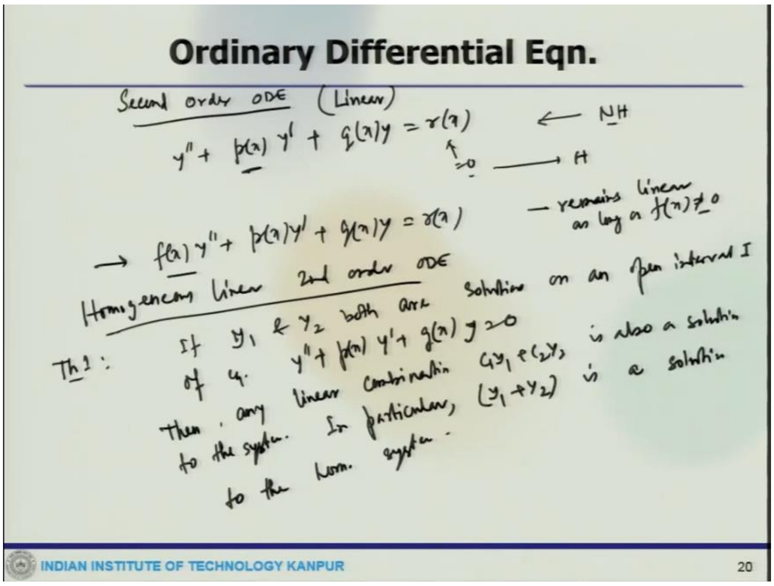
$$u = \frac{B}{A} + c_3e^{-Ax}$$

and

$$y = \frac{1}{u} = \frac{1}{\frac{B}{A} + c_3e^{-Ax}}$$

So, now you can put like x = t and $y(t) = \frac{1}{\frac{B}{A}+c_3e^{-At}}$ then you can plot the system like for t. So, these are the curve which is going to be looking like that. So, here $y' > 0$ if $y < \frac{A}{B}$, $y' < 0$ if $y > \frac{A}{B}$. So, you get this kind of curve in the solution. Now this is where it is a first order nonlinear ODE.

(Refer Slide Time: 08:23)



Now we can look at the second order ODE which is for linear. So, this will give us system like the

$$y'' + p(x)y' + q(x)y = r(x)$$

So, this is we straight way to take an example which is non homogeneous. And if $r(x)$ going to be zero this will be to homogeneous system. So, as long as this let us say if we multiply to the function

$$f(x)y'' + p(x)y' + q(x)y = r(x)$$

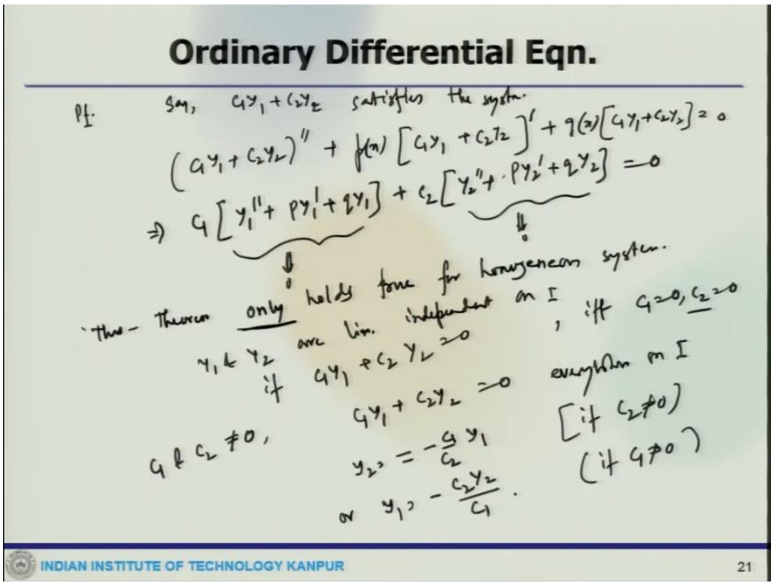
So, this kind of system also called or rather remains linear as long as $f(x)$ not equals to 0.

Now when you go the homogeneous system or non-homogeneous system, in this system this p and q are called the coefficients. So, let us start with the first the homogeneous linear second order ODE. So, we will take some, so here we start with some start of in theoretical understanding like let us say one theorem. Where if y_1 and y_2 both are solutions on an open interval I of equation

$$y'' + p(x)y' + q(x)y = 0$$

Then any linear combination $c_1y_1 + c_2y_2$ is also a solution to the system. In particular $y_1 + y_2$ is a solution to the homogeneous system since it is a second order system it is has to have two solutions. So, this show ever the two options solutions have to be linearly independent. Now we say that if there are $y_1 \, and \, y_2$ are two solutions then the linear combination of that is going to be also solution we can prove that.

(Refer Slide Time: 12:08)



How we will look at that let us say $c_1y_1 + c_2y_2$ satisfy or this combination, satisfy the system then what will have

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) = 0$$

So, what we get here

$$c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] = 0$$

Now this equality holds since y_1 and y_2 are individually solution to the system.

So, this will also contribute to either they will also satisfy the system. So, this theorem only holds true for homogeneous system. Now these two functions y_1, y_2 define on an open interval I call the linearly independent. So, y_1 and y_2 are linearly independent on I. If $c_1y_1 + c_2y_2 = 0$ everywhere on I. If and only if $c_1 = 0, c_2 = 0$. Now assuming both c_1 and c_2 are not 0 then what we will have c_2y_2 is 0 everywhere on I. So, we can have

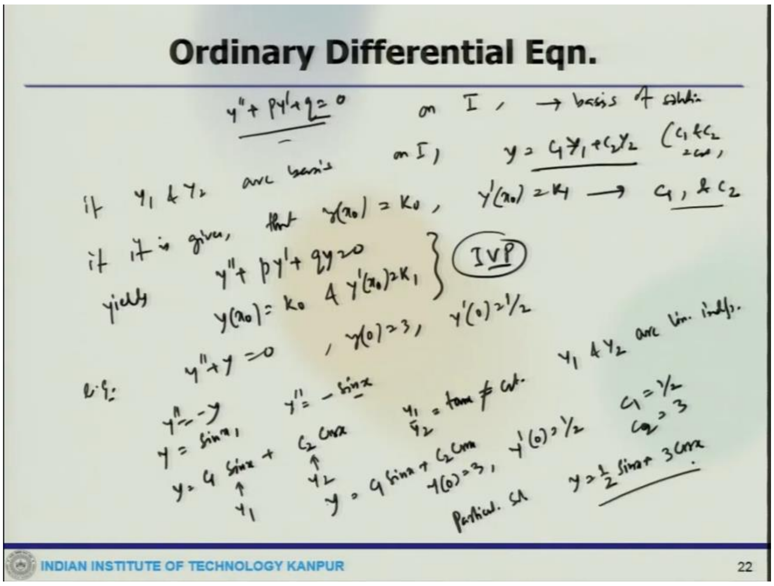
$$y_2 = -\frac{c_1}{c_2}y_1$$

if c_2 not equals to 0 or

$$y_1 = -\frac{c_2}{c_1}y_2$$

if c_1 not equals to 0, so either of that case. If y_1, y_2 are not linearly independent then they will be proportional to each other.

(Refer Slide Time: 15:52)



So, we can define that a pair of solutions of

$$y'' + py' + qy = 0$$

which is linearly independent on an open interval I is called a basis of solution of ODE on I. So, you see where we are coming from, if they are not linearly independent then they are going to be proportional to each other. So that means they are linearly independent and we define that a pair of solutions of this particular system which are linearly independent on an open interval I they are called the basis of solution of that ODE and I.

Now if y_1 and y_2 are the basis solution bases of the solution of ODE on I

$$y = c_1y_1 + c_2y_2$$

is called the general solution of the ODE on I. And where c_1 and c_2 are constant when c_1 and c_2 have absolute value, it gives the particular solution. So, like you can say if it is given that

$$y(x_0) = k_0$$

and

$$y'(x_0) = k_1$$

it will actually give us value for c_1 and c_2. So, which finally yields that

$$y'' + py' + qy = 0$$

And

$$y(x_0) = k_0$$

and

$$y'(x_0) = k_1$$

So, these are called initial value problem. So, first of all either there could be generic solution, so that means if y_1 and y_2 are the solution of this kind of system which are the basis solution of that ODE then this is called this combination of $c_1y_1 + c_2y_2$ would be the generic solution. Or for a particular value we can find out the particular value of c_1, c_2 that is called the particular solution.

And the system becomes an initial value problem. We can just explore that we just in simple example

$$y'' + y = 0$$

were given $y(0) = 3$ and $y'(0) = \frac{1}{2}$. Here

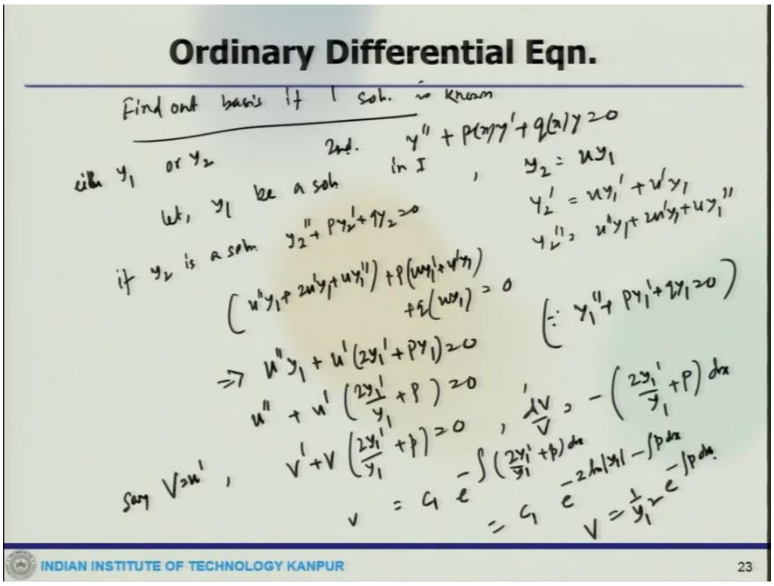
$$y'' = -y$$

So, y = sin x, so

$$y'' = -\sin x$$

which is true. So, the solution will be $c_1 \sin x + c_2 \cos x$, so this is y_1 this is y_2 and $\frac{y_1}{y_2} = \tan x$ which is not constant. So, what we can say y_1 and y_2 are linearly independent. So, the general solution would be $c_1 \sin x + c_2 \cos x$. Now when we use this value of $y(0) = 3$ and $y'(0) = \frac{1}{2}$, we get $c_1 = \frac{1}{2}$ and $c_2 = 3$. So, the particular solution would be $y = \frac{1}{2}\sin x + 3\cos x$. So, this is the particular.

(Refer Slide Time: 20:26)



Now second point is that find out the basis, if one solution is known. So, that means if we know either of y_1 or y_2 then we can reduce the second order system to your first order system. Like the second order system is given

$$y'' + p(x)y' + q(x)y = 0$$

So, let y_1 be a solution in I then we can then y_2 would be uy_1. So, here

$$y_2' = uy_1' + u'y_1$$

$$y_2'' = u''y_1 + 2u'y_1 + uy_1''$$

So, if y_2 is a solution, now if y_2 is a solution then

$$y_2'' + py_2' + qy_2 = 0$$

So, if you put it back

$$(u''y_1 + 2u'y_1 + uy_1'') + p(uy_1' + u'y_1) + q(uy_1) = 0$$

So, which gives us back that

$$u''y_1 + u'(2y_1' + py_1) = 0$$

Here y_1 is already a solution. So, this is going to be 0 since y_1 is solution. So, this means

$$u'' + u'\left(2\,\frac{y_1'}{y_1} + p\right) = 0$$

Let us say $V = u'$ then we can write

$$V' + V\left(2\,\frac{y_1'}{y_1} + p\right) = 0$$

So, the

$$\frac{dV}{V} = -\left(2\,\frac{y_1'}{y_1} + p\right)dx$$

So,

$$V = c_1 e^{-\int\left(2\frac{y_1'}{y_1}+p\right)dx}$$

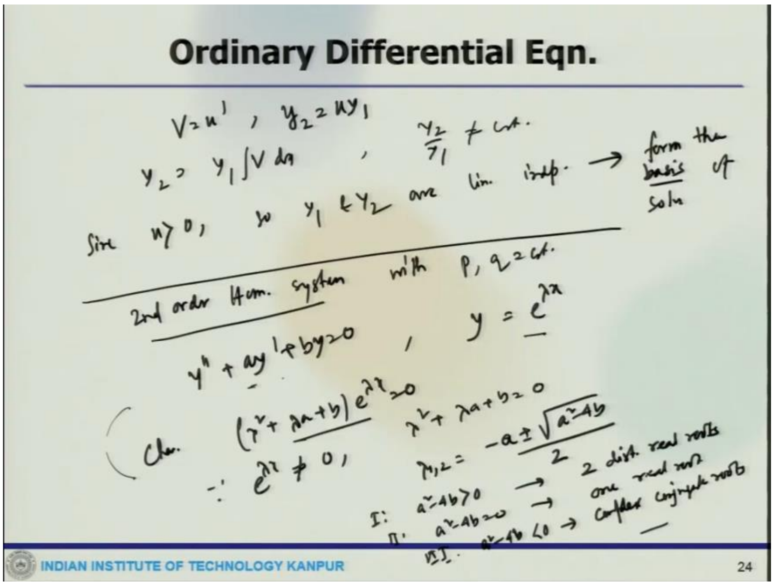
So, we can write that like this can be written

$$V = c_1 e^{-2\ln|y_1| - \int p dx}$$

So, which we can write

$$V = \frac{1}{y^2} e^{-\int p dx}$$

(Refer Slide Time: 24:17)



Now we have

$$V = u'$$

and

$$y_2 = u y_1$$

So, we get

$$y_2 = y_1 \int V dx$$

So, $\frac{y_2}{y_1}$ is not constant, since u greater than 0. So, y_1 and y_2 are linearly independent and that form the basis of solution. Now homogeneous, so here we can see that if we know one solution, we can find out the second solution. Now there would be this is where we have not talked about p or q which I mean the coefficient p and q.

Now this could be possible that this second order homogeneous system with p, q constant. Then what we can say

$$y'' + ay' + by = 0$$

Now since it is homogeneous then ab is constant. So, the solution would be in the form of

$$y = e^{\lambda x}$$

And then we can get an characteristics equation like

$$(\lambda^2 + \lambda a + b)e^{\lambda x} = 0$$

Now since $e^{\lambda x}$ not equals to 0, this is going to be the $\lambda^2 + \lambda a + b$. So, we can find out λ_1 and λ_2 would be

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Now λ is decided on discriminant part of $a^2 - 4b$. So, what we can have that we can have a situation where $a^2 - 4b$ greater than 0 then we will get two distinct real roots, case second where this could be 0 then we get one real root or third which is possible that you can have $a^2 - 4b$ which is less than 0 then we will get complex conjugate roots.

And now this is what you can see. So, first I mean when p and q are constant then this solution would be in this form and you get these characteristics equation on λ. And this depends on this $a^2 - 4b$ discriminant and whether it is a positive 0 or negative, whether we will end up getting real root or one real root or complex conjugate root. So that would be determined. Now we can look at the kind of detail of these different case studies in the next session.