Computational Science in Engineering
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Lecture – 02
Linear Algebra

So, we are looking at the Gauss Elimination process, and we just started discussing about the elementary matrix and let us continue what we were talking about the elementary matrix here.

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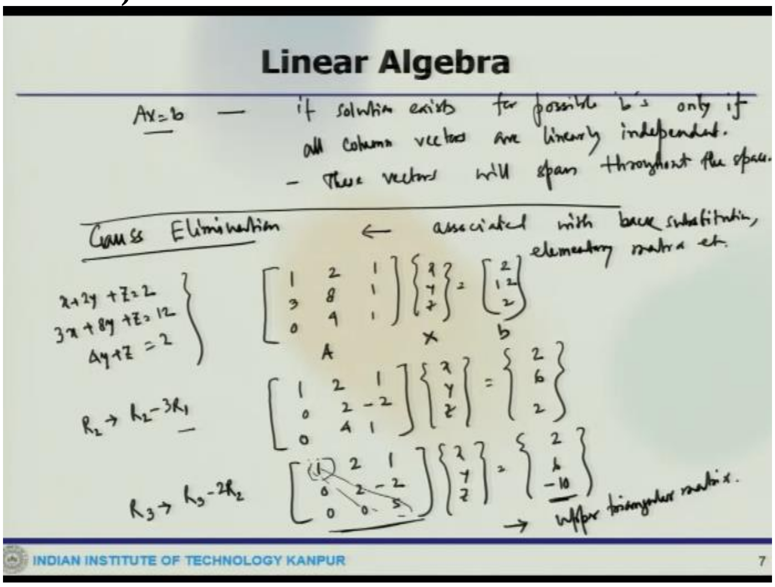

So, here you can see when you do the row operation so, the process actually what you have done. This is the row operation that the first row operation that we have done. And that is why you get the first second row and then when you do the operation of the third row that is what you get it. So, now what we can do?

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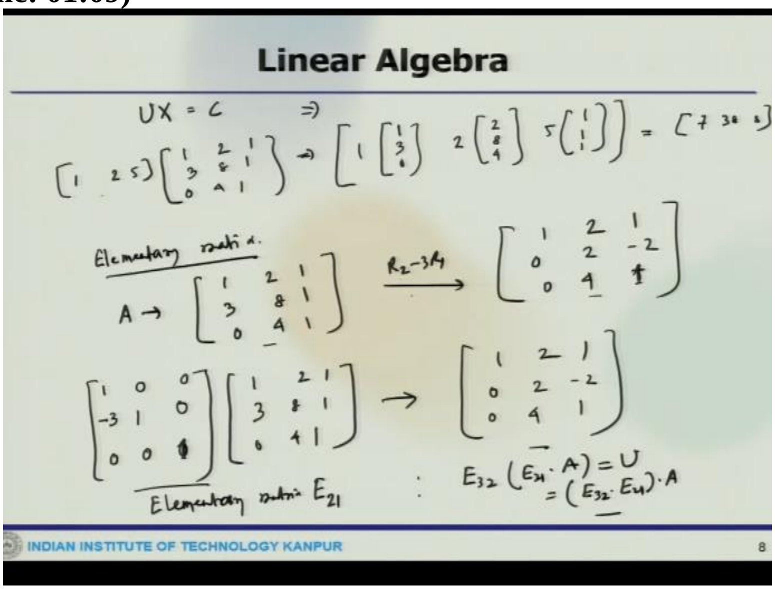

We can similarly find out these things by doing row operations of the system. So, this is the matrix and when you did that operation, first row operation we get this and so, this is the elementary matrix which E_{21}. So, that is what it gives the first things. So, this is E_{21}. So, what we can have essentially that means the other one would be $E_{32}(E_{21}.A) = U.$ which will give us the upper triangular matrix or one can write $E_{32}(E_{21}.A) = U = (E_{32}.E_{21})A$. So, this is called the associative property of the matrix.

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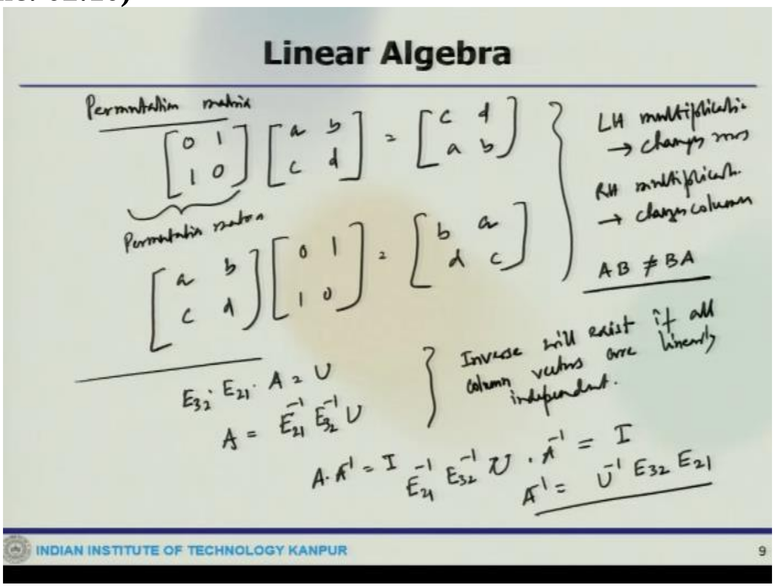

So, now, the other matrix which will come on the way is the permutation matrix, so what is the permutation matrix? It just like let us say we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so it can be multiplied with the permutation matrix, so, that will get $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$. So, let us say we can then multiply to these another 2× 2 system, which could be $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So, this guy is called the permutation matrix. Now, this can be multiplied on the other side of the system for example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

So, point here is that the left hand multiplication changes row and right hand multiplication changes columns. So, this is important here that left hand multiplication changes rows and right hand multiplication changes columns. So, this follows non commutative nature of matrix multiplication which is AB ≠ BA, this is called the non-commutative nature of matrix multiplication.

So, the process of elementary matrix generally goes from left to right, which makes it easier to find the inverse. For like we had $E_{32}.E_{21}.A = U$ is upper triangular matrix. So, if we try to find out A this would be A,

$$A = {E_{21}}^{-1}{E_{32}}^{-1}U$$

So, inverse will exist as all column vectors are linearly independent, this is important inverse will exist if all column vectors are linearly independent and we have seen through some simple examples of 2 dimension and 3 dimensions, how they span whether they kind of independent or not.

So, in that case what will happen is that $A.A^{-1} = I$ identity matrix. So, what we will write

$${E_{21}}^{-1}{E_{32}}^{-1}U \cdot A^{-1} = I$$

So, A inverse would be

$$A^{-1} = U^{-1}E_{32}.E_{21}$$

So, this is how you can find out that thing

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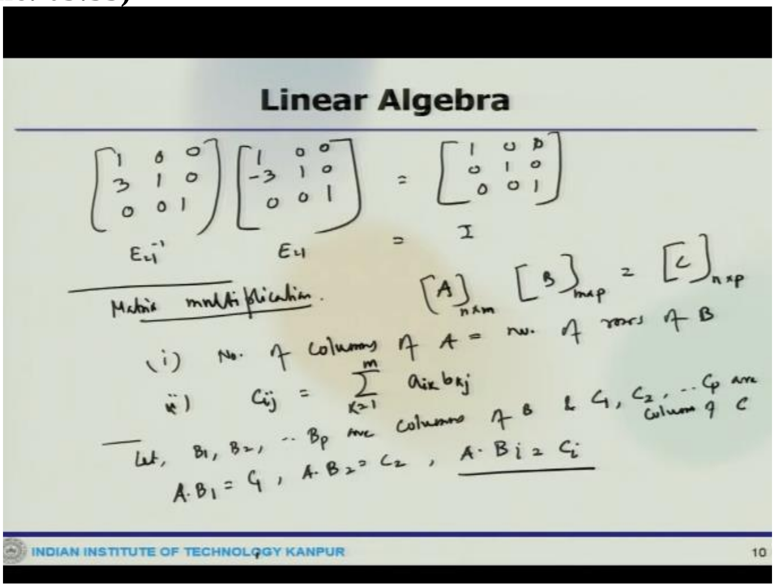

Or other hand you had, so our E_{21} was $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, this is our E_{21}. Now, to get the ${E_{21}}^{-1}$ which would be straightforward,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

this will give us an identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, you can see this is how it works. Now, we look at the matrix multiplication. So, matrix multiplication what we do is that

$$[A]_{n\times m}\,[B]_{m\times p} = [C]_{n\times p}$$

So, least number of conditions that need to be satisfied:

i) Number of columns of A would be number of rows of B.

ii) $C_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$

Other possibilities to do matrix multiplications column multiplications approach where let us say let B_1, B_2,… B_p are columns of B and C_1, C_2,… C_p are columns of C. Then we can write $A\cdot B_1 = C_1$, $A\cdot B_2 = C_2$ that means $A\cdot B_i = C_i$ in generic form one can write. So, here are the columns of C or some linear combination of columns of A.

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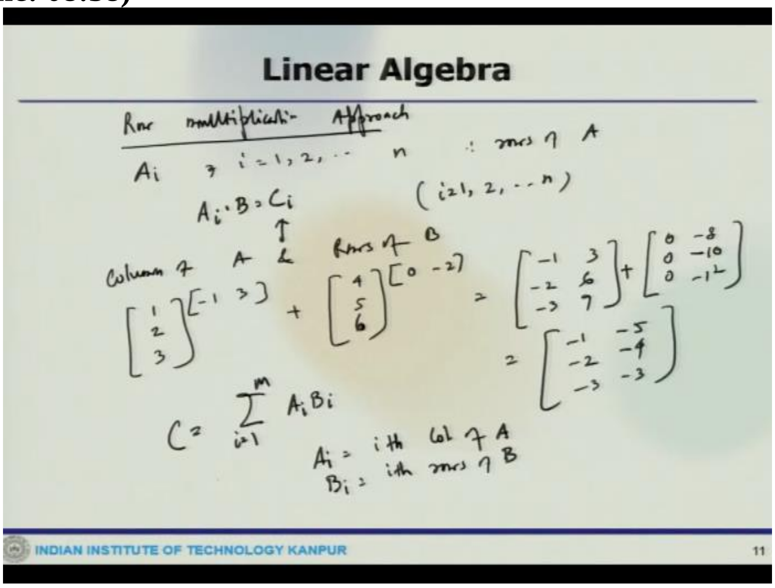

Now if we go row multiplication approach which is like let us say A_i where i goes 1 2 to n these are the rows of A. Then we can write $A\cdot B_i = C_i$ where i goes 1 to n where C_i is the ith row in C matrix. So, you can see rows of C are linear combination of rows of B. Now if you let us say columns of A and rows of B, that is the way if you multiply it, so like

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}-3 & 1\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix}\begin{bmatrix}0 & -2\end{bmatrix} = \begin{bmatrix}-1 & 3\\-2 & 6\\-3 & 9\end{bmatrix} + \begin{bmatrix}0 & -8\\0 & -10\\0 & -12\end{bmatrix} = \begin{bmatrix}-1 & -5\\-2 & -4\\-3 & -3\end{bmatrix}$$

So, that means C would be summation

$$C = \sum_{i=1}^{m} A_i B_i$$

where A_i ith column of A, B_i is ith rows of B. So, now, there are other means that one can find out is the block multiplication approach.

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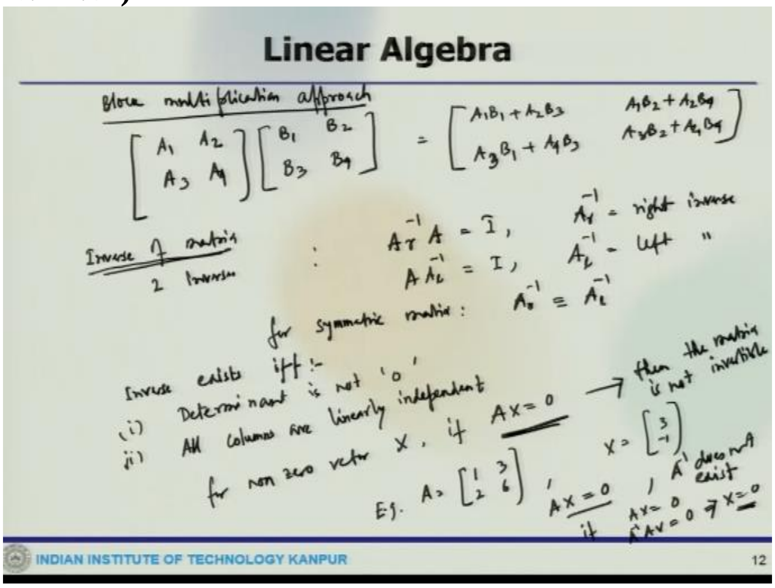

So, here in the block multiplication approach, you can do the multiplication in that fashion in block wise like

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

So, now, this is what you call it a block multiplication. Now, if we talk about inverse of A matrix there are in general there are 2 inverses one is the right inverse which we call like ${A_r}^{-1}A = I$ is identity where ${A_r}^{-1}$ is the right inverse.

Or you can have left inverse where ${A_l}^{-1}A = I$ is identity where ${A_l}^{-1}$ is left inverse. Now, one can note here for symmetric matrix would the inverses are same. So, for symmetric matrix ${A_r}^{-1} = {A_l}^{-1}$. Now, when you talk about inverse, so, there will be another question which may come how do we guarantee that inverse actually exists. So, one can say that inverse exists if and only if number

i) determinant is not 0 that means is not equal to 0.

ii) all columns are linearly independent.

Now, for a non-zero vector x if Ax = 0 then the matrix is not invertible. So, this is what that means the columns are not linearly independent. So, you can take an example like let us say

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

So, there exists A vector which will like $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, then if we do $Ax = 0$, so which means A inverse does not exist. So, now, if Ax = 0 then $A^{-1}Ax$ would be also 0 which means x is 0. So, the main logic is that x has to be a non zero vector. So, that is what is important.

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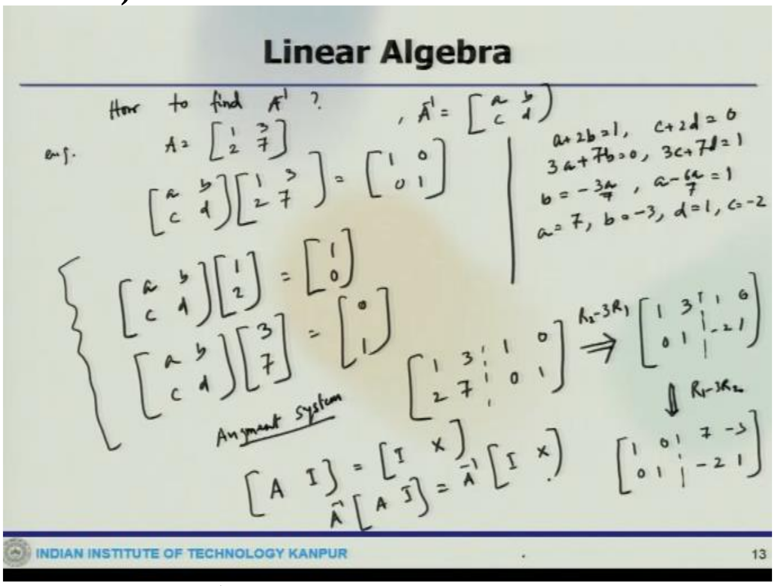

Now, one can say how to find A^{-1} that is important, so let us take an example again for where the matrix is invertible. So, let us say

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

So, here the matrix is invertible then we can find out the inverse and let us say the inverse is in terms of A matrix which is $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So, we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which would be identity matrix. So, just using the property of the matrix that inverse A equal to identity.

So, from here what you get

$$a + 2b = 1$$

$$c + 2d = 0$$

$$3a + 7b = 0$$

$$3c + 7d = 1$$

$$b = -\frac{3a}{7}$$

$$a - \frac{6a}{7} = 1$$

$$a = 7$$

$$b = -3$$

$$c = -2$$

$$d = 1$$

So, you can solve this is a simple algebra one can solve it and you can get these things. So, the point here is that if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if this is multiplied with $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ first column this will get us $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if it is multiplied with $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ this should be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, this is what it actually does. Now, how do we get back that this is one way to do that.

Now, there is alternative way one can also handle the situation like let us see if you augment the system. So, we augment the system what do we have here $\left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$. So, augment the identity matrix here and we do row operation which is let us see R_3 -3R_1. So, here what we get $\left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$ then again, we do R_1 -3R_2. So, we get $\left[\begin{array}{cc:cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$, so which means, one can write here $[A \quad I] = [I \quad X]$. So, if I both side multiplied with A inverse, so this is

$$A^{-1}[A \quad I] = A^{-1}[I \quad X]$$

that would give us.

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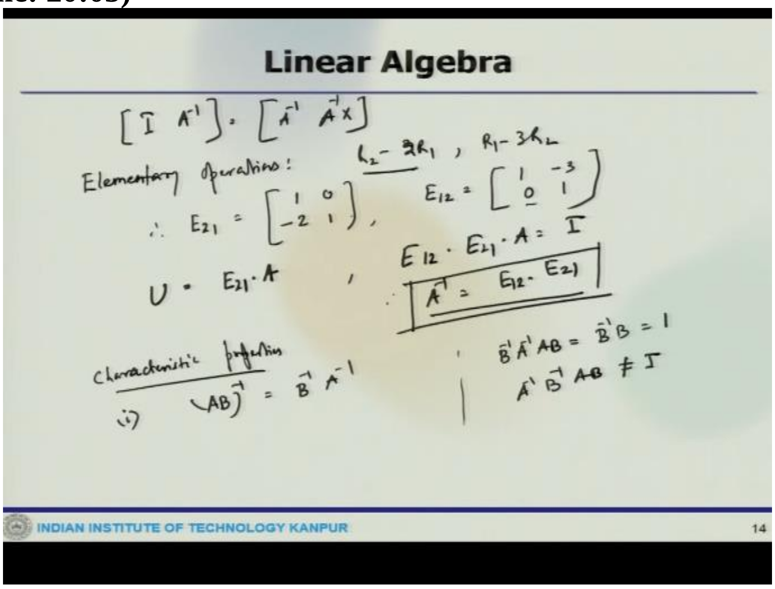

$$[I \quad A^{-1}] = [A^{-1} \quad A^{-1}X]$$

So, what we have got here this is the inverse of the system and you can see when we do the other way around it is the same. So, what are the elementary operations that has been carried out, the elementary operations which are R_2 - $2R_1$ and $R_1 – 3R_2$. So, this guy then my E_{21} would be $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and E_{12} would be $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$. So, these are my elementary matrices.

So, my upper triangular matrix U would be, $U = E_{21}A$. So, $E_{12}E_{21}A = I$, So, $A^{-1} = E_{12}E_{21}$. So, this is how you can also so, if you multiply the so, this is an important property and one can cross check here if you multiplied this is 1 2. So, this will give us A inverse. So, if you multiply the elementary matrices that also give you the inverse. Now, there are some characteristics properties which are there,

i) $(AB)^{-1} = B^{-1}A^{-1}$ which means $B^{-1}A^{-1}AB = I$.

And $A^{-1}B^{-1}AB \neq I$.

so provided A is invertible it can be denoted as product of lower triangular and upper triangular matrices. So, we can look at the same thing from the previous example.

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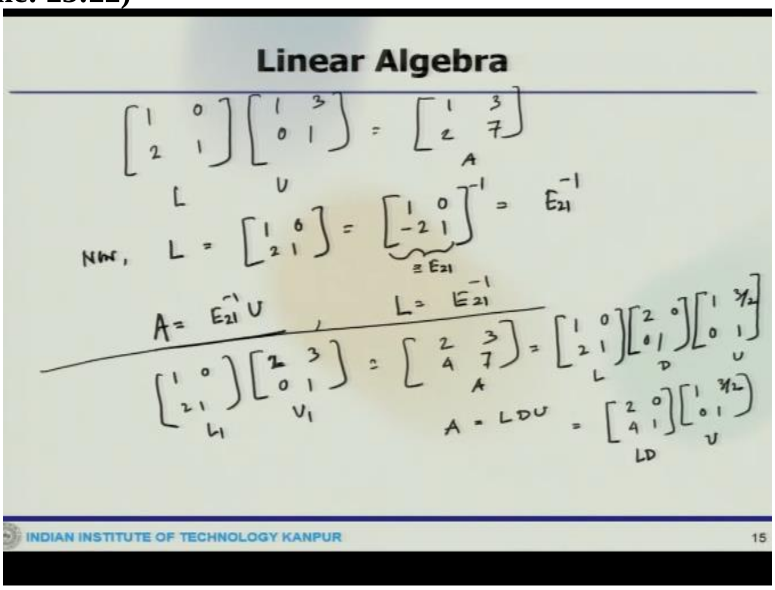

So, we had lower triangular matrix which is $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, so you can see that and then upper triangular is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ this is L this is U not only give us $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ which is A. Now, what is L?

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = {E_{21}}^{-1}$$

So, this is my elementary matrix E_{21}. So, I can write $A = {E_{21}}^{-1}U$. So, my L would be ${E_{21}}^{-1}$ is the lower triangular matrix.

Now, an invertible matrix can be decomposed into a lower triangular, diagonal matrix and upper triangular matrix. Let us say we have this guy like the same example here, we have

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

So, this should be lower triangular matrix, this is diagonal matrix and this is upper triangular matrix, so what essentially one is writing A is L D U. Now, here again little bit more algebra one can do which will get you $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$ this guy is L D, this is U. So, the conservative solutions they do not work for millions of unknowns. So, thus for those kinds of problems, these kinds of decompositions are highly required because we cannot do this kind of conservative approach.

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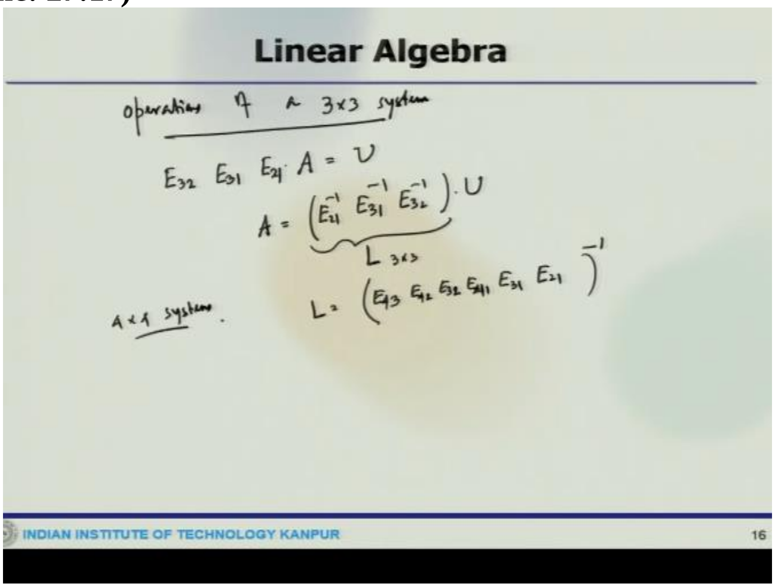

So, for example, if you look at the operations of a 3× 3 system, what you get? You get here is A then finally, lower triangular to upper triangular matrix. So, first thing would be elementary matrix of E_{21} then elementary matrix of E_{31} then elementary matrix of E_{32}. So, this is what is going to give us U. What it will give us A would be now, to find out A we get

$$A = ({E_{21}}^{-1}{E_{31}}^{-1}{E_{32}}^{-1})U$$

So, this one can think about is the lower triangular system for 3× 3 matrix system.

Now, if you have 4× 4 systems then our system then the lower triangular would be you can think about the lower triangular like first would be

$$L = (E_{43}E_{42}E_{32}E_{41}E_{31}E_{21})^{-1}$$

So, now, you see when there is an increase in the unknown so, the matrix sizes also go up and when the matrix sizes go up so, this lower and upper triangular decompositions are important.

So, the point here one has to note that when the system becomes really large or a number of unknowns are quite high, then the conservative decompositions actually is not a very viable solution. And either this kind of decomposition is through lower triangular and upper triangular matrix is quite handy and this is what is desired. And this exactly this situation you will see when we will be looking at the numerical point of view, how to decompose the matrix and get these systems and how to get actually the solution. So, we will stop the discussion here and continue the discussion in the next session.