Computational Science in Engineering
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Lecture - 16

Okay. So, we have looked at this non homogeneous solution, how we can find out. I mean either there could be simple way one can find out like using the undetermined coefficients or variation of parameter which is more generic.

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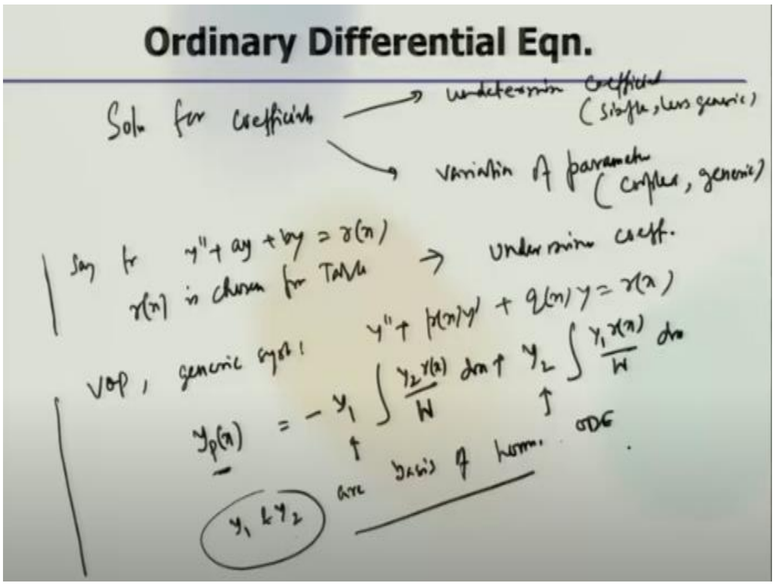

Now with this where we can now move to the higher order linear system.

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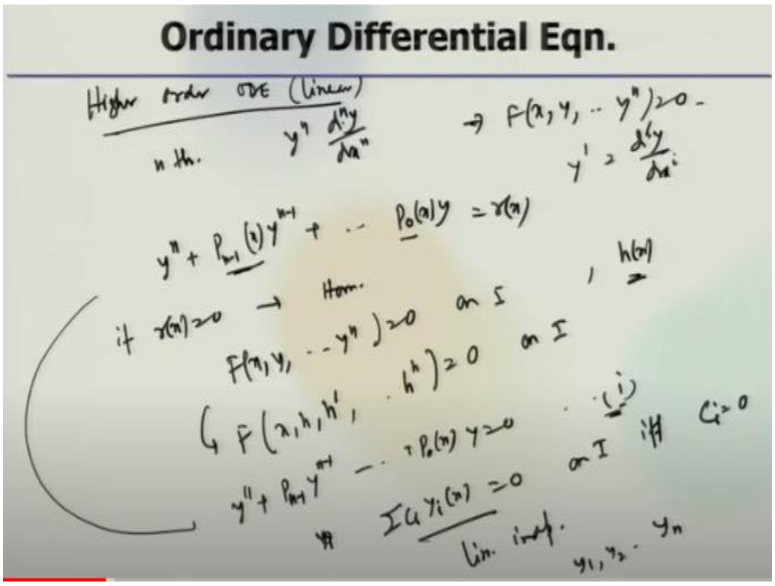

So now we will look at higher order ODE which are linear okay, again. So, there would be n-th derivative which is existing. So,

$$y^n = \frac{d^n y}{dx^n}$$

So, like this would be n-th order ODE. So, the system can be written as that

$$F(x, y, \dots y^n) = 0$$

Where, $y^i = \frac{d^i y}{dx^i}$, okay. So, this is let us say $\frac{d^n y}{dx^n}$. That is the way it should be.

So now on expanding the system one can write

$$y^n + P_{n-1}(x)y^{n-1} + \cdots + P_0(x)y = r(x)$$

So, you can expand this n-th order system here. $P_0(x)$ to $P_{n-1}(x)$. These terms are called the coefficients. And in this form ODE is linear function of $F(x, y, \dots y^n) = 0$, can take individually. So, if $r(x)$ here is 0, which will lead to the homogeneous part, which will be n-th order homogeneous system.

Now how do you find out the solution? So, the solution of this n-th order ODE, which is of this particular form, the functional form on I is a function $h(x)$ which is defined in time, so the function $h(x)$ is defined which is in time differentiable on I such that we replace y by $h(x)$ and the function can be written that

$$F(x, h, h', \dots h^n) = 0$$

on everywhere in I.

Now what is superposition principle? So, we have this particular system, homogeneous system

$$y'' + P_{n-1}(x)y^{n-1} + \cdots + P_0(x)y = 0$$

So, any linear combination of solution of I is again a solution on I. So, this indicates that all the solution of this particular equation, homogeneous equation will form a vector space and this would form the basis and the vector space.

So, say $y_1, y_2, \dots y_n$ are the solution then for y_1, y_2 at the solution then this would be

$$\sum c_i y_i(x) = 0$$

on I if and only if c_i are 0. So that means they are linearly independent. So that is what they will form that vector space. Now the basis of solution of this ODE, of this homogeneous ODE is a set of n solution starting of $y_1, y_2, \ldots y_n$ they are linearly independent.

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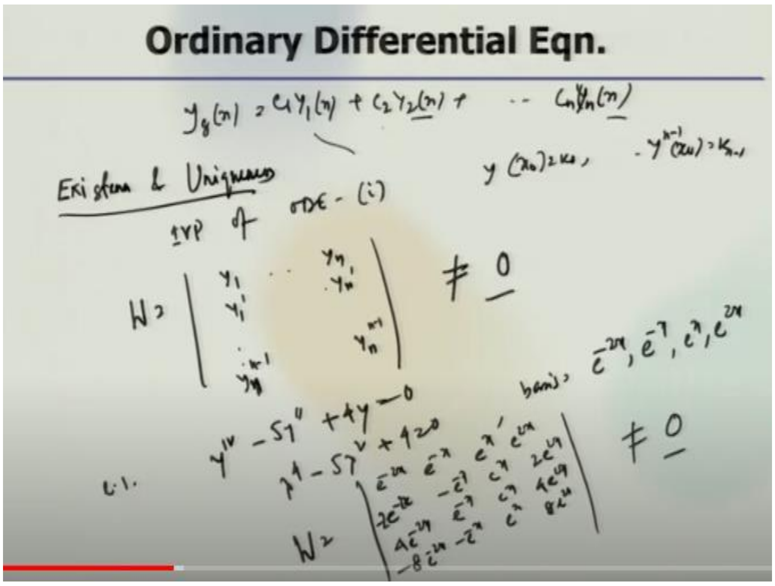

Now the general solution of this homogeneous part would be

$y_g(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ and this y_n, y_1, y_2 these are the basis function or solution on I. And c_1, c_2, c_n are the constants. Now we can see the existence and uniqueness. So, an IVP or initial value problem for let us say IVP of ODE obtained the homogeneous n-th order system and there would be n initial conditions like

$$y(x_0) = K_0$$

Like

$$y^{n-1}(x_0) = K_{n-1}$$

then IVP has a unique solution. So, the Wronskian would be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \vdots & \vdots & & \vdots \\ {y_n}^{n-1} & {y_n}^{n-1} & \cdots & {y_n}^{n-1} \end{vmatrix}$$

So, the solution of one with constant coefficients are linearly independent if this Wronskian is not equals to 0.

That means, this particular system, this homogeneous system it will have n solutions and they are linearly independent then the Wronskian would be not 0. And for this initial value problem, they are also having initial conditions and has unique solution. Now we can see some quick situation like let us say, if you have an

$$y'''' - 5y'' + 4y = 0$$

So, the characteristics would be

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

So, the basis are e^{-2x}, e^{x}, e^{2x}. So, when you look at the Wronskian this would be

$$W = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix} \neq 0$$

So, this if you look at that this would be not zero. so, if you calculate that. So, the solutions are all linearly independent.

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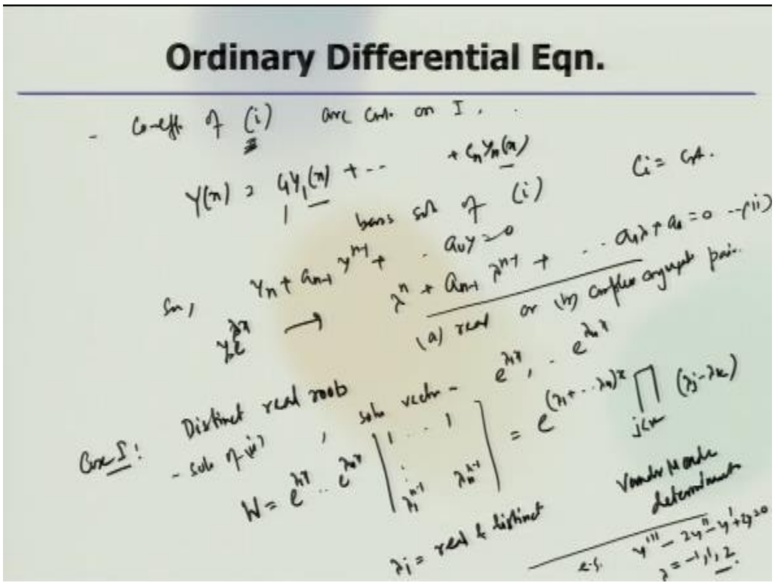

Now another situation where the coefficients of (i) are constants on I then this (i) has general solution on I. Similarly, a general solution all solutions includes all solutions. So, if this ODE has constant coefficients on I, then every solution of i is of the form. So, we can have

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where this y_1, y_2 and y_n are the basis solution. They are basis solution of equation (i).

And the c_1 to c_n are constant. So, c i's are constants. Let us say

$$y^n + a_{n-1}y^{n-1} + \cdots + a_0y = 0$$

then $y = e^{\lambda x}$ it will give us an n-th order characteristics equation which will be

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

So, this is the characteristics polynomial which you will get if we put back $y = e^{\lambda x}$.

So, this is an n-th order polynomial with roots. Either roots could be real or complex conjugate pair. Now when you have this n-th order polynomial and if you have different roots which could be either real or distinct. So, you can look at similarly like what we have seen in the second order. There could be distinct real roots.

Now this solution of this ii, solution of ii will be distinct and real then the solution vector should be, solution vectors would be $e^{\lambda_1 x}$ to so on $e^{\lambda_n x}$. And Wronskian would be

$$W = e^{\lambda_1 x} \dots e^{\lambda_n x} \begin{vmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ {\lambda_1}^{n-1} & \dots & {\lambda_n}^{n-1} \end{vmatrix}$$

So,

$$W = e^{(\lambda_1 + \cdots + \lambda_n)x} \prod_{j<k} (\lambda_j - \lambda_k)$$

So, this is called Vandermonde determinant. The solutions up y_1, y_2 and y_n from the basis solution obviously, and all roots this λ_i's are real and distinct, okay. So, we can see quickly a system where we can have this like

$$y''' - 2y'' - y' + 2y = 0$$

So, what you see is that λ would be -1, 1, 2. So you can see these are all real and distinct.

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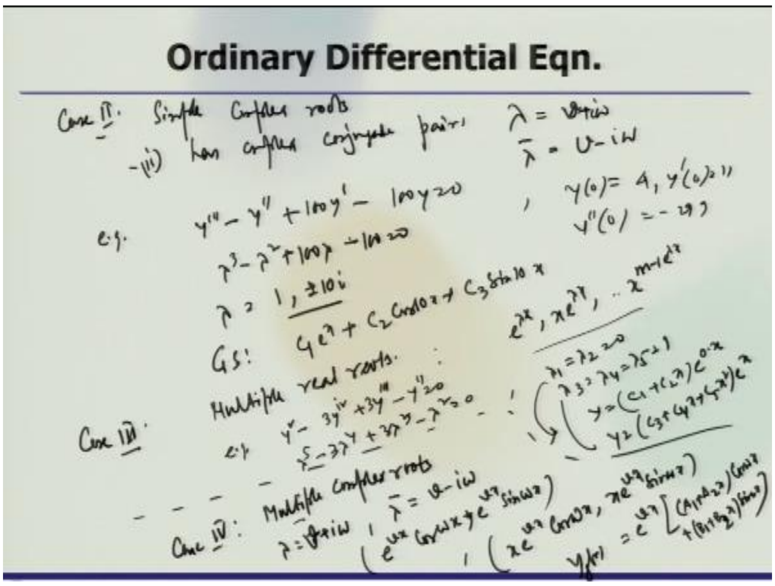

Now case II where you can have simple complex roots. So now if (ii) has complex conjugate pairs then $\lambda = v + i\omega$ is a root and $\bar{\lambda} = v - i\omega$ is also a root which indicates they are linearly independent real value solution. So, you can again see an example like

$$y''' - y'' + 100y' - 100y = 0$$

where $y(0) = 4,\ y'^{(0)} = 1, and\ y''^{(0)} = -299.$

So, you get here you get

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

$$\lambda = 1, \pm 10i$$

So, these are the three roots. So, the general solution would be in form of

$$c_1 e^x + c_2 \cos 10x + c_3 \sin 10x$$

Now case (iii) you can have multiple real roots.

Generally, if λ is a real root of order m then the you will have so m corresponding linear independent solution like $e^{\lambda x}, xe^{\lambda x}, \dots x^{m-1}e^{\lambda x}$. So, these are the solution. For quick example, you can see

$$y''''' - 3y'''' + 3y''' - y'' = 0$$

So, this is

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

So, you get $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = \lambda_5 = 1$. So, corresponding to this you can have solution which is

$$y = (c_1 + c_2 x)e^{0x}$$

or corresponding to this, this one will have

$$y = (c_3 + c_4 x + c_5 x^2)e^x$$

This is how you can. Or you can have case iv where you have multiple complex roots.

So, if $\lambda = v + i\omega$ and $\bar{\lambda} = v - i\omega$ corresponding linear. So let us say $v - i\omega$ corresponding linear independent solutions are would be $(e^{vx}\cos\omega x, e^{vx}\sin\omega x)$. And $(xe^{vx}\cos\omega x, xe^{vx}\sin\omega x)$.

So, the general solution would be in terms of like

$$y_g(x) = e^{vx}[(A_1 + A_2 x)\cos\omega x + (B_1 + B_2 x)\sin\omega x]$$

So, if there are multiple roots, then this is how the solution is going to look like.

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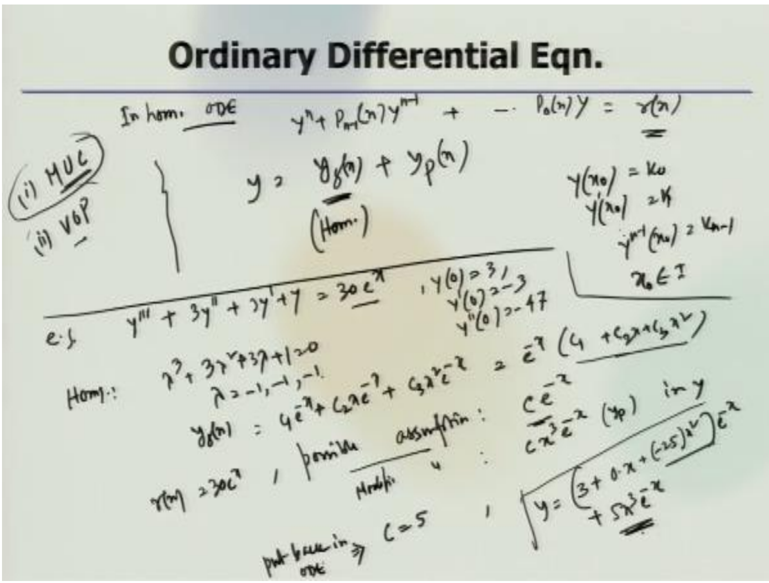

Now similarly, for this n dimensional problem we can have in homogeneous ODE, in homogeneous ODE. So, the equation system would be pretty much similar. You will have n-th order derivative and instead of now you have $r(x)$. So, if $r(x)$ is 0, it goes to homogeneous. It is non zero then the solution would be in two forms. One would be homogeneous plus particular. It is general or this is homogeneous part.

Now while converting to that initial value problem, where you will have $y(x_0) = k_0$, $y'(x_0) = k_1$. Similarly, $y^{n-1}(x_0) = k_{n-1}$. All x_0 belongs to I. One initial condition is known, we obtain the, for c i constants for the general solution. Now for this kind of higher order system the obviously what we have seen in the second order system there are two ways.

One is the method of undetermined coefficient, method of undetermined coefficient or variation of parameters. So, both can be used. But obviously, as we have earlier discussed even in the second order system, the method of undetermined coefficients is simple, but it cannot be too generic. Variation of parameter is generic but bit complex. So, when you have higher dimensional problem, probably this could be a better choice.

But one is free to choose whatever he or she wants to do that. We can take an example, then it would be quite clear how one can handle such systems. So let us take this

$$y''' + 3y'' + 3y' + y = 30e^{x}$$

and we have all initial conditions. These are $y(0) = 3, y'(0) = -3, y''(0) = -47$. So, the obviously, if we take the homogeneous first, homogeneous part first then it would be

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

So, λ is -1, -1, -1. So, our generic solution would be

$$y_g(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} = e^{-x}(c_1 + c_2 x + c_3 x^2)$$

Now $r(x) = 30e^x$. So, the possible $r(x) = 30e^x$. So, the possible assumption could be ce^{-x}. However, e^{-x} now here is in check.

This could be a possible solution. This is how you doing method of undetermined coefficients. Now e^{-x} is kind of a solution, which is part of the homogeneous. So, we need to modify this factor by a factor of x^3. The reason is that already the homogeneous solution has the highest order of x^2. So, the modified assumption is cx^3e^{-x}.

Now if you put back this y_p in y so we get like, so we can put back in the equation here. And then what we will get that constant would be 5. So put back in ODE and we get C 5. So, our complete solution would be like

$$y = (3 + 0 * x + (-25)x^2)e^{-x} + 5x^3 e^{-x}$$

So, this is the, this is homogeneous.

This is non homogeneous. So, using all the initial conditions you can get these values. Now if one wants to use the variation of parameter then he has to like use the particular.

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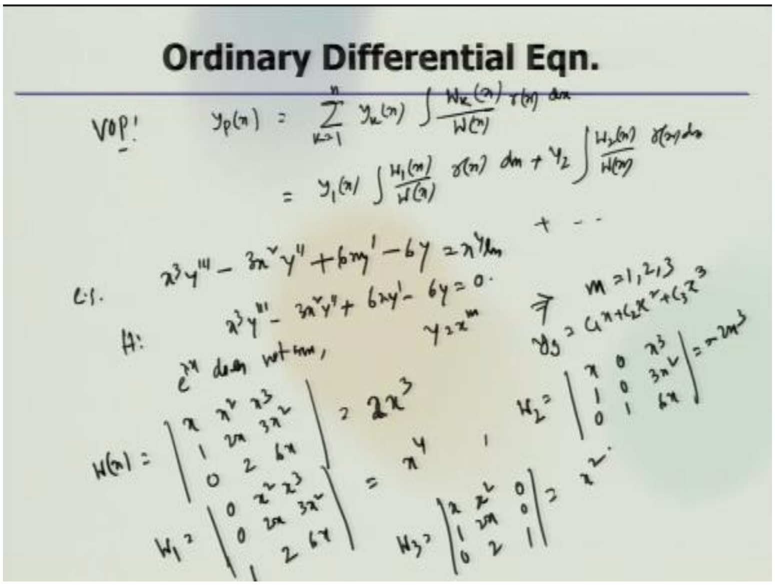

So, variation of parameter the particular solutions would be like

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)dx$$

So, which means it would be

$$y_p(x) = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)dx \ and\ so\ on$$

So, this is where you can find out that thing. And like we can look at another example like

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4dx$$

So homogeneous component you get like

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0$$

So, we put that back. So here e to the power lambda x does not work. So, we have to use

$$y = x^m$$

So, and then putting that we get m equals to 1, 2 and 3. So our homogeneous solution would be

$$y_g = c_1x + c_2x^2 + c_3x^3$$

Now our Wronskian is

$$W(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

Now

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

Similarly,

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = 2x^3$$

W 2 which is x, 1, 0; 0, 0, 1. So you get x cube, 3x square, 6x which is 2 x cube. And

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

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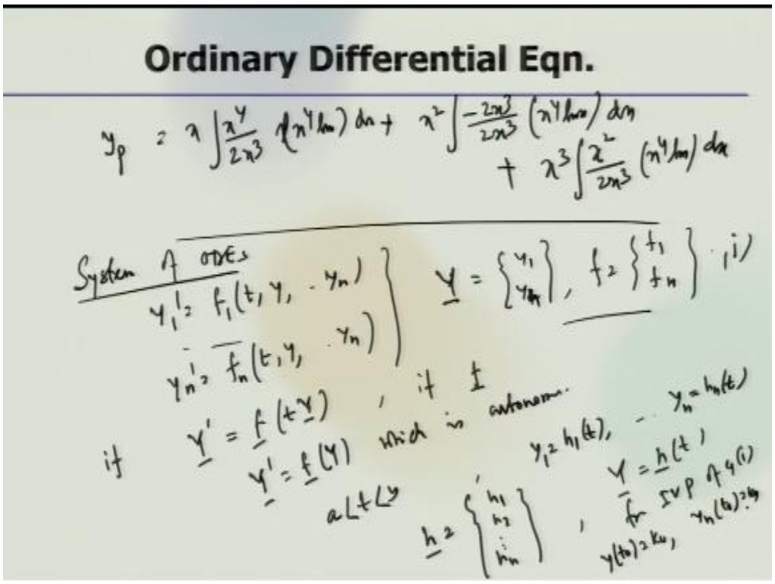

So, our solution of particular solution would be

$$y_p = x\int \frac{x^4}{2x^3}(x^4 \ln x)dx + x^2\int \frac{-2x^3}{2x^3}(x^4 \ln x)dx + x^3\int \frac{x^2}{2x^3}(x^4 \ln x)dx$$

So, this is how you can use the Wronskian to find out that solution, okay. Now what you can do like now we can look at some system of ODEs. So, system of ODEs what you can have, let us say you have

$$y_1' = f_1(t, y, \dots y_n)$$

Similarly, you can have like

$$y_n' = f_n(t, y, \dots y_n)$$

Then we can have a like if you have

$$y' = f(t, y)$$

it is a non-autonomous system if it does not depend explicitly on t and $y' = f(y)$ which is autonomous system. Now solution of this particular system of ODEs on an open interval where t lies between a to b is in set of n differentiable functions which are

$$y_1 = h_1(t), y_n = h_n(t)$$

Then in vector form we can see these are $h_1, h_2 \; to \; h_n$, okay. So now the solution would be

$$Y = h(t)$$

for the initial value problem of equation (i) which will have n initial conditions like k_0 to k_n.

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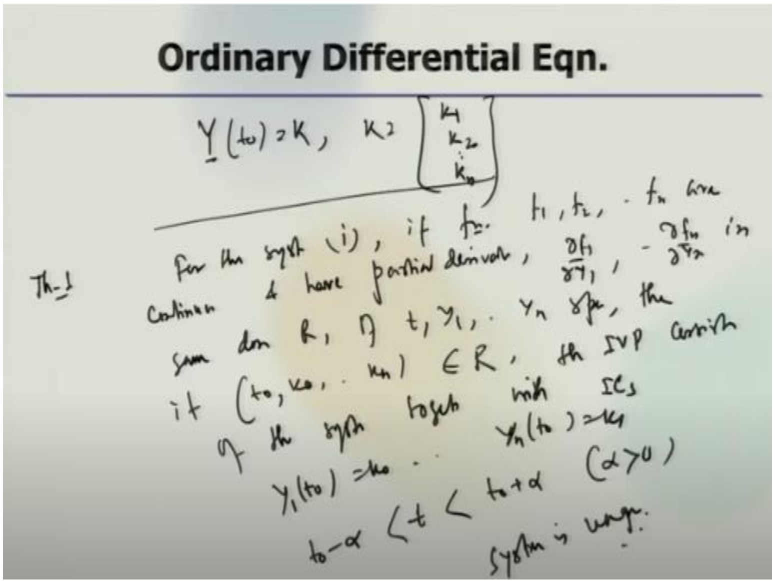

So, what we can say that this guy would be k where

$$K = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix}$$

So, this is where you can find out the solution. Now we have a theorem which says for the system which is defined as (i) if function $f_1, f_2 \; to \; f_n$ are continuous and have partial derivatives like

$$\frac{\partial f_1}{\partial y_1}, \dots \frac{\partial f_n}{\partial y_n}$$

in same domain R of t, $y_1 \; to \; y_n$ space.

Then if $(t_0, K_0, .. K_n)$ belongs to R the IVP consisting of the systems together with the initial conditions $y_1(t_0) = K_0$ to $y_n(t_0) = K_n$ will have a solution on same open interval where t_0 lies for t_{0i}, t lies between $t_0 - \alpha$ to $t_0 + \alpha$ where α greater than 0 containing t_0 further the system is also, system is also unique.

So, we will stop here and continue this system of ODEs in the next session.