Computational Science in Engineering

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Lecture - 19

Okay, so let us continue our discussion on the Fourier analysis now. So, we are looking at the Fourier series and now we will look at the Fourier analysis and then touch little bit on the Laplace.

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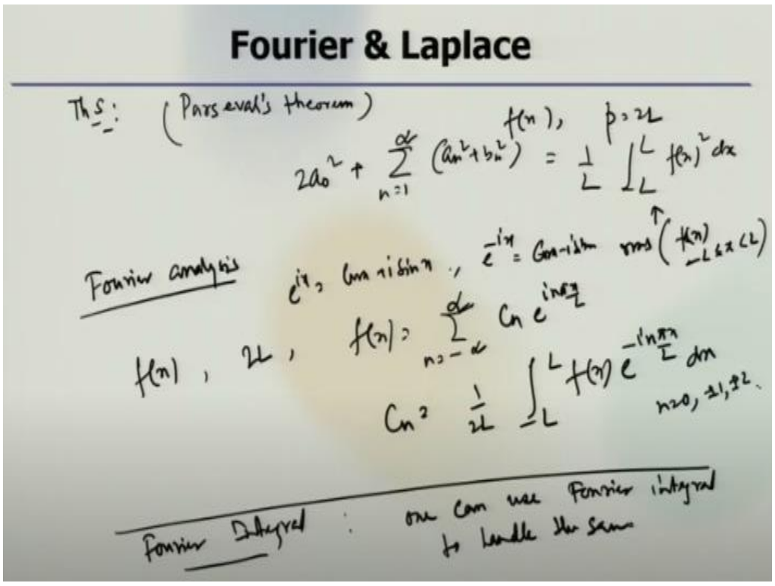

So now moving ahead, so this is what we are going to look at is the theorem of Parseval's theorem. So, what it talks about for a periodic function of $f(x)$, which is in the period of 2L that it is written as

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx$$

where a_n and b_n of the Fourier coefficients and the right hand side is the root mean square of $f(x)$ in one period.

So, this is RMS of $f(x)$ in period of minus L to L. So that is in within one period. Now we would move to the Fourier analysis. Now the Fourier series can be written in complex form, which simplifies calculation in many cases. And this complex form can be obtained because in complex analysis. The exponential function is related to the sine and cosine function by Eulerian formula like

$$e^{ix} = \cos x + i \sin x$$

Similarly, one can have

$$e^{-ix} = \cos x - i \sin x$$

Now the formal definition of the complex Fourier series for a period for a function $f(x)$ with a period of 2L is written

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{n\pi x}{L}}$$

And

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}} dx$$

where n is 0, $\pm 1, \pm 2$ ... and so on.

So, the derivation of complex Fourier series also one can look at. These are I mean basically quite standard thing that one can find in a textbook. So now we will just move to the Fourier integral. Now the Fourier series to Fourier integral, now Fourier series is a powerful tool for problem solving, problems which are having periodic functions within a finite interval.

Now it may be possible there are functions which are non-periodic in nature. And then in that case one can use Fourier integral to handle the same. Now this is an extension of Fourier series.

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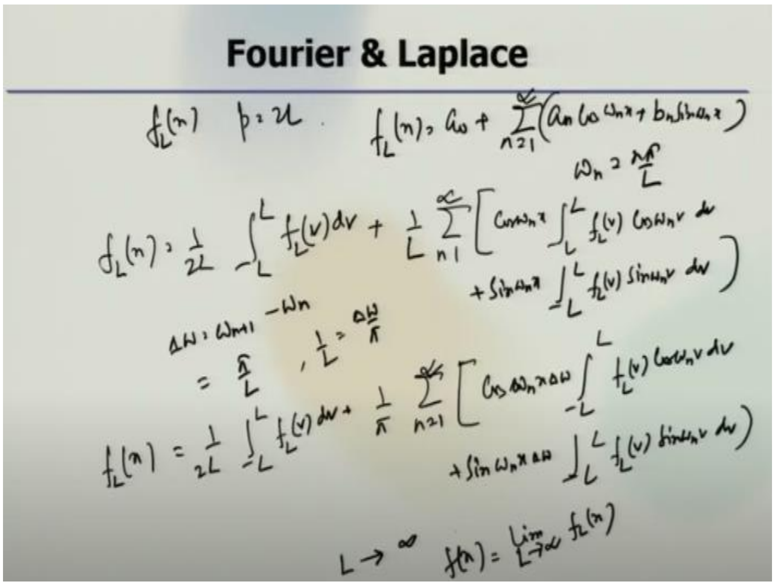

So let us say we consider a function $f_L(x)$ in period of 2L, then it can be represented as

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \omega_n x + b_n \sin \omega_n x)$$

where $\omega_n = \frac{n\pi}{L}$. Now if we insert Eulerian formula here, then the coefficients we can write this guy as

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(V)\, dV + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos \omega_n x \int_{-L}^{L} f_L(V) \cos \omega_n V\, dV + \sin \omega_n x \int_{-L}^{L} f_L(V) \sin \omega_n V\, dV\right]$$

So, V is the integration variable here. And

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$$

which is now one can write

$$\frac{1}{L} = \frac{\Delta\omega}{\pi}$$

So, this can be rewritten as

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(V)\, dV + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos \omega_n x\, \Delta\omega \int_{-L}^{L} f_L(V) \cos \omega_n V\, dV + \sin \omega_n x\, \Delta\omega \int_{-L}^{L} f_L(V) \sin \omega_n V\, dV\right]$$

So, this representation is valid for fixed L. Now for a non-periodic function L tends to infinity.

So, this above equation we investigate that what happens when L tends to infinity. So when L tends to infinity this would be

$$f(x) = \lim_{L\to\infty} f_L(x)$$

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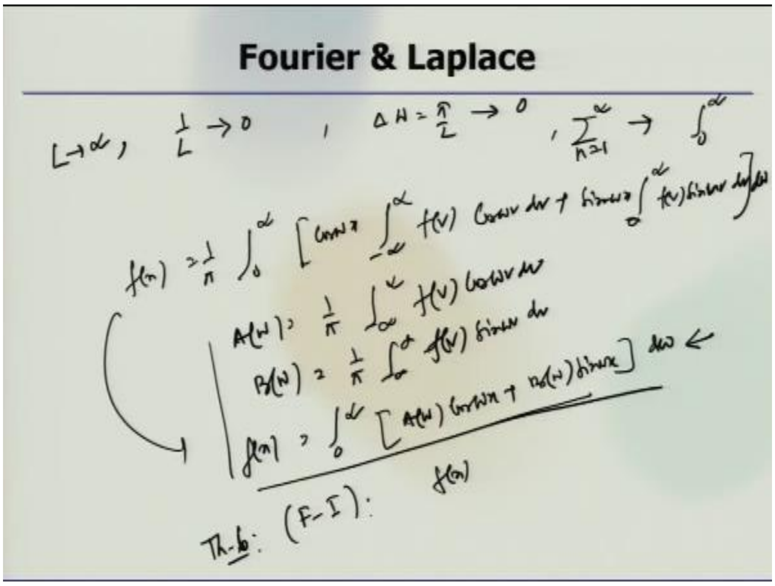

So now $\frac{1}{L}$ then when L tends to infinity $\frac{1}{L}$ tends to 0. So, if you see the value of the first term here, that vanishes in the integral. So, this guy goes off. Further we have

$$\Delta\omega = \frac{\pi}{L}$$

which is also tends to 0. So, which indicate that w becomes continuous hence the summation here like n equals to 1 to infinity would be replaced by 0 to infinity.

And then we get

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos\omega x\int_{-\infty}^{\infty} f(V)\cos\omega V\,dV + \sin\omega x\int_{-\infty}^{\infty} f(V)\sin\omega V\,dV\right]d\omega$$

So that is what you guess which reduces that

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(V)\cos\omega V\,dV$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(V)\sin\omega V\,dV$$

So, you can rewrite this

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

So, this is a representation of the Fourier integral.

Now there is another theorem which is associated which is for the Fourier integral, Fourier integral is that if $f(x)$ is a piecewise continuous in every finite interval and has a right and left hand derivatives at every point and the integrals represented here is finite, then $f(x)$ can be represented by a Fourier integral given here. At any point $f(x)$

is discontinuous, the value of the Fourier integral would be average of the left hand right hand side derivative. Now this has also again application to ODEs and PDEs.

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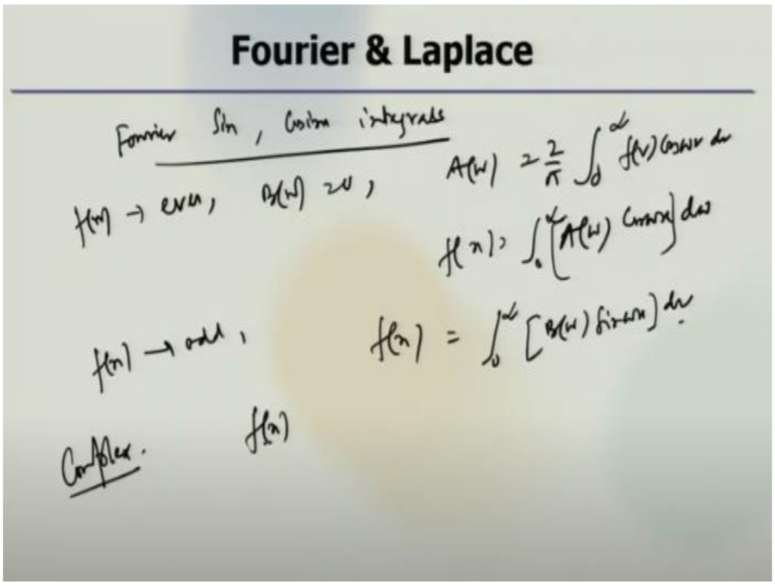

Now similarly one can look at like Fourier sin and cosines integrals like one can have Fourier sin cosine integrals these are quite a bit of standard textbook material. So let us say $f(x)$ is even then $B(\omega)$ is 0. So

$$A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(V)\cos\omega V \, dV$$

So,

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x] d\omega$$

Similarly, $f(x)$ is odd function then this becomes

$$f(x) = \int_0^{\infty} [B(\omega)\sin\omega x] d\omega$$

Now similarly we can have complex integral, complex Fourier integral.

Now $f(x)$ if it is having this standard these things, the standard function like this and Fourier integral not necessary even or odd, then we have all these $A(\omega), B(\omega)$ in this fashion, okay. And now we can write these things and we can get the complex integral for the same.

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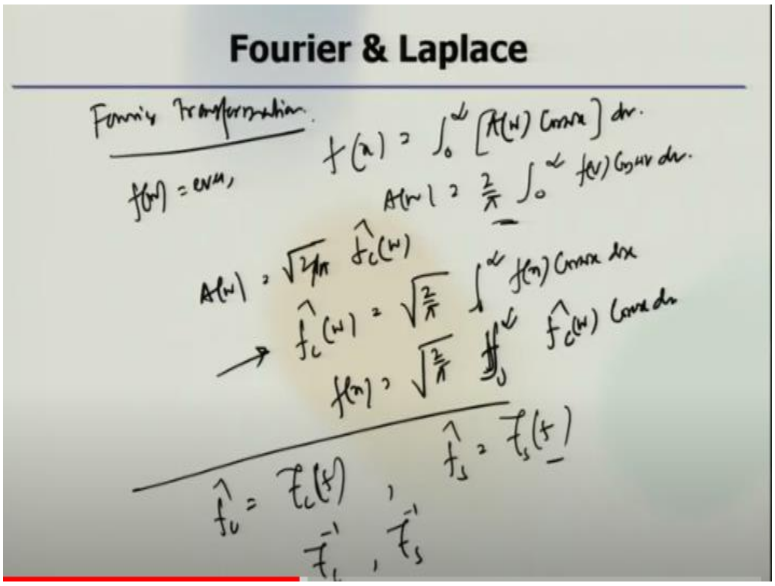

Now just going with the another one is the Fourier transformation, okay. Now the integral transformation is the form of integral that produces from eigenfunctions. Then the Fourier will also have Fourier cosine, so let us say $f(x)$ is even. Then the cosine integral we have already got like

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x] d\omega$$

And

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(V)\cos\omega V\, dV$$

Now we now

$$A(\omega) = \sqrt{\frac{2}{\pi}}\widehat{f_c}(\omega)$$

c indicates the cosine.

Then changing the integration variable v to x, so you can write this guy

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\, dx$$

And

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \widehat{f_c}(\omega)\cos\omega x\, dx$$

Now this guy is called the Fourier cosine transformation. So similarly, one can find the Fourier sine transformation also, okay.

Now some common terminology that let us say like for Fourier cosine transformation one could like this and Fourier sine transformation one could write like this. And these are the inverse transformation of these two values. So, we can now again there are series of Fourier sine and cosine transformation, which one can find in like standard textbook.

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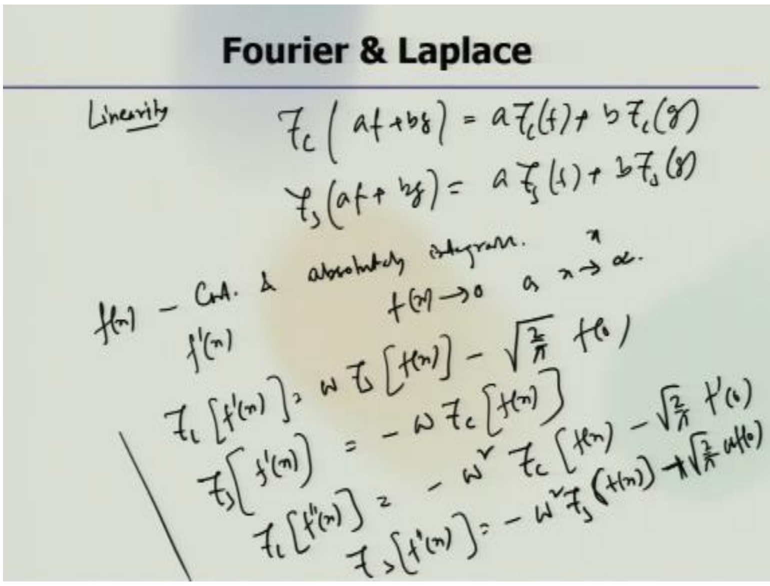

But we talked about some of the linearity like say let us say

$$F_c(af + bg) = aF_c(f) + bF_c(g)$$

Similarly,

$$F_s(af + bg) = aF_s(f) + bF_s(g)$$

So this is a linear property of, similarly we can do the derivative like let $f(x)$ is continuous and absolutely integrable on x axis and $f'(x)$ is piecewise continuous on a finite interval says that $f(x)$ tends to 0 as x tends to infinity. Then we can write

$$F_c[f'(x)] = \omega F_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0)$$

Similarly, we can write that

$$F_s[f'(x)] = -\omega F_c[f(x)]$$

Similarly

$$F_c[f''(x)] = -\omega^2 F_c[f(x)] - \sqrt{\frac{2}{\pi}} f'(0)$$

And

$$F_s[f''(x)] = -\omega^2 F_s[f(x)] - \sqrt{\frac{2}{\pi}}\omega f(0)$$

So, these are pretty much used. So, these derivatives and for solving ODE or PDE.

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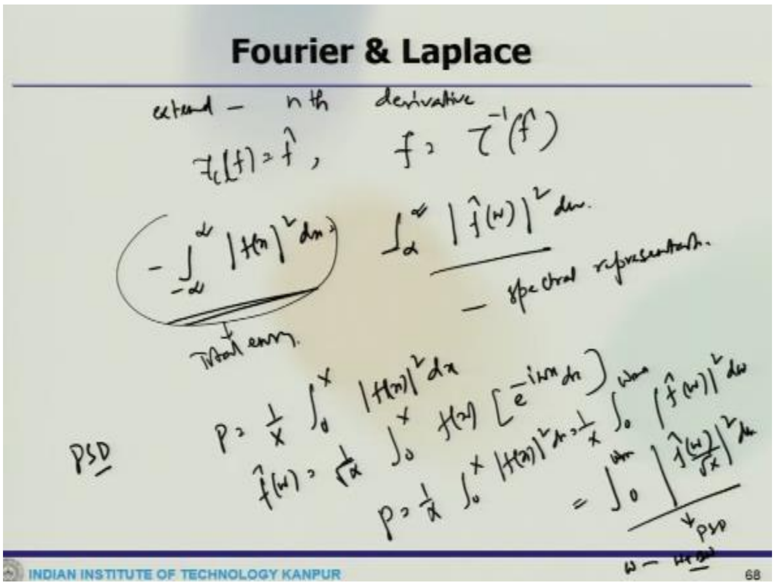

Now similarly, one can write this for the n-th derivative also. And now so you can extend this for n-th derivative and also write the complex Fourier transform. Now some of the other notation like we have already written

$$F_c[f] = \hat{f}$$

So,

$$f = F^{-1}(\hat{f})$$

So now one interesting thing is that from the Parseval's theorem Fourier transform, which refers to that

$$\int_{-\infty}^{\infty} f(x)^2\, dx = \int_{-\infty}^{\infty} \hat{f}(\omega)^2\, d\omega$$

So, the left hand side represent the square of the $f(x)$, which represents fluctuations mathematically modeled by the physical system. So, these also represent the total energy of the fluctuation. And the right hand side this guy which can be interpreted as the total energy of the physical system. Hence the integral of this contributes to the frequencies of ω between A to B and so these are called the spectral representation, okay.

And there is another term which is called power spectral density or PSD. So, this let us say, so here the quantity of this left hand side minus so represents the this represents the total energy and this integrate will converge only when there is a transient signal. So, if x goes to 0 sufficiently at X large X. So, in the signal $f(x)$ for example is form of an acoustic sensor monitoring.

So, one can say this

$$P = \frac{1}{X}\int_0^X f(x)^2\,dx$$

And the Fourier transform can be defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{X}}\int_0^X f(x)\,e^{-i\omega x}dx$$

Now using this guy, one can write

$$P = \frac{1}{X}\int_0^X f(x)^2\,dx = \frac{1}{X}\int_0^{\omega_{max}} \hat{f}(\omega)^2\,d\omega = \int_0^{\omega_{max}} \left(\frac{\hat{f}(\omega)}{\sqrt{X}}\right)^2 d\omega$$

So, this guy is called the power spectral density which measures the power available in the signal from frequency ω to $\omega + \Delta\omega$.

So, between that so it measures this. So, this is an important measurement these things. So, one can have power spectral density or energy spectral density which are very useful in analyzing some of the physical system which shows that.

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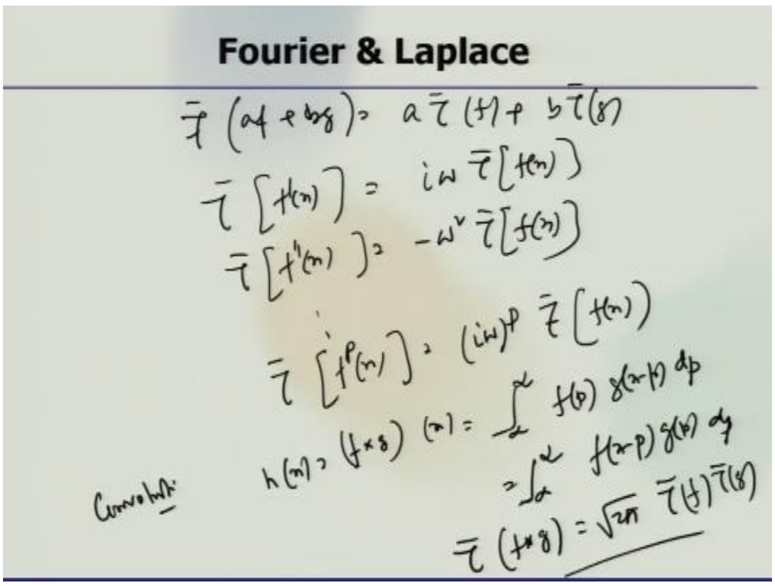

Now similarly, your Fourier transform also shows the linearity like that. And also, it follows some of the derivative law which is

$$F(af + bg) = aF(f) + bF(g)$$

$$F[f'(x)] = i\omega F[f(x)]$$

$$F[f''(x)] = -\omega^2 F[f(x)]$$

and so on. So, one can have t-th derivative like this

$$F[f^p(x)] = (i\omega)^p F[f(x)]$$

So, you can have this. Now the convolution theorem which says that

$$h(x) = (f \times g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

So, which says that

$$F(f \times g) = \sqrt{2\pi}\, F(f)F(g)$$

So, this is what it says. So, these are the some of the important information which one can use for solving some of the physical system.

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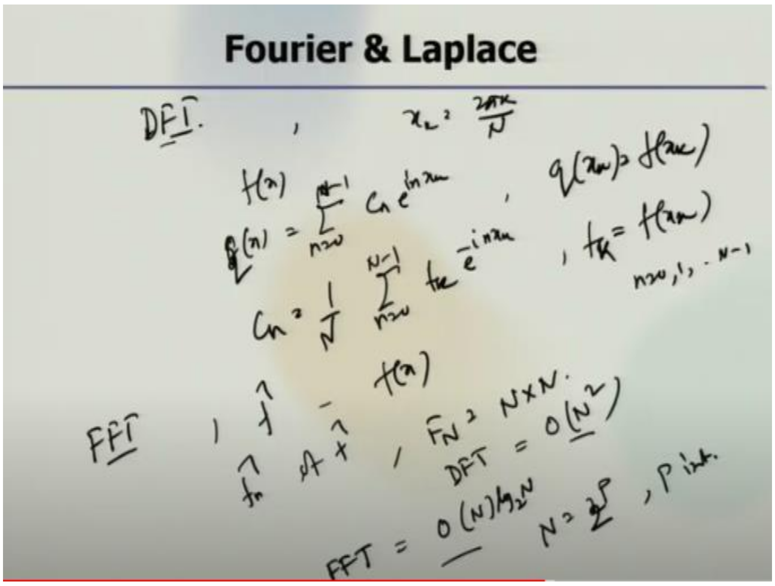

So, another thing is called the discrete Fourier transform. So, because in Fourier transform the function f has to be given over some interval. But in any most of the practical cases when the data comes from simulations or experience which represent the physical problem $f(x)$ is given only intervals of value of finite number or many points. So that time we do discrete Fourier transform.

So, where what we let us say if we take n samples or snaps where the

$$x_k = \frac{2\pi k}{N}$$

So let us say $f(x)$ is sampled over these points. Then we can write

$$q(x) = \sum_{n=0}^{n-1} c_n e^{inx_k}$$

So, this interpolates $q(x_k)$= $f(x_k)$. And the coefficient

$$c_n = \frac{1}{N} \sum_{n=0}^{n-1} f_k e^{-inx_k}$$

where $f_k = f(x_{k_n})$, n=0, 1, . . . and then so on.

So, the discrete Fourier transform is also helpful to analyze some of these practical systems. So, and then we can have first Fourier transform which is called FFT. So, where $\hat{f}$ is the frequency spectrum of function or signal $f(x)$. So, the component $\widehat{f_n}$ of $\hat{f}$ give a resolution of 2π periodic function. So, the Fourier matrix F_N would be of size N × N. And the discrete Fourier transform would be order of n square.

$$DFT = O(N^2)$$

But this kind if we do FFT then we will have order of

$$DFT = O(N) \log_2 N$$

So, for the FFT has this much so we can see how DFT becomes so important. So, to in FFT one has to choose N is

$$N = 2^P$$

where P is an integer. So, one can see that why this DFT becomes important because of the large size of the system.

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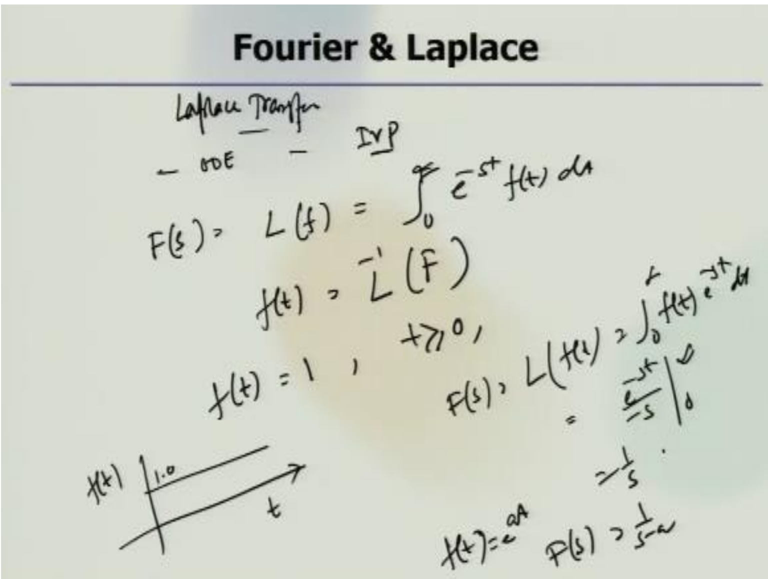

Now that what I would like to touch up on Fourier and quickly, we will go to Laplace transform. What we will do is that, so this is a method is a powerful method for solving linear ordinary differential equation. So, and corresponding initial value problem. So, we can use Laplace transformation, is generally solve for the initial value problem.

So let us say for a function of $f(s)$ the Laplace transform would be given as

$$f(s) = L(f) = \int_0^\infty e^{-st} f(t)dt$$

We assume ft such that above integral exist. So, and where $f(t)$ would be Laplace of F inverse. So, we can see a simple let us say a function $f(t)$ equals to 1 for t greater than 0. That means if I plot ft which is 1. Here is the t's axis.

Then

$$f(s) = L(f) = \int_0^\infty e^{-st} f(t)dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

So similarly, if we say

$$f(t) = e^{at}$$

then Laplace of that would become

$$f(s) = \frac{1}{s-a}$$

So now there are standard functions where one can actually find out this.

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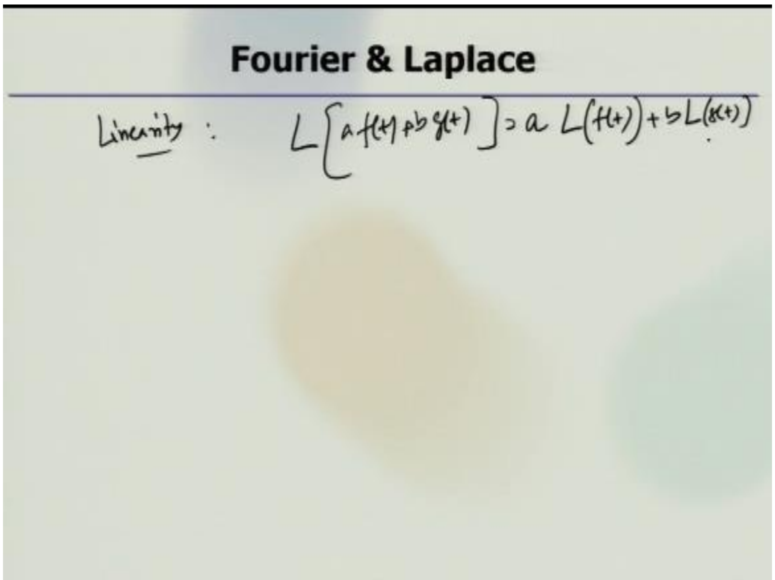

Now similarly Laplace transformation also there is linearity or linearity exists like if we have

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

like that. So that linearity also exists. Then we can shift, shifting is there.

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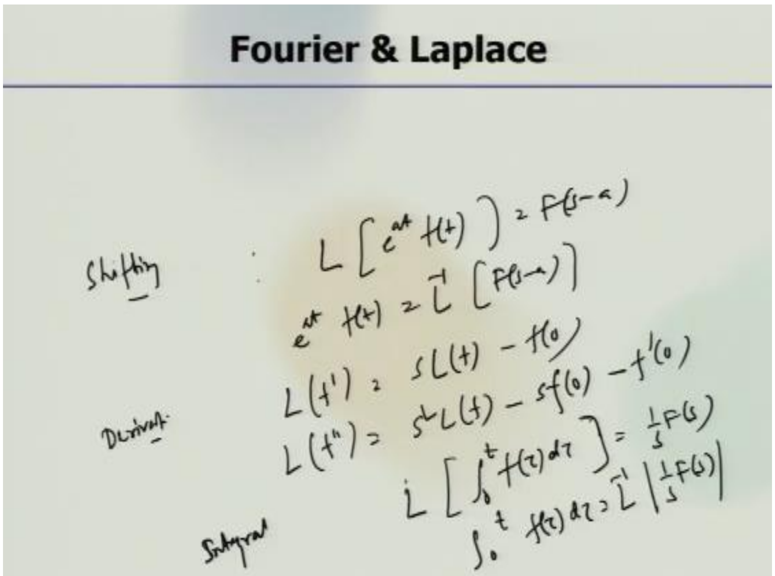

So, shifting also we can have let us say ft as an transformation phase then we can write

$$L[e^{at}f(t)] = f(s-a)$$

When you take the inverse of that so this would be

$$e^{at}f(t) = L^{-1}[f(s-a)]$$

So, like this we will close the, now similarly the derivative also exists. So, like

$$L(f') = sL(f) - f(0)$$

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

and so on.

So, because these are important to solve the initial value problem. Similarly, you can have integrals where you can say that

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\,f(s)$$

So,

$$\int_0^t f(\tau)d\tau = L^{-1}\left[\frac{1}{s}\,f(s)\right]$$

Now I am just giving you another idea about how to use this.

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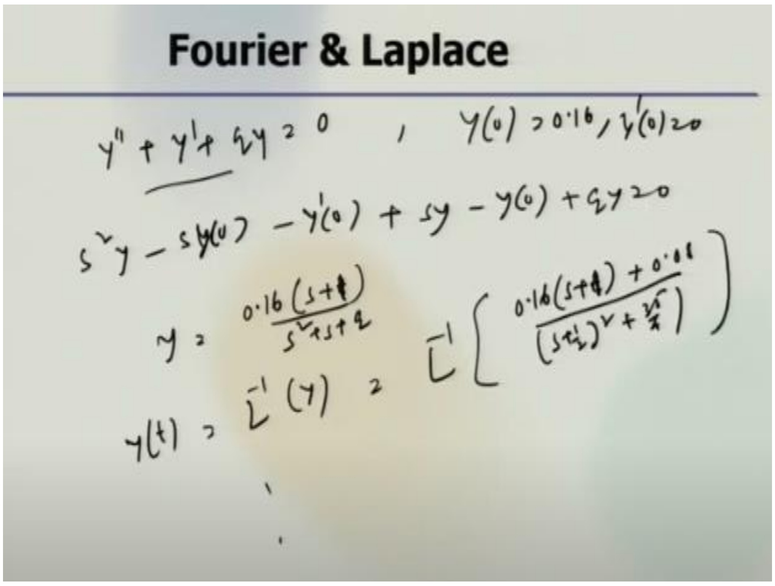

Now we can look at some of quick example of application to ODE, how one can use this Laplace transformation. For example, you have

$$y'' + y' + qy = 0$$

for initial value let us say $y(0) = 0.16, y'(0) = 0$. So, if you take the Laplace transformation this will become

$$s^2y - sy(0) - y'(0) + sy - y(0) + qy = 0$$

And if we do this using the values, we get

$$y = \frac{0.16(s+1)}{s^2 + s + 9}$$

So,

$$y(t) = L^{-1}(y) = L^{-1}\left[\frac{0.16(s+1)+0.08}{\left(s+\frac{1}{2}\right)^2+\frac{35}{4}}\right]$$

So, you can finally get the solution for this. Just to give you an idea how to have this kind of initial value problem and you can transform using the Laplace.

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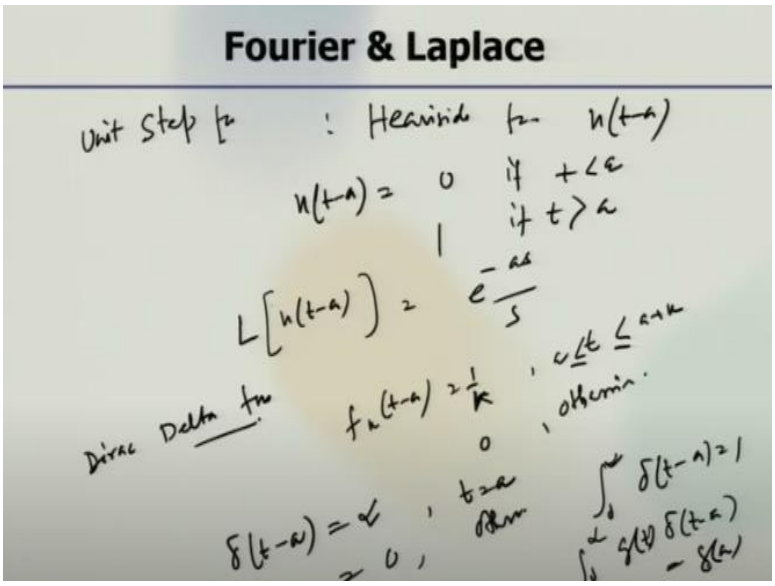

Now the advantage of this kind of Laplace method is that it is a non-homogeneous ODE, which does not require for solving the so homogeneous ODE. We can directly use this Laplace transformation of this. Now there is one could be, there are couple of function, one is the step function. You need step function which is called the Heaviside function.

So, which is defined $f = u(t-a)$, which is defined as

$$u(t-a) = \begin{cases} 0 & if\ t < a \\ 1 & if\ t > a \end{cases}$$

So, if we take the Laplace of this function so we will get essentially

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

So typically, this function used in lot of engineering applications. And similarly, one more function is there which is called the direct delta function. So direct delta function is sort of a defined like

$$f_k(t-a) = \begin{cases} \frac{1}{K} & if\ a \le t \le a+k \\ 0 & otherwise \end{cases}$$

So, if we, the integral of this function over an interval would give so typically we write that direct delta function is

$$\delta(t-a) = \begin{cases} \infty & if\ t = a \\ 0 & otherwise \end{cases}$$

So,

$$\int_0^{\infty} \delta(t-a) = 1$$

$$\int_0^{\infty} g(t)\delta(t-a) = g(a)$$

in general.

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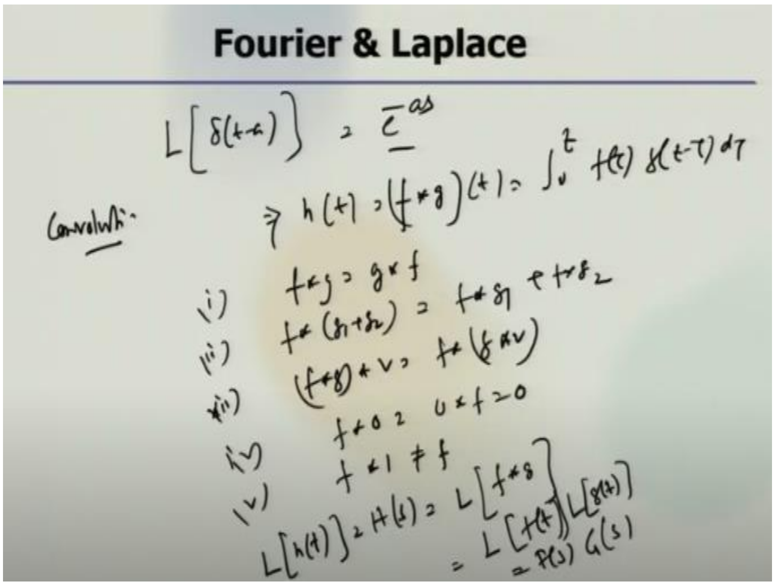

So, if we do the Laplace of delta function, so we get

$$L[\delta(t-a)] = e^{-as}$$

So, this is another important function and similarly we can look at the convolution. Convolution is the integral in solving also non-homogeneous linear ODEs where we say

$$h(t) = (f * g)(t) = \int_0^{\infty} f(\tau)g(t-\tau)\, d\tau$$

So, one can say

$$f * g = g * f$$

So, this is a commutative law. Secondly

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

So, this is distributive law. Then we can have

$$(f * g) * V = f * (g * V)$$

so, this is associative law. And

$$f * 0 = 0 * f = 0$$

One can have like

$$f * 1 \neq f$$

And the Laplace transform of

$$L[h(t)] = H(s) = L[f * g] = L[f(t)] + L[g(t)] = F(s)G(s)$$

So, these are different functions where one can do this Laplace transform and already we have seen that this Laplace transform is essentially quite handy when you are trying to solve these initial value problems. And especially when you have non-homogeneous equation it does not require that you transform or the split the equation into homogeneous part and then solve the non-homogeneous part straightaway.

Or other you can consider the homogeneous equations and taking Laplace transform and try to solve it. So, this would be handy in solving this kind of ODEs and we see these applications more when you go to the numerical part. So, we will stop the discussion here and continue with the other things in the next session.