Computational Science in Engineering
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Lecture - 17

Let us continue the discussion on the ODE and we are pretty much on the towards the last part of the discussion where we are looking at the system of ODEs.

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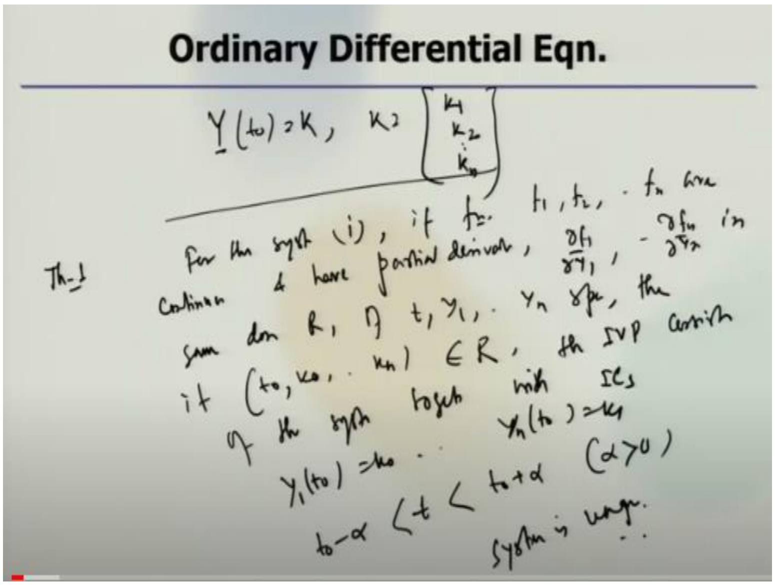

And where we have stopped, we stopped with this theorem 1. So now we will look at the system of ODEs for sort of a linear system.

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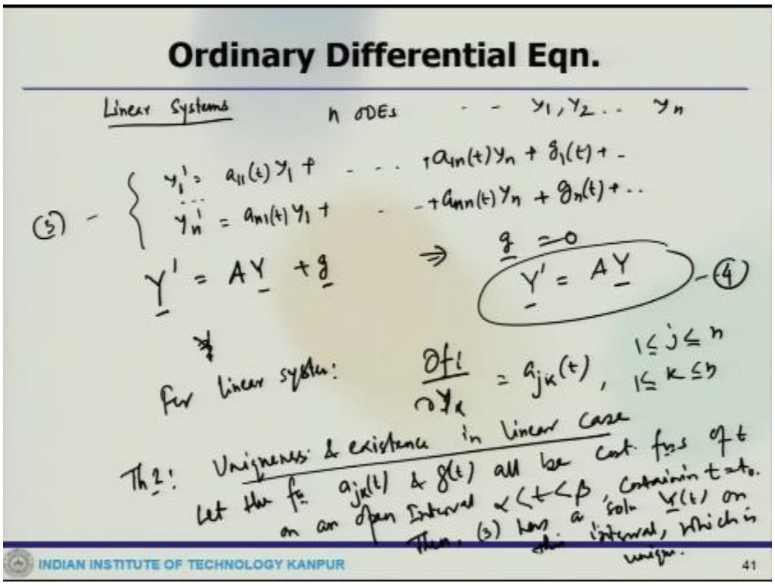

And what we are going to look at, let us say we have system of, so we will look at linear systems. Let us say we have n ODEs form a linear system. If it is linear then we have like this,

$$y_1' = a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t)$$
$$y_n' = a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t)$$

So, we can write

$$y' = Ay + g$$

So then now if you see this guy here if $g = 0$ then this will lead to, so this will lead to $y' = Ay$ which is a homogeneous system. So let us say this is equation 4. This is equation 3. So otherwise, it remains non homogeneous. So, for linear system we write

$$\frac{\partial f_i}{\partial y_k} = a_{jk}(t)$$

where $1 \leq j \leq n, and\ 1 \leq k \leq n$, okay.

Now the second theorem will talk about uniqueness and existence in linear case. So, theorem 2, which is talks about uniqueness and existence in linear case, okay. So, I would say that let the function $a_{jk}(t)$ and $g(t)$ all be constant function of t on an open interval where t goes between $\alpha\ to\ \beta$ containing t equals to t_0. Then 3 or equation 3 has a solution which is $Y(t)$ on this interval which is unique, okay.

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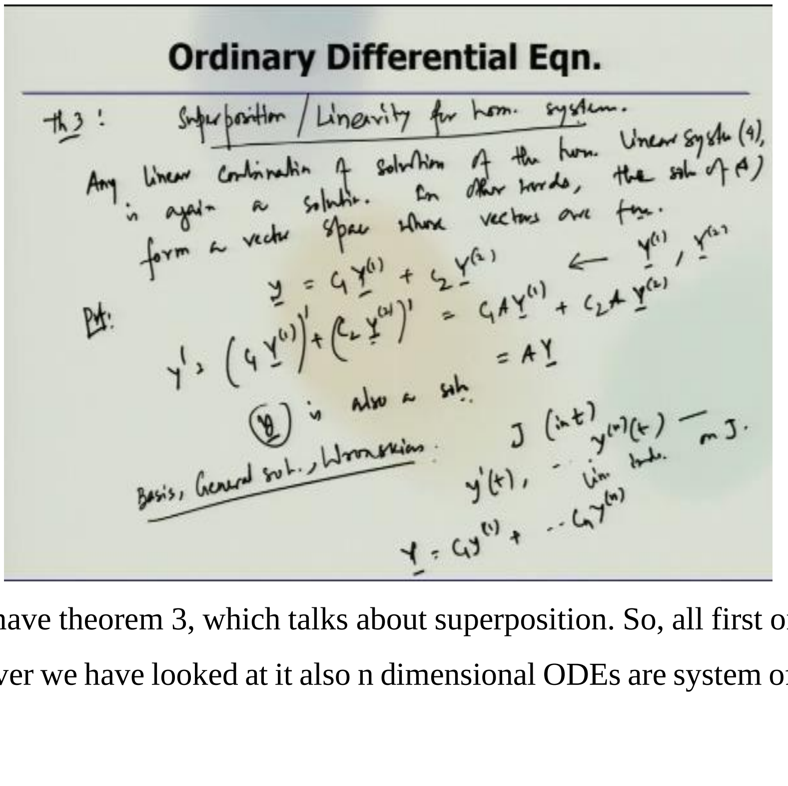

So, then we have theorem 3, which talks about superposition. So, all first order, second order, whatever we have looked at it also n dimensional ODEs are system of ODEs they are valid like the linearity for homogeneous system. It is at any linear combination, any linear combination of solution of the homogeneous linear system, which is 4 is again a solution.

In other words, the solutions of, the solution of 4 form a vector space whose vectors are functions. So, we can look at the proof quickly. Let us say for a linear combination like what we can say

$$y = c_1 y^{(1)} + c_2 y^{(2)}$$

which are the coming from two solutions $y^{(1)}, y^{(2)}$. So, we get

$$y' = \left(c_1 y^{(1)}\right)' + \left(c_2 y^{(2)}\right)'$$

So, which will again lead to

$$y' = c_1 A y^{(1)} + c_2 A y^{(2)}$$

which would be

$$y' = Ay$$

So, y is also a solution.

Now we can have also for this system of ODEs will have basis, general solution, Wronskian, all we have. So, a basis of solution of a homogeneous linear system on an interval let us say J, which is in t is a set of in solutions like $y^{(1)}(t), y^{(2)}(t), \dots y^{(n)}(t)$ which are they are linearly independent on J.

So, a linear combination of all these which is let us say this kind of

$$Y = c_1 y^{(1)} + c_2 y^{(2)} + \cdots + c_n y^{(n)}$$

is called the general solution where this c_i's are arbitrary constant. So, if we fix the values of the constant, we obtain a particular solution.

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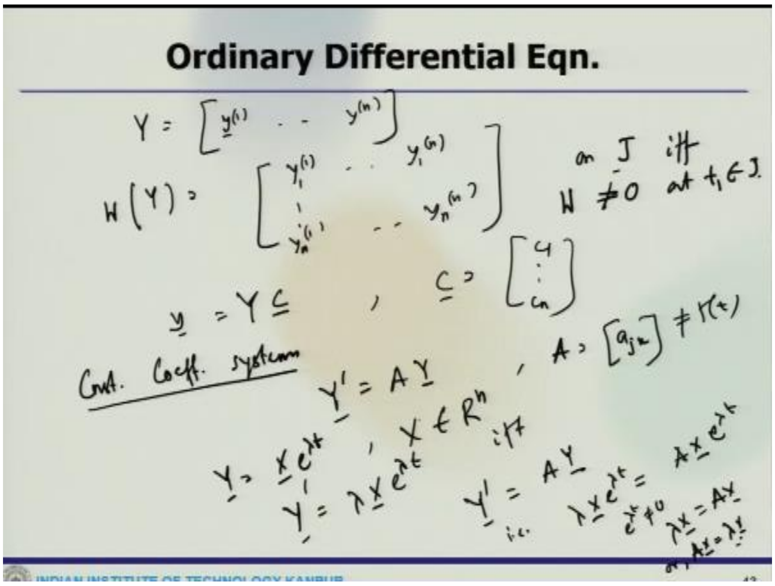

So, if we put n solutions like this together as an column matrix so we have

$$Y = \left[y^{(1)} \dots y^{(n)}\right]$$

Then the determinant of this matrix would be the Wronskian. And so, the Wronskian of Y would be

$$W(Y) = \begin{bmatrix} y_1^{(1)} & \cdots & y_1^{(n)} \\ \vdots & \ddots & \vdots \\ y_n^{(1)} & \cdots & y_n^{(n)} \end{bmatrix}$$

So, the solution forms a basis, so this solution forms a basis on J if and only if W is not equals to 0 at any t_1 which belongs to J. Also, either W equals to 0 and J or W is nonzero everywhere on J, okay.

So finally, the solution one can write this $y = Yc$ where c is kind of

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

which are the constant. Now let us see the constant coefficient system. So, whatever we have looked at for the second order ODEs and all these so they are going to be the similar for. So again, we consider any linear system which is in the form of $Y' = AY$ where the entries of n $\times$ n coefficient matrix are A which is

$$A = \left[a_{jk}\right]$$

okay.

So. They are constant so they do not depend on so this is not a function of t. Now why we bother about with eigenvalues and eigenvector. So, suppose we try a solution for this system like $Y = Xe^{\lambda t}$ and where x belongs to R^n. Then $Y' = \lambda X e^{\lambda t}$. And we have a solution if and only if $Y' = AY = \lambda X e^{\lambda t} = AXe^{\lambda t}$.

So, since $e^{\lambda t}$ not equals to 0 we get $\lambda X = AX$. So, or in the other way around or one can write $AX = \lambda X$. So, λ is the eigenvalue and X is going to be the eigenvector of this system.

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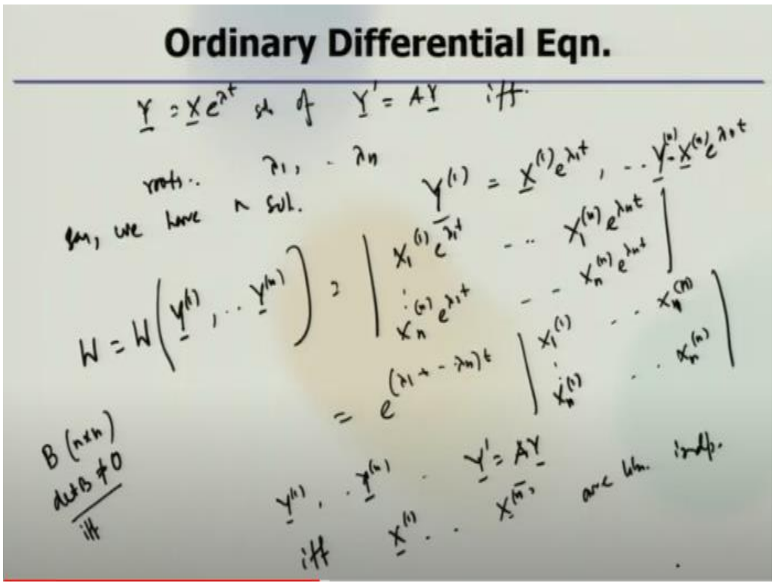

So, one can say $Y = Xe^{\lambda t}$ is a solution of solution of $Y' = AY$ if and only if λ is an eigenvalue of A and X is an eigenvector with eigenvalue λ. So let us say if we recall this n × n matrix the characteristics equation is going to be a polynomial of degree n where the root should be $\lambda_1, \dots \lambda_n$. Let us say sum may be the same if we have multiple roots.

So let us say now say we have n solutions now. Then $Y^{(1)} = X^{(1)} e^{\lambda_1 t}, Y^{(n)} = X^{(n)} e^{\lambda_n t}$. So, we have like that. Now you want to know if we have the basis of solutions. To see this the solutions, if the solutions are linearly independent, we can look at the Wronskian where this would be Wronskian of $Y^{(1)}\ to\ Y^{(n)}$.

So, this we can write

$$W = W\left(Y^{(1)}, \dots, Y^{(n)}\right) = \begin{vmatrix} X_1^{(1)} e^{\lambda_1 t} & \cdots & X_1^{(n)} e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ X_n^{(1)} e^{\lambda_1 t} & \cdots & X_n^{(n)} e^{\lambda_n t} \end{vmatrix}$$

So, if we remember that for n × n the determinant of for n × n matrix, let us say B is an n × n matrix and determinant of B is not equals to 0. And that would be possible if the columns of matrix B are linearly independent.

Then we have the solutions. These are the solutions.

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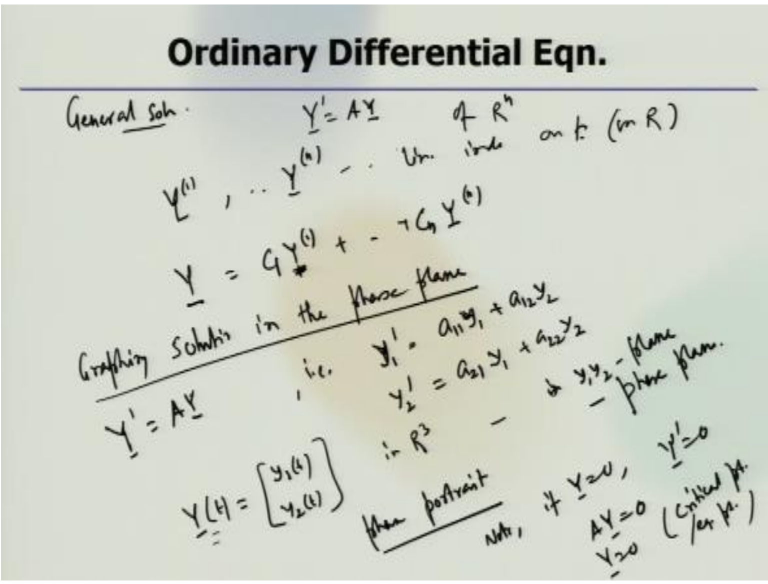

And so now we can look at the general solution. So again, the general solution to find out let us say if the matrix A in the system $Y' = AY$ has a basis of eigenvectors of R^n in linearly independent eigenvectors, then the corresponding solutions are $Y^{(1)}\ to\ Y^{(n)}$. So, they are also linearly independent as function on R and the general solution is given by

$$Y = c_1 Y^{(1)} + \ ... + c_n\, Y^{(n)}$$

So now we do the graphing solution in the phase plane, okay. So, consider a system of two linear ODEs with constant coefficients. So, like $Y' = AY$. So, these are two linear ODEs. That is what we are considering. So that means we will have

$$y_1' = a_{11}y_1 + a_{12}y_2$$

And

$$y_2' = a_{21}y_1 + a_{22}y_2$$

Now we can graph this

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

in the usual way in R^3 or we can graph these as a parametric curve in y and y_2 plane which is so y_1 on y_2 plane. This is called the phase plane. So, the curve this $Y(t)$ in the phase plane is called the trajectory or orbit. Several such curves which indicate the behavior of the system are called the phase portrait. So now note if Y is 0 then Y' is 0. So, AY is also 0.

We have a constant solution. And so, this Y equals to 0 is called the known as critical point or equilibrium point. In a linear system with constant coefficient there can be one critical point. But in general, there could be more than one such point.

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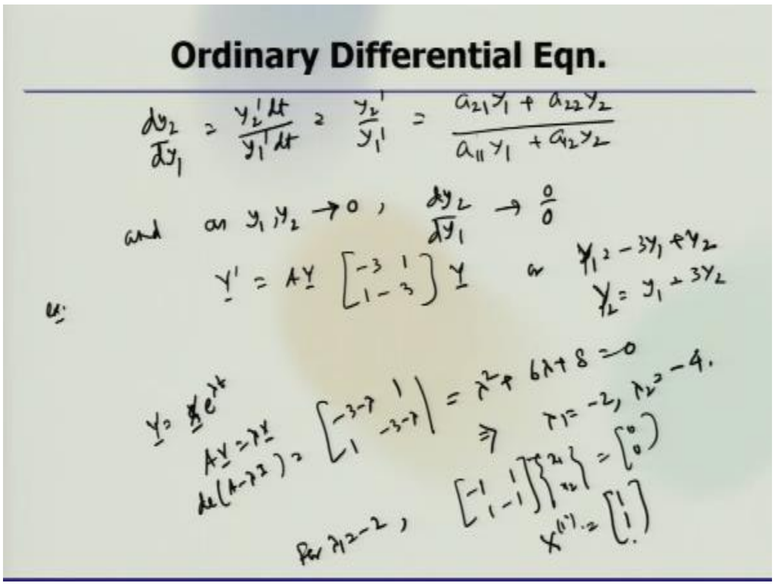

So, from calculus what we know

$$\frac{dy_2}{dy_1} = \frac{y_2' dt}{y_1' dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

And as y_1, y_2 tends to 0, $\frac{dy_2}{dy_1}$ tends to 0 by 0 which is undefined. Now we can take an example and then close the discussion of this particular section.

So let us say we have $Y' = AY = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} Y$ or what we have

$$y_1 = -3y_1 + y_2$$

$$y_2 = y_1 - 3y_2$$

So that can be capital so that we can write in that fashion. So let us say we substitute

$$Y = Xe^{\lambda t}$$

And then we get $AX = \lambda X$. So, once we substitute that this is what we get.

And determinant of $(A-\lambda I)$, which gives us characteristics polynomial which is 0. So doing this

$$\lambda^2+6\lambda+8=0$$

$$\lambda_1=-2, \qquad \lambda_2=-4$$

Now we consider first $\lambda_1=-2$. Then the eigenvector we have the linear system. So, we will get

$$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

These are the component of $X^{(1)}$. So, we get $X^{(1)}=\begin{bmatrix}1\\ 1\end{bmatrix}$ as eigenvector.

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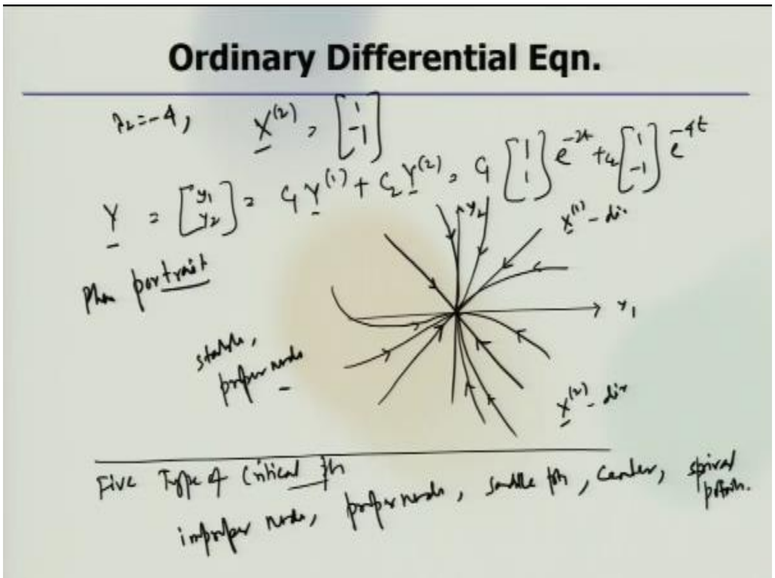

Now similarly for $\lambda_2=-4$ we get $X^{(1)}=\begin{bmatrix}1\\ -1\end{bmatrix}$. So, this gives a general solution Y is

$$Y=\begin{bmatrix}y_1\\ y_2\end{bmatrix}=c_1y_1+c_2y_2=c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-4t}$$

So, if we look at the phase portrait that would looks like so our system goes like this.

This is direction and guys would look like and so this will go through like this. So, like this. So that comes like that. So, the phase portrait would so which is sort of a stable and proper node. So that will look like that. Now here we have five types of critical points. Now like we have improper nodes, proper nodes, saddle points, centers and spiral nodes, sorry points, spiral points.

So, these five types of critical points depending on the geometric shape of the trajectories near them.

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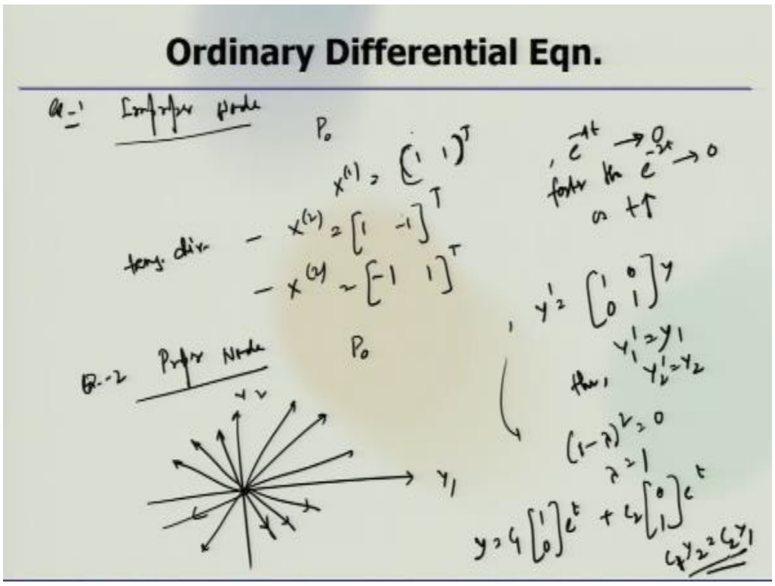

And we can just have a quick look of them. So let us say we take in first one is the improper node. So, an improper node is a critical point P_0 at which all trajectories or all the trajectories except for two of them have the same limiting direction of the tangent. The two exceptional trajectories also have a limiting direction of the tangent at P_0 which however is different.

So just like we can see this particular plot here and we can see this has an improper node at 0 in its phase portrait. The common limiting direction at 0 is that the eigenvector of $X^{(1)} = [1 \quad 1]^T$. Because e^{-4t} goes to 0 faster than e^{-2t} goes to 0 as t increases. The two exceptional limiting tangents directions are, so the tangent directions are $X^{(2)} = [1 \quad -1]^T$ and $-X^{(2)} = [-1 \quad 1]^T$, okay.

Now we take second one is the proper node. So proper node is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction dP_0 there is a trajectory having d as a limiting direction. Like one can see

$$Y' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Y$$

where $y_1' = y_1$, $y_2' = y_2$. So, we can see this case if this is y_1, this is y_2, then these are all going this direction like this.

So, like that, so this is y_2. So indeed, the matrix is in unit matrix. Its characteristic equations would be

$$(1-\lambda)^2 = 0$$

has root λ equals to 1. We can take this

$$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$$

So here it will give you

$$c_1 y_2 = c_2 y_1$$

That is what it gives you.

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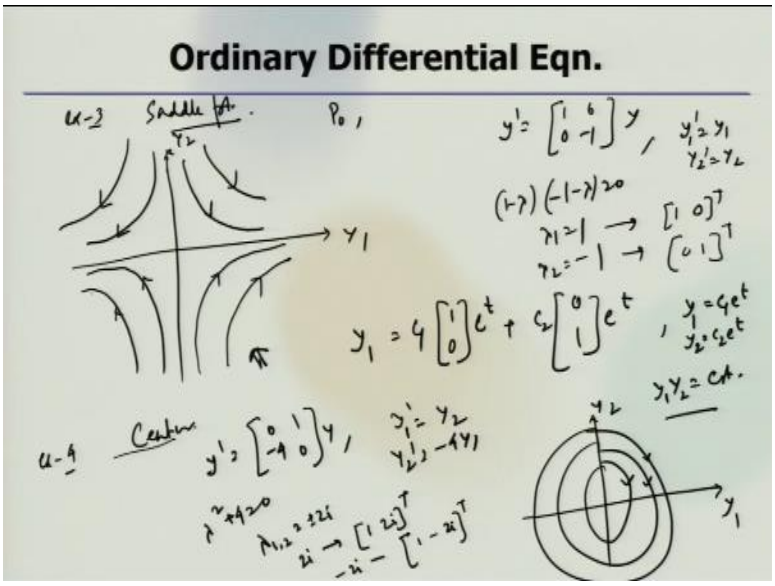

Now we come to 3 is the saddle point. A saddle point is a critical point P_0 at which there are two incoming trajectories, two outgoing trajectories. And all other trajectories in a neighborhood of P_0 bypass P_0, okay. So, we have then

$$Y' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Y$$

So, $y_1' = y_1$, $y_2' = y_2$. It has a saddle point at the origin. So, if you look at the plot y_1, y_2. so, the curves go like this. This goes like this.

This goes like, sorry this goes in this direction. So, this goes in this direction. Now the trajectories of the systems here. Now this guy's characteristics equation would be

$$(1-\lambda)(-1-\lambda) = 0$$

So, $\lambda_1 = 1$ and $\lambda_2 = -1$. For this guy the eigenvector would be $[1 \quad 0]^T$ and for this guy $[0 \quad 1]^T$. So, the general solution one can write

$$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$$

So, $y_1 = c_1 e^t$, $y_2 = c_2 e^t$. So, y_1, y_2 is essentially constant. So, this is a family of hyperbola as shown here. Now we look at the center. Now Centre is critical point that is enclosed by infinite many close trajectories. So, the system would be

$$Y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} Y$$

$y_1' = y_2$, $y_2' = -4y_1$. So, the picture if we draw, this would look like y_1. This is y_2.

So, these are the curve. So, this would be and the characteristics here would be

$$\lambda^2 + 4 = 0$$

So, $\lambda_{1,2} = \pm 2i$. So, for 2i the eigenvector is $[1 \quad 2i]^T$ and for -2i this is $[1 \quad -2i]^T$. So, it has a complex general solution.

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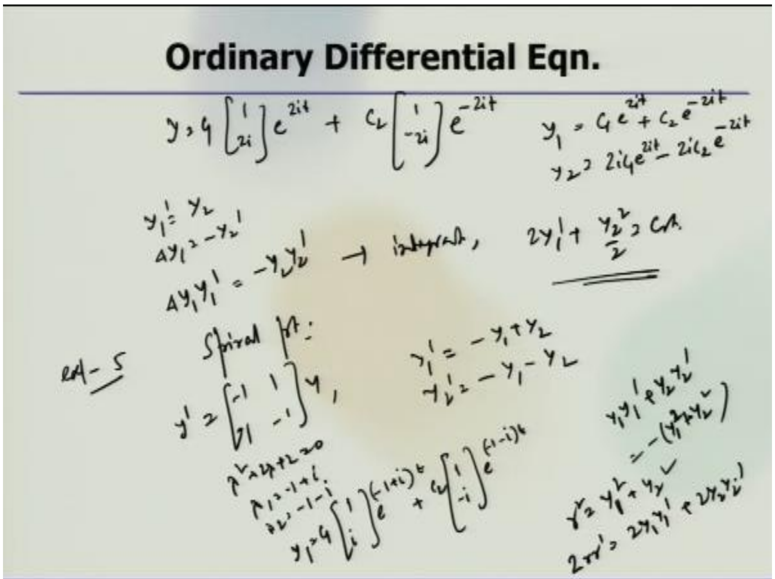

So, the solution one can write

$$y = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}$$

So, you will have

$$y_1 = c_1 e^{2it} + c_2 e^{-2it}$$

$$y_2 = 2ic_1 e^{2it} - 2ic_2 e^{-2it}$$

So, the transformation of the solution to the real form by Euler formula but we may just be curious to see what kind of eigenvalues we obtain in the case of center. Accordingly, we can start and so if we derive the equation then what we write $y_1' = y_2$, $-y_2' = 4y_1$.

Then the product of the left side must be equal to the product of the right side. So,

$$4y_1 y_1' = -y_2 y_2'$$

So, by integration we get

$$2y_1' + \frac{y_2^2}{2} = constant$$

So, this is a family of ellipse I mean enclosing the center at the origin which is like this. Now the other example which is example 5 is the spiral point.

So, the spiral point is a critical point P_0 about which trajectories are spiral approaching P_0 as t_0 tends to infinity. So, we will get

$$Y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} Y$$

$y_1' = -y_1 + y_2, y_2' = -y_1 - y_2$, has a spiral point at his origin. So, the characteristics equation would be

$$\lambda^2 + 2\lambda + 2 = 0$$

So, $\lambda_1 = -1 + i, \lambda_2 = -1 - i$.

So, we can find out the corresponding eigenvector. And so, the eigenvectors we will find out and we can write down the general solution. So, this is

$$y = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

So again, if you transform this complex solution to the real general solution by Euler formula. So now, we can simply multiply the first equations here by y_1 and the second by y_2.

So, what we get. So, we now introduce polar coordinates, and let us say

$$r^2 = y_1^2 + y_2^2$$

So, differentiating we will get

$$2rr' = 2y_1y_1' + 2y_2{y_2}'$$

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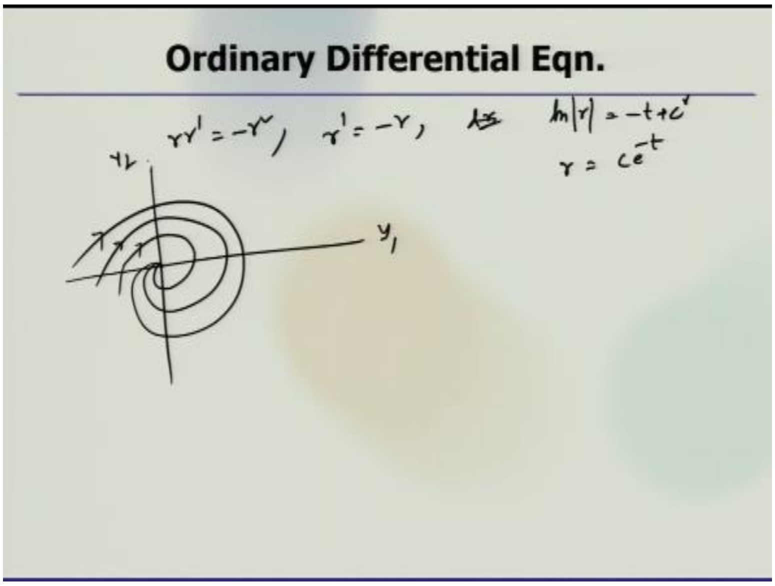

Hence the previous equations can be written as that our $rr' = -r^2$. So, you get

$$\ln|r| = -t + c^*$$

So, this is essentially which will give $r = ce^{-t}$. So, if you draw that this is y_1, this is y_2. So, we kind of start like that. So, some sort of a spiral going in that kind of directions. Now so this is how we can have these kinds of things for system of ODEs and all this.

So, like you can see when you have the system of ODEs also that satisfy all the properties that we have talked about second order or first order system. So, we will stop the discussion on ODEs pretty much here. I will conclude the discussion on ODE. Now we will go to the next session.