Computational Science in Engineering
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Lecture – 15
Ordinary Differential Equation.

So, let us continue the discussion on ODE. Now we have started discussing the second order ODEs. And here we have first looked at how they look like and what to do with the situation and the thing which we started is the second order homogeneous system where p and q could be the constant

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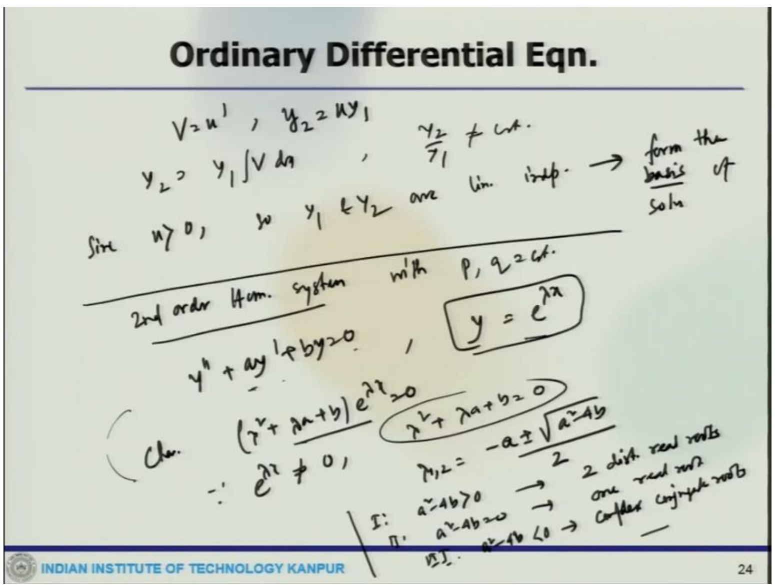

So, there could be which are constant. So, this is second order homogeneous system these are constant coefficients. And when that happens it gives us some characteristics system because the solution would be in this particular form. And there are different situations which may arise either one may have since it is second order obviously there has to have two solutions. So, one could have two distinct real roots or other case you may have one single root or it could be complex conjugate roots. So, these are situations.

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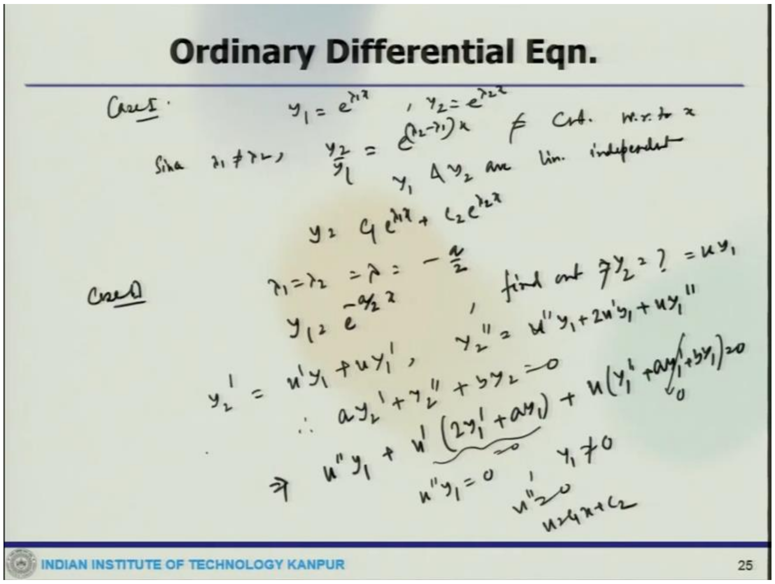

Now let us see what happened in case-I? So that means you have to look, so it will be two solutions

$$y_1 = e^{\lambda_1 x}$$

and

$$y_2 = e^{\lambda_2 x}$$

Since λ_1 is not equal to λ_2,

$$\frac{y_2}{y_1} = e^{(\lambda_2 - \lambda_1)x}$$

which is not constant with respect to x, so which means y_1 and y_2 are linearly independent. So, the general solution would be

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Now situation where you can have case II, one single root then you can find out that $\lambda_1 = \lambda_2 = \lambda = -\frac{a}{2}$. So, we only know that

$$y_1 = e^{-\frac{1}{2}x}$$

So, we need to find out y_2. So, we can use reduction of order like

$$y_2' = u'y_1 + uy_1{}'$$

And

$$y_2'' = u''y_1 + 2u'^{y_1} + uy_1{}''$$

So, it falls down to the one we put it back we get

$$ay_2' + y_2'' + by_2 = 0$$

So,

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0$$

So, this is the system, so where you can basically this is putting back in the y_2 into the system, so you get these guys 0. So, what do we get? Now this term is also 0. So, what do we get is that

$$u''y_1 = 0$$

since y_1 is not equal to 0, you get $u'' = 0$. So, $u = c_1 x + c_2$.

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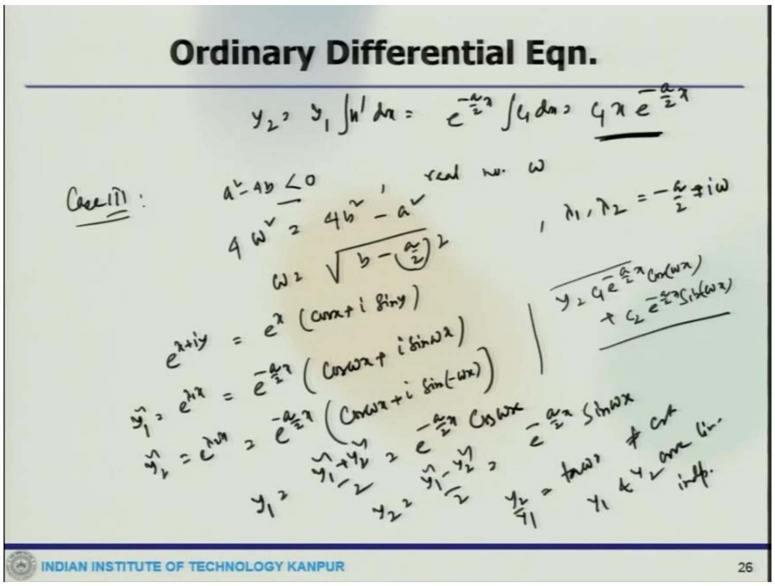

So,

$$y_2 = y_1 \int u' dx = \ e^{-\frac{a}{2}x} \int c_1 dx = c_1 x e^{-\frac{a}{2}x}$$

So, that would be y_2. That means if one solution is known for common real root.

then case III where these are imaginary roots and $a^2 - 4b$ less than 0. So, let us assume a real number omega then

$$4\omega^2 = 4b^2 - a^2$$

So, omega would be

$$\omega = \sqrt{b - \left(\frac{a}{2}\right)^2}$$

So,

$$\lambda_1, \lambda_2 = -\frac{a}{2} \pm i\ \omega$$

So,

$$e^{x+iy} = e^{x}(\cos x + i \sin y)$$

now our solution

$$\widetilde{y_1} = e^{\lambda_1 x} = \ e^{-\frac{a}{2}x}(\cos \omega x + i \sin \omega x)$$

then we have

$$\widetilde{y_2} = e^{\lambda_2 x} = e^{-\frac{a}{2}x}(\cos \omega x + i \sin(-\omega x))$$

Now for the real solution we have

$$y_1 = \frac{\widetilde{y_1} + \widetilde{y_2}}{2} = e^{-\frac{a}{2}x} \cos \omega x$$

And

$$y_2 = \frac{\widetilde{y_1} - \widetilde{y_2}}{2} = e^{-\frac{a}{2}x} \sin \omega x$$

So,

$$\frac{y_2}{y_1} = \tan \omega x$$

so that means y_1 and y_2 are linearly independent. So, our solution would be in the form like

$$y = c_1 e^{-\frac{a}{2}x} \cos \omega x + c_2 e^{-\frac{a}{2}x} \sin \omega x$$

this would be the form of the I mean the complete solution. Now this is what can happen when you have different roots.

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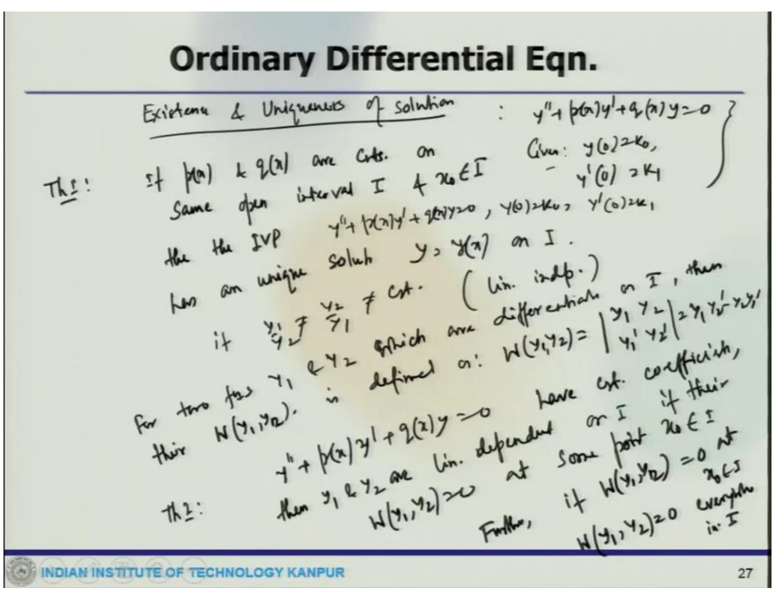

But one important thing one has to check whether the existence or the uniqueness of the solution. So that means existence and uniqueness of solution, so that one has to kind of check. So, to do that we write down the second order system $q(x)y = 0$. So, let us again part into the homogeneous ODE and given conditions are like $y(0) = K_0$, $y'(0) = K_1$. So, like to discuss if solution existence if they are unique or not.

So, this leads to the sort of when any particular solution for the given condition. So, now here we have some theorem to talk about let us say if $p(x)$ and $q(x)$ are constants on same open interval I and x_0 belongs to I then the initial value problem which is some given as

$$y'' + p(x)y' + q(x)y = 0$$

for like $y(0) = K_0$, $y'(0) = K_1$ has a unique solution $y = y(x)$ on I. Now linearly independent solutions y_1 and y_2 an open interval I now if $\frac{y_1}{y_2} \neq \frac{y_2}{y_1}$ not equal to constant.

So, they are linearly independent that time linearly independent, now for two functions y_1 and y_2 which are differentiable on I then they are Wronskian y_1, y_2 is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix} = y_1 y_2' - y_2 {y_1}'$$

So, we can see how this IVP your initial value problem has an unique solution. Now subsequent to that the theorem two who says

$$y'' + p(x)y' + q(x)y = 0$$

have constant coefficients then y_1 and y_2 are linearly dependent on I if they are Wronskian is 0 at some point x_0 belongs to I.

Further to that if Wronskian of y_1 and y_2 is 0 at some point x_0 which belongs to I, then $W(y_1, y_2) = 0$ everywhere on I. So, if there is an x_1 belongs to I at which are on Wronskian is not equals to 0.

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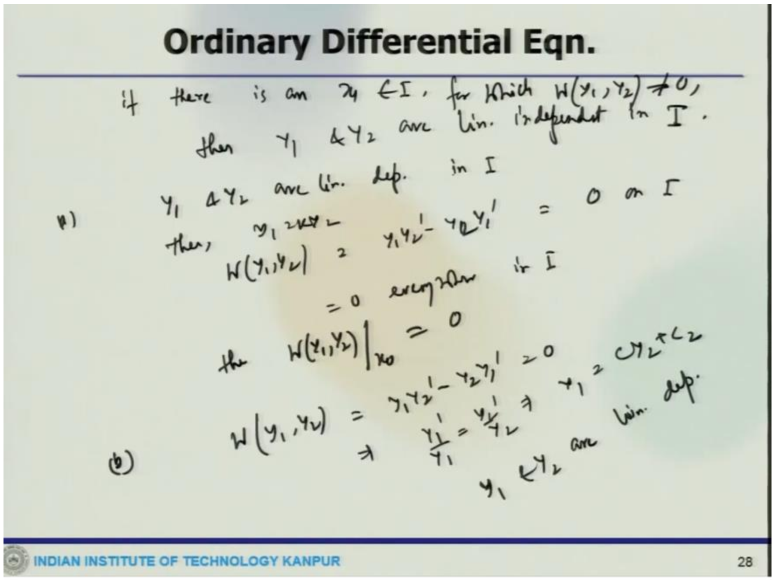

So that means if there is a point x_1 which belongs to I for which Wronskian is not equals to 0, then y_1 and y_2 are linearly independent in I. So, these are basically talking about two conditions whether linear dependency or independency. So, let us say y_1 and y_2 are linearly dependent in I then what happened $y_1 = ky_2$, Wronskian would be

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$$

Hence this guy would be zero everywhere in I.

So, if we let x_0 be any point in I then $W(y_1, y_2)$ and x_0 will be also 0. Now the other part b

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$$

then

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$$

which is

$$y_1 = cy_2 + c_2$$

hence y_1 and y_2 are linearly dependent.

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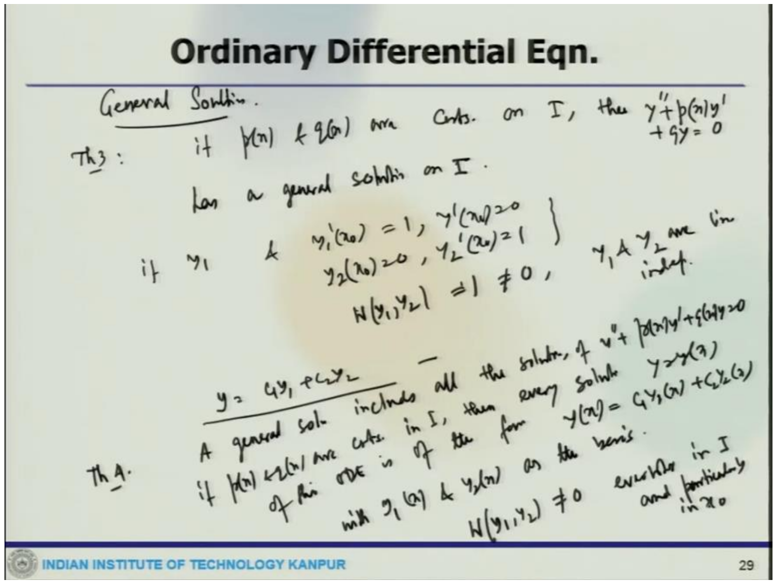

Now we can see how we can find out the general solution. So, this must also include all the solution. So, again some theorem which is in existence of a general solution, so if $p(x)$ and $q(x)$ are constants on I then

$$y'' + p(x)y' + q(x)y = 0$$

has a general solution on I. Now using theorem one if y_1 is a solution and

$$y_1'(x_0) = 1$$

And

$$y'(x_0) = 0$$

then x_0 is an arbitrary point.

So, which is

$$y_2(x_0) = 0$$

$$y_2{}'(x_0) = 1$$

Where, x_0 is an arbitrary point. So, what will happen the Wronskian of this would be $W(y_1, y_2) = 1$ which is not 0. So that means y_1 and y_2 are linearly independent. Hence the linear combination which includes all solutions which could be written as $c_1y_1 + c_2y_2$. So, this includes all solution, now theorem four which says that a general solution includes all the solutions of equation

$$y'' + p(x)y' + q(x)y = 0$$

If $p(x)$ and $q(x)$ are constants in I, then every solution $y = y(x)$ of this ODE is of the form at

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where with $y_1(x)$ and $y_2(x)$ as the basis. So, what happened the Wronskian would not be 0 everywhere in I and particularly in x_0.

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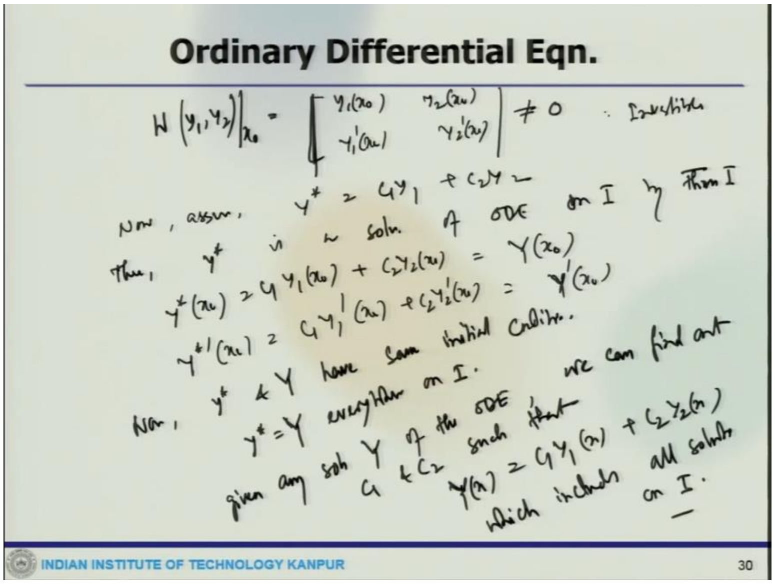

So, what will have

$$W(y_1, y_2)|_{x_0} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{vmatrix} \neq 0$$

So, it is invertible, so if we now assume that

$$y^* = c_1 y_1 + c_2 y_2$$

then y^* is a solution of the ODE on I by theorem one which is a superposition of solution. So that means

$$y^*(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = Y(x_0)$$

and

$$y^{*\prime}(x_0) = c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) = Y'(x_0)$$

Now y^* *and* Y have same initial condition, so that is by virtue of uniqueness property of theorem 1 we must have $y^* = Y$ everywhere on I, so given any solution y of the ODE we can find out the constant we can find out c_1 and c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

which includes all solutions on I. So that means we can have all the solutions which also includes the system.

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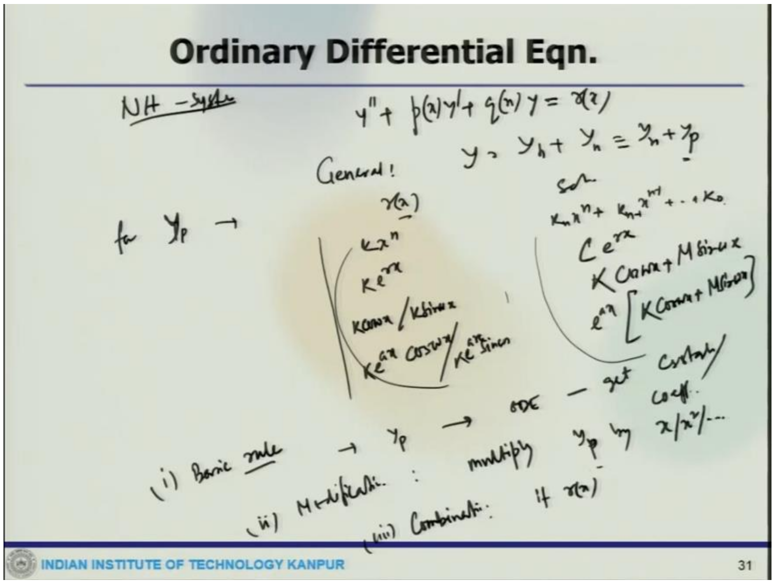

Now we can move to on non-homogeneous system. So, in the non-homogeneous system we can have

$$y'' + p(x)y' + q(x)y = r(x)$$

So, $r(x) = 0$ means it will go into the homogeneous. So, the general solution would be y would be y homogeneous plus y non homogeneous or one can say y homogeneous plus y particular solution whatever one can prefer. For the non-homogeneous solution where we can find out this particular solution.

So, $r(x)$ for y_p the $r(x)$ has certain form like let us see if $r(x)$ would be the then the solution form like that if it is Kx^n then the solution would be $K_n x^n + K_{n-1}x^{n-1} + \cdots + K_0$ such a polynomial equation. If it is Ke^{rx}. So, this solution would be in pattern Ce^{rx} if it is $K \cos \omega x$ or $K \sin \omega x$ then here it would be $K \cos \omega x + M \sin \omega x$ if this is $Ke^{ax} \cos \omega x$ or $Ke^{ax} \sin \omega x$, then it would be $e^{ax}[K \cos \omega x + M \sin \omega x]$.

So, these are the possible ways that I whatever form you have in non-homogeneous part. So, there are some basic rules. So, let us say $r(x)$ has some standard function and accordingly y_p can also be chosen and determine coefficients can be determined by substituting this y_p in the ODE. So, the first basic rule is that so if $r(x)$ has some standard pattern like this then we can have a solution of this pattern and once we have y_p we can put it back in ODE to get constants or the coefficients.

Second is the rule is the modification where the choice is y_p happens to be the solution of the homogeneous form, then we need to multiply y_p by some x or x square or depending on the polynomial. So, if y_p let us say the homogeneous solution is for order form of some same form whether polynomial or these then we have to multiply like that third which is combination what is that if $r(x)$ is same combination function then y_p has to be also a combination of that kind. So, let us say if it is in sin x + cos x then it has to be also in that form.

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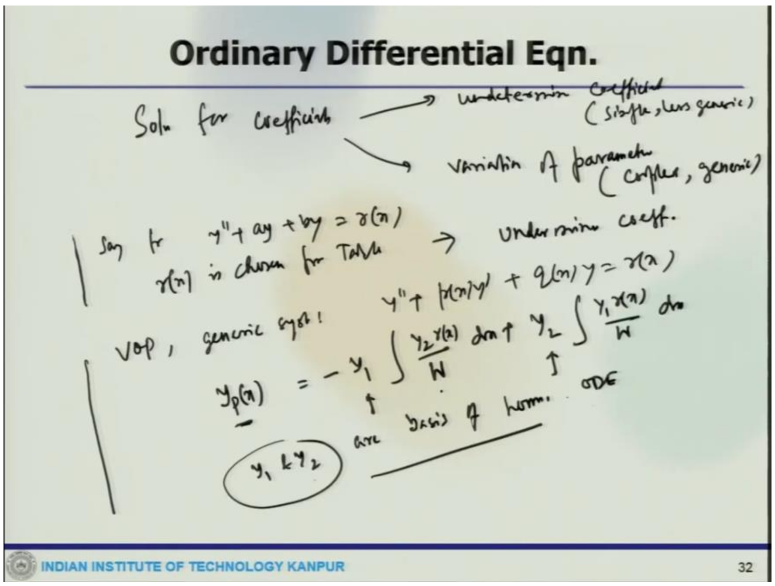

So, what do you have solution for coefficients either you can have undetermined coefficients, this process is simple but less generic or we can have variation of parameters which is complex process but generic? So, these are the way one can do that so undetermined coefficient the process of undetermined coefficient means we have a pattern of $r(x)$ and using the pattern of $r(x)$ we can assume some sort of solution and then we can find out the coefficients or in the variation of parameter what we can do?

Let us say for this

$$y'' + ay + by = r(x)$$

and $r(x)$ is provided some of this table. So, for $r(x)$ is chosen from table then this system is called undetermined coefficient. So, this is the way or other way for variation of parameter which is a generic system

$$y'' + p(x)y' + q(x)y = r(x)$$

which is used then the particular solution would be in this particular form where we can say

$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

So, this is how the part, now what is y_1 and y_2 here, here y_1 and y_2 are basis of homogeneous ODE. So, which means once you have the basis of the homogeneous ODE then this is more like a generic approach this is not very generic. This is but there are obviously advantage and disadvantage to that. One case is quite I would say simple. The other case is more like a generic but little bit complex and involved. So, you get this y_1, y_2 which are the basis of the homogeneous ODE and using that we can find out the particular solution and using the Wronskian. So, we will continue this discussion in the next session and stop it here.