Computational Science in Engineering

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Lecture - 18

And on ODEs, now just before moving to the partial differential equation, we will just touch upon quickly the Fourier and the Laplace so that this would be a bit of making some connectivity between this ODEs and while looking at the numerical part.

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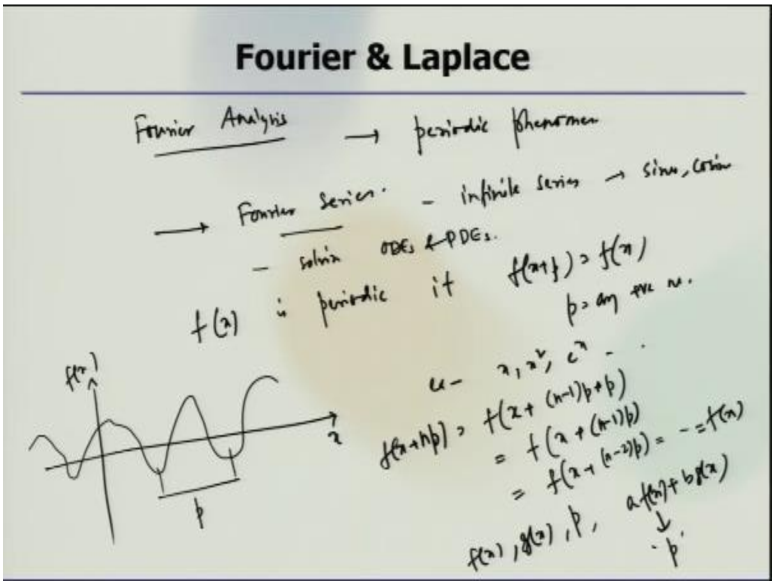

So, what we are going to look at here, I mean, as I said just quickly touch upon the Fourier. So now first thing is that we will just look at the Fourier analysis. Now this is, this Fourier analysis actually convince periodic phenomena. So, this is a periodic phenomena. And they occur quite frequently in engineering applications. So, like vibrations, alternative currents, rotating part of machine such that.

Now the when we have that kind of physical problem the related periodic function from the Fourier series. So, representing this complicated function in terms of simple periodic and sine and cosine functions gives many insights into the phenomenon and that representation is called the Fourier series. And sequence of investigation made in this way is called the Fourier analysis.

Now the basic concepts remain that the Fourier series are infinite series. So, these are infinite series designed to represent a generic periodic function in terms of simple ones like sines, cosines, something like that. Also apart from this analysis, this they constitute an important tool for solving ODEs and PDEs. So that is also important.

So now so this is essentially Fourier series the basic tool for representing periodic function which play an important role in applications. Now let us say if you say the function $f(x)$ is called periodic, is periodic $f(x+p)$ is $f(x)$. So, p is any positive number, any positive number. So, p is also called the period of the function like one can see, let us say if I have a function like that. So, this is called p.

So, this is a $f(x)$ and this is x. So now the smallest possible period is called the fundamental period. Periodic functions are sines and cosines. I mean like examples of x, x^2, e^x such that. Now some properties like a p is in period then 2p, 3p or np is period of the function when we can write

$$f(x+np) = f(x+(n-1)p) = f(x+(n-2)p) = f(x)$$

Now if $f(x)$ and let us say $g(x)$ have period of p then $af(x) + bf(x)$ where a, b are constants also has period p. So, this is sort of an some properties.

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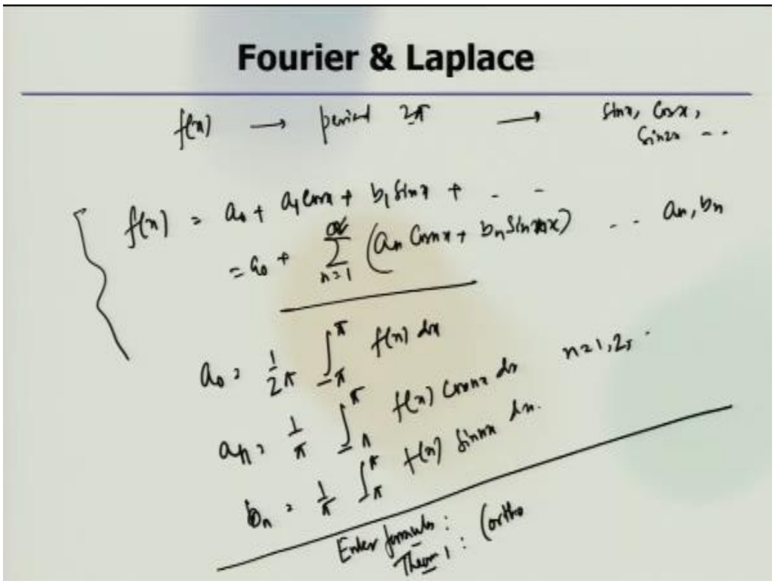

Now typically the periodic function $f(x)$ is defined in a period of 2π interval and simple periodic functions like sin x, cos x, sin 2x so on. And they have a period of 2π. So, these functions are called the trigonometric functions and they are kind of trigonometric system of function. Now $f(x)$ through the if you represent this $f(x)$ through the linear combination then we can write

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots$$

And so, like that which is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

So here the assuming these coefficients a_n and b_n are such that the summation converges to $f(x)$. Now as you see this right hand side expression this is a linear combination with the sum having a period of 2π, okay. So, the this is what gives you an idea of the definition of a Fourier series.

And the coefficients one can find out if the period is a 2π

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

where n goes 1, 2 like that. And

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

So, one can look at this different function like cos x, cos 2x and they are I mean this is very simple, one can plot these things even any simple mathematical tools or.

Now we get down Euler formula. So, the key to the Euler formula is the orthogonal of the trigonometric system. So, this is a very basic and important which actually appears quite often in many places. Using this orthogonality, we can derive this. Now there is a theorem, let us see theorem 1 which we say that this is an orthogonality. So, we can write it in the next page.

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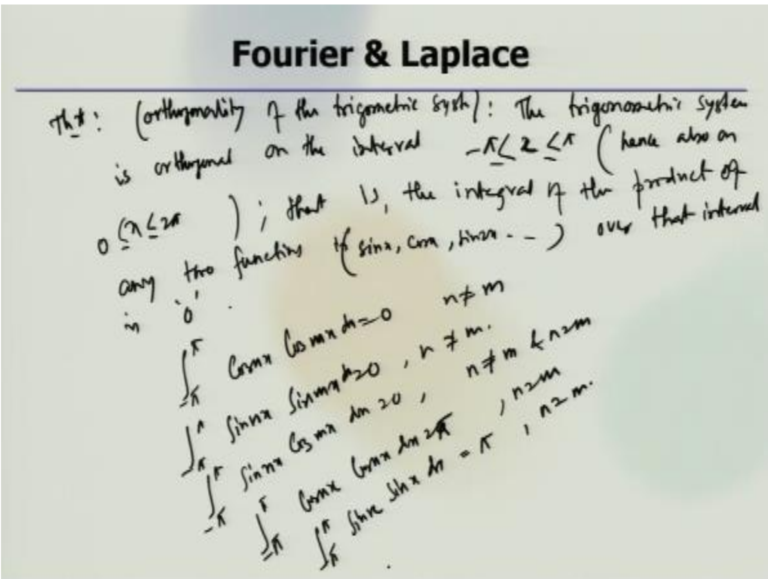

Let us say theorem 1, which talks about orthogonality of the trigonometric system. We said the trigonometric system which is sin x, cos x, sin 2x, cos 2x something like that is orthogonal on the interval which is $-\pi \leq x \leq \pi$. Hence also on $0 \leq x \leq 2\pi$; that is the integral of the product of any two functions, any two functions like sin x, cos x, sin2x and so on over that interval is also 0.

So, which means let us say

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0$$

it would be 0 for n not equals to m. Similarly

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$$

equals to 0 for n not equals to m. So similarly, you can have

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$

which is zero and n not equals to m and n equals to m. So, there are like other would be

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \pi$$

where n equals to m.

Similarly,

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi$$

where n equals to m. So, this is the orthogonality condition. And so, this proof is quite straightforward. You can look at any of the books which are given. So, we are not going into the details of that. Now how do you find the Euler formula, so that is important.

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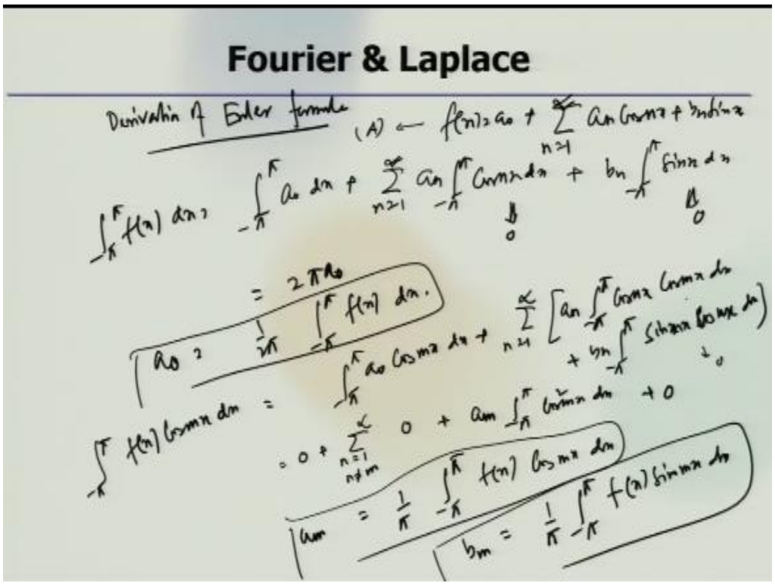

Now the derivation of that. So, we have the function effects, which is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

So, if we integrate both sides

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \, dx$$

So, this guy is contributing to 0, this guy will contribute to 0. So, this would be

$$\int_{-\pi}^{\pi} f(x) dx = 2\pi a_0$$

So, we get

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

Now similarly, if this let us say if we multiply both side by cos mx and integrate it, so what do we get

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= \int_{-\pi}^{\pi} a_0 \cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx$$

So, these guys contribute to 0, so last guy will be there.

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

And similarly, one can show

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

So, these are the values that one can find by deriving this Eulerian formula.

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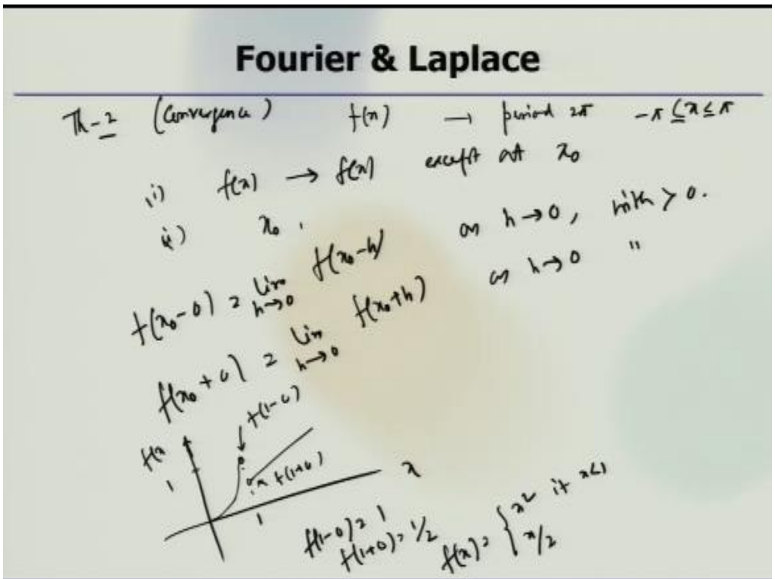

Now we can have another theorem which is called the convergence theorem, theorem 2 which is on convergence. So here let $f(x)$ be periodic with a period of 2π and piecewise continuous interval which is $-\pi \leq x \leq \pi$. So furthermore, let $f(x)$ have a left hand derivative and right hand derivative at each point of that interval, then the Fourier series $f(x)$ convergence to $f(x)$ except at point x_0 where $f(x)$ is discontinued.

Second, at the point of discontinuity at x_0 Fourier series converges to the average of the left hand right hand limits of $f(x)$ at x_0. So that means so for example left hand right hand limits let us say

$$f(x_0 - 0) = \lim_{h \to 0} f(x_0 - h)$$

here h tends to 0 with h greater than 0. So, the right hand limit would be

$$f(x_0 + 0) = \lim_{h \to 0} f(x_0 + h)$$

as h tends to 0 with h positive.

So, one of the example one can see here is that $f(x)$ let us say this is 1, so the function goes like here and this point if $(1 - 0)$ and this is the point if $(1 + 0)$, this is also 1. So,

$$f(1 - 0) = 1$$

$$f(1 + 0) = \frac{1}{2}$$

So, the function the $f(x)$ is defined as

$$f(x) = \{ \begin{matrix} x^2 & if\ x < 1 \\ \frac{x}{2} & \end{matrix}$$

So, this is the function where you can see the left hand and right hand limit.

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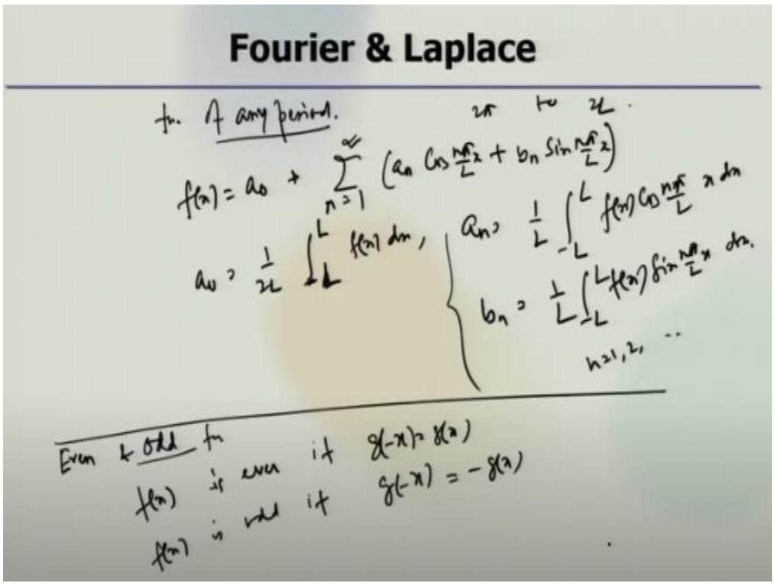

Now the function of any period so it just like let us say the so it would be easier function of any period which means that the transition from period 2π to 2L and the Fourier series we can write

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}\ x + b_n \sin \frac{n\pi}{L} x \right)$$

So, this is going for and the coefficients would be

$$a_0 = \frac{1}{2L} \int_{-L}^{\pi} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx$$

So here both these cases n goes from 1 to 2. So, I mean one can again like the previous one, one can derive this. So now we talk of some even and odd functions. Like a function $f(x)$ it can be said even, is even if $g(-x) = g(x)$ or a function $f(x)$ is odd if $g(-x) = -g(x)$.

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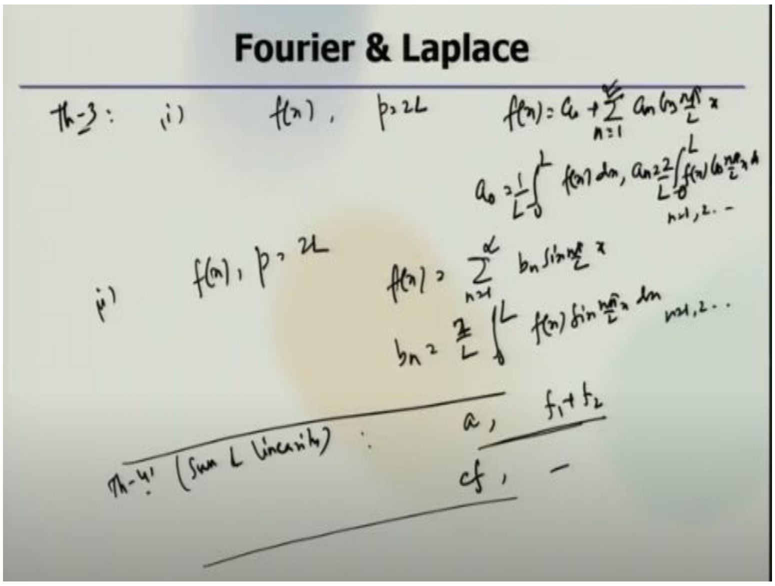

So, like similarly we can have a theorem like Fourier cosine and sine series like the Fourier series of an even function $f(x)$ for a period of 2L is a Fourier cosine series containing only the cosine terms like $f(x)$ would be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}\ x$$

$$a_0 = \frac{1}{L} \int_{0}^{L} f(x) dx$$

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi}{L} x\, dx$$

n is 1, 2 like that.

Second the Fourier series of an odd function $f(x)$ in a period of 2L is a Fourier sine series containing only the cosine terms which containing only the sine terms. So, the $f(x)$ would be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

And

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

So similarly, linearity of this Fourier series the Fourier coefficients $a,$ of a sum $f_1 + f_2$. So, this is sum and linearity.

So, the Fourier coefficients a and the of a sum $f_1 + f_2$ are the sum of the corresponding Fourier coefficients of f_1 *and* f_2 where both f_1, f_2 are function. Or second the Fourier coefficients c into f are the c times of the corresponding Fourier coefficients of f. So essentially you can I mean have the linear combinations of the Fourier functions and the scalar multiplication of that.

Now Fourier series we can apply to system and look at how we get solution for this kind of system.

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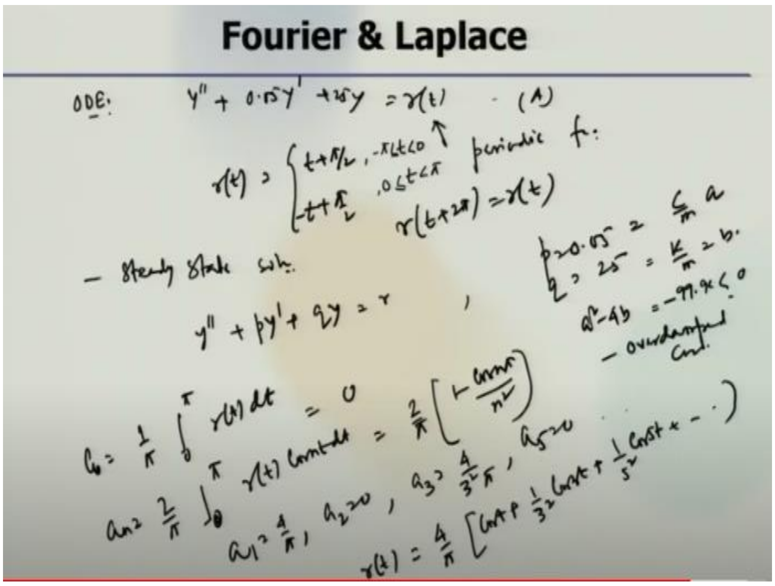

And like for example let us say let us consider an ODE which is

$$y'' + 0.05y' + 25y = r(t)$$

So, this is an ODE where we can, here $r(t)$ is in periodic function which is given as

$$r(t) = \{ \begin{array}{ll} t + \frac{\pi}{2} & -\pi \leq t < 0 \\ -t + \frac{\pi}{2} & 0 \leq t < \pi \end{array}$$

So,

$$r(t + 2\pi) = r(t)$$

So, we are going to find out the steady state solution, okay.

So here if you compare this equation like

$$y'' + py' + qy = r$$

then here p is 0.05, q is 25 and if you see this is $\frac{c}{m} = a$ and this is an $\frac{K}{m} = b$. so,

$$a^2 - 4b = -99.99 < 0$$

So, this is an over damped case. Now and the steady state solutions are particular integral of y_p of a. So, we can find out the Fourier series now.

So, series, cosine series we will get;

$$a_0 = \frac{1}{\pi} \int_0^{\pi} r(t) dt$$

So here after integration we will get

$$a_0 = \frac{2}{\pi} \int_0^{\pi} r(t) \cos nt \, dt$$

which after integration and doing all this, so we will get

$$a_0 = \frac{2}{\pi} \left[\frac{1 - \cos n\pi}{n^2} \right]$$

So,

$$a_1 = \frac{4}{\pi}$$

$$a_2 = 0$$

$$a_3 = \frac{4}{3^2 \pi}$$

$$a_5 = 0$$

So

$$r(t) = \frac{4}{\pi} \left[\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right]$$

and like that. So, we can replace this one in the ODE. Then we can apply the superposition principle and find out the steady state solution for this one, okay.

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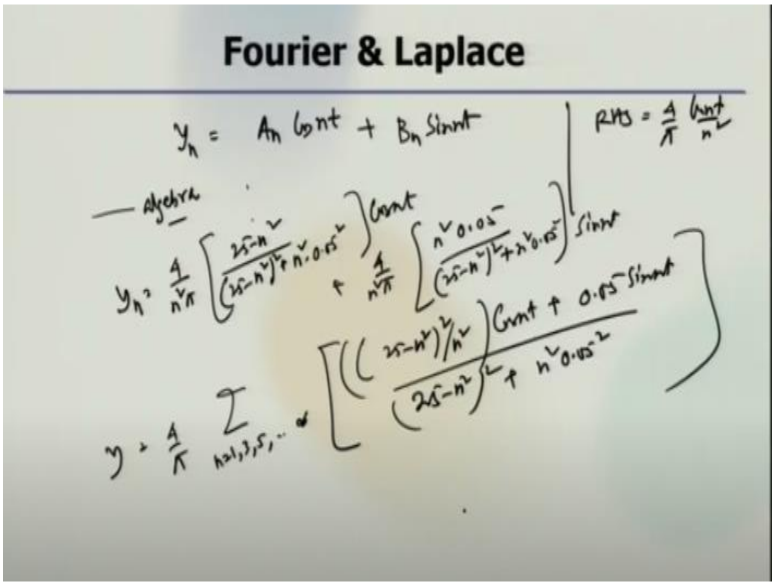

So now if you put that $r(t)$ and then you can have the generic solution of the system like

$$y_n = A_n \cos nt + B_n \sin nt$$

which is a generic form of

$$RHS = \frac{4}{\pi} \frac{\cos nt}{n^2}$$

So, the assumed solution would be in this particular form. Now if you take the derivative of this and put it in back, so what we will finally get that we will get, So, there are lot of algebra here, which one has to do. I mean, basically you just take the derivative of the solution and put it back in the ODE.

$$y_n = \frac{4}{n^2\pi}\left[\frac{25-n^2}{(25-n^2)^2+n^2-0.05^2}\right]\cos nt$$
$$+\frac{4}{n^2\pi}\left[\frac{n^2 0.05}{(25-n^2)^2+n^2-0.05^2}\right]\sin nt$$

So, the solution would

$$y = \frac{4}{\pi}\sum_{n=1,3,5,\dots\infty}\left[\frac{\left(\frac{(25-n^2)^2}{n^2}\right)\cos nt + 0.05 \sin nt}{(25-n^2)^2+n^2 0.05^2}\right]$$

So, this would be the, so once you get this particular solution from here, one can find out the steady state solution. So, we will stop here and continue this discussion in the next session.