Computational Science in Engineering

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Lecture - 20

Okay, so we have now looked at the ODE and just little bit of quickly we have touched upon some of the properties of Laplace and Fourier analysis because they are probably handy in solving ODEs. Now we are moving to the discussion on partial differential equation.

So far, we have looked at Eulerian differential equation, now we are going to look at the partial differential equation and once we have these discussions which are completed then we will move to the different techniques or the numerical ways of solving this kind of system, so the linear system or ODEs are like that.

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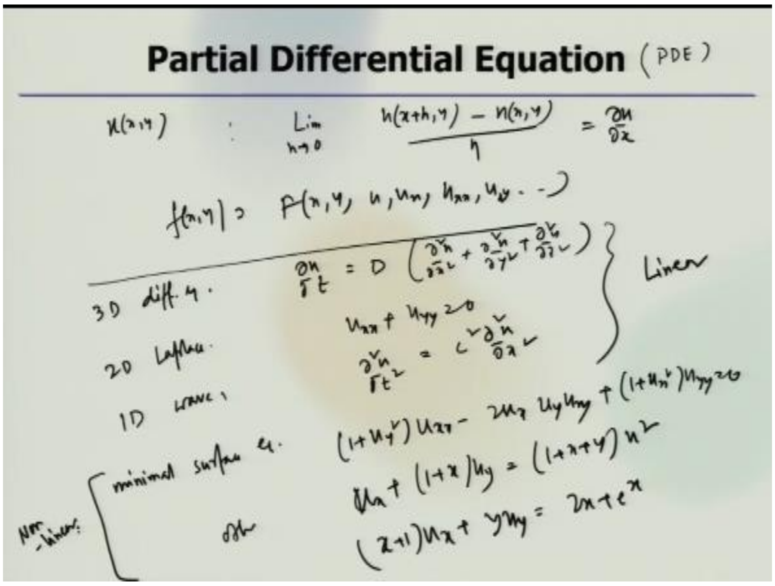

So how you define essentially let us say when you talk about this PDE, if you have a function which is $u(x, y)$ which has this partial derivative, so the basic function of the partial derivative which would be given as

$$\lim_{h\to 0}\frac{h(x+h,y)-h(x,y)}{h}$$

which is giving you the partial derivative of x. Now typically the partial derivative functions which are

$$f(x,y)=F(x,y,u,u_x,u_{xx},u_{xy},\dots)$$

and so on, okay.

So, this is the kind of. Now partial derivatives from the engineering problems like if somebody is solving solid mechanics problem, vibrational problem or fluid mechanics problem he will encounter these partial derivatives. Now for example if you have three-dimensional diffusion equation which looks like

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

So, this is 3D differential equation. Then you can have 2D Laplace equation which is

$$u_{xx} + u_{yy} = 0$$

Then one can have 1D wave equation, which is again

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

So, all these or one can have minimal surface equation which is

$$\left(1 + u_y^2\right)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$$

okay. Or some other function like some other functions which are let us say

$$u_x + (1 + x)u_y = (1 + x + y)u^2$$

Or one can have

$$(x + 1)u_x + yu_y = 2x + e^x$$

something like that. So, this set of again these equations what you look at here now they are going to be kind of these are linear system. These guys are all nonlinear partial differential equation. So again, like the Eulerian differential equation here also you have linear and nonlinear system.

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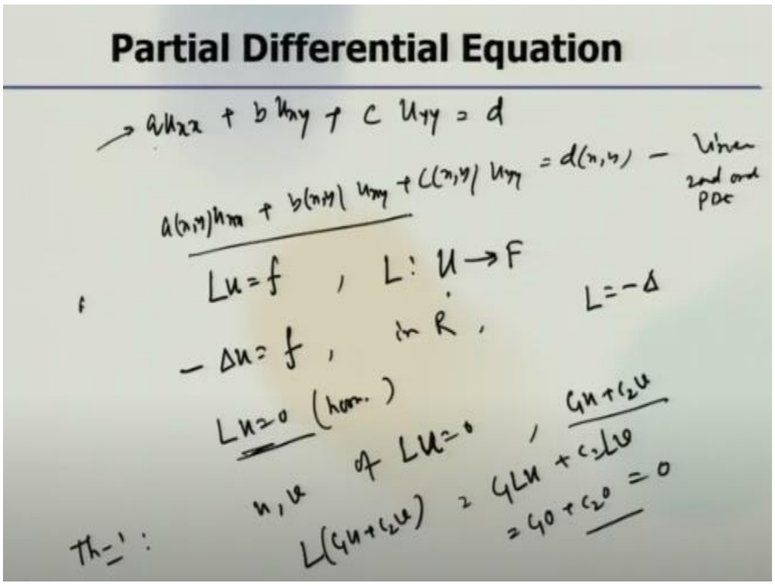

And typically, I mean a system in general class of equation one can write that

$$au_{xx} + bu_{xy} + cu_{yy} = d$$

This is a second order system. a, b, c, d are the basically some functions, continuous functions for variables and if these are depending on x and y only then these guys would become a linear second order system, which is

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y)$$

This is linear second order PDE. Now the here coefficients here do not depend on the derivatives u. So, you obtain a quasilinear second order PDE. And then from that we can characterize these things. And now in general a linear PDE which can be expressed in a form $Lu = f$ where L is u tend to F is a linear differential operator for a linear space. Now u of differential function of linear space. F is a continuation.

So, for example, one can say $-\Delta u = f$ where in R where L equals to $-\Delta$ which is a linear operator and if this is Lu equals to 0 this becomes the homogeneous linear system. So, this kind of system one can solve, which is very useful property nodes in the principle of superposition and we can solve like that. So there, what is the theorem there?

The theorem is that let u and v denote two solutions of a homogeneous PDE of kind $Lu = 0$. Then the any constant c_1 and c_2, $c_1u + c_2v$ is also a solution. So that means

$$L[c_1u + c_2v] = c_1Lu + c_2Lv$$

So, this also

$$c_1 0 + c_2 0 = 0$$

So that means, this linear combination is also a solution to this particular PDE.

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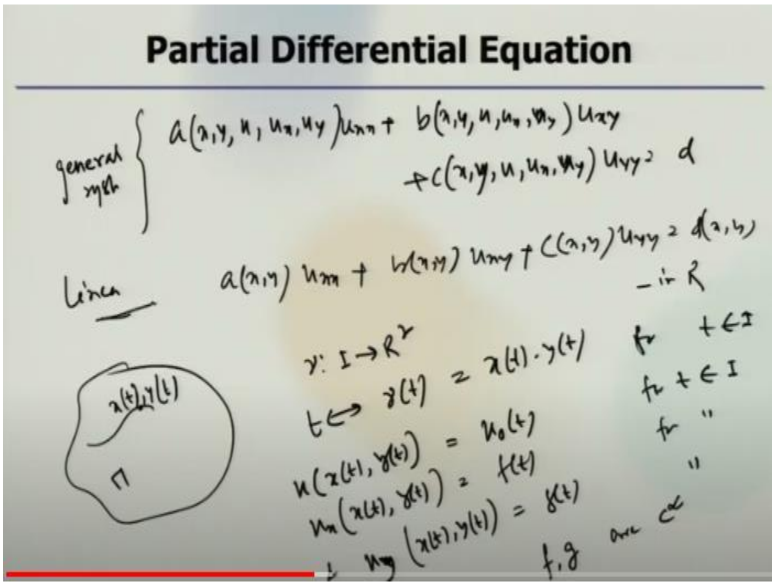

Now once you look at this particular curve, I mean like we have this in general we can write

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d$$

Now we can begin with a special case of the linear system. So, the linear system can be written as that. So, this is a generic general system. Now the linear system we can write

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y)$$

Now here a, b, c, d are continuous function defined on some interval R. Now the classification of this equations which are defined here based on the properties of the curve one R which is associated with this curve called the characteristics curve. Now we begin with a let us say the characteristics curve on γ.Now we begin with a curve in this is in R, so this is also in R parameters by a map.

So,

$$\gamma(t) = x(t)y(t)$$

for t belongs to I. Now I is some interval in real numbers like as shown here. So, suppose we are trying to solve this PDE linear one on this curve gamma specifically suppose we have given values of u and then, so we can specify these conditions on u like

$$u\big(x(t)y(t)\big) = u_0(t)$$

for t belongs to I. You can have

$$u_x\big(x(t)y(t)\big) = f(t)$$

That is also for t belongs to I and we have

$$u_y\big(x(t)y(t)\big) = g(t)$$

for I belongs to I. So here f, g are given continuous function on I. So, if we assume in addition that f, g are c^∞ functions we can also obtain the c infinity function then we can obtain the second derivative like u_{xx}, u_{xy} and u_{yy} and like that on this γ. So now we can attempt to construct a solution for this particular system here.

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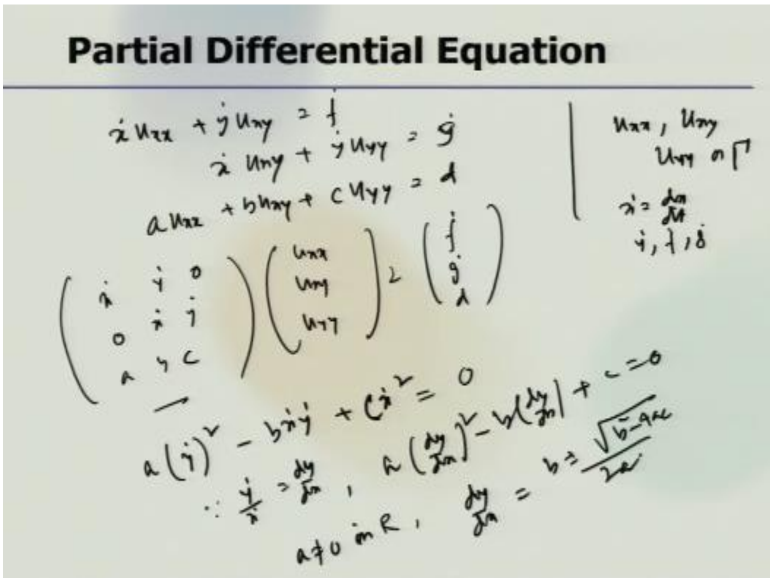

Now we will write like the first step in this construction taking the derivative and once we do that what we get

$$\dot{x}u_{xx} + \dot{y}u_{xy} = \dot{f}$$

Then we have

$$\dot{x}u_{xy} + \dot{y}u_{yy} = \dot{g}$$

$$au_{xx} + bu_{xy} + cu_{yy} = d$$

So here the unknowns are u_{xx}, u_{xy}, u_{yy} on this Γ. Or a dot on the top of the variable denotes the with respect to so here

$$\dot{x} = \frac{dx}{dt}$$

Similarly, $\dot{y}, \dot{f}$, $\dot{g}$ like that.

So, if we construct this, this is

$$\begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & y & c \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} \dot{f} \\ \dot{g} \\ d \end{pmatrix}$$

So, the matrix here can be solved for second derivative u in terms. And the determinant of this matrix which we can write that

$$a(\dot{y})^2 - b\dot{x}\dot{y} + c(\dot{x})^2 = 0$$

So now what we can write that since

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$$

we write

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

So, the assuming $a \neq 0$ on R or in R and we can solve for

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

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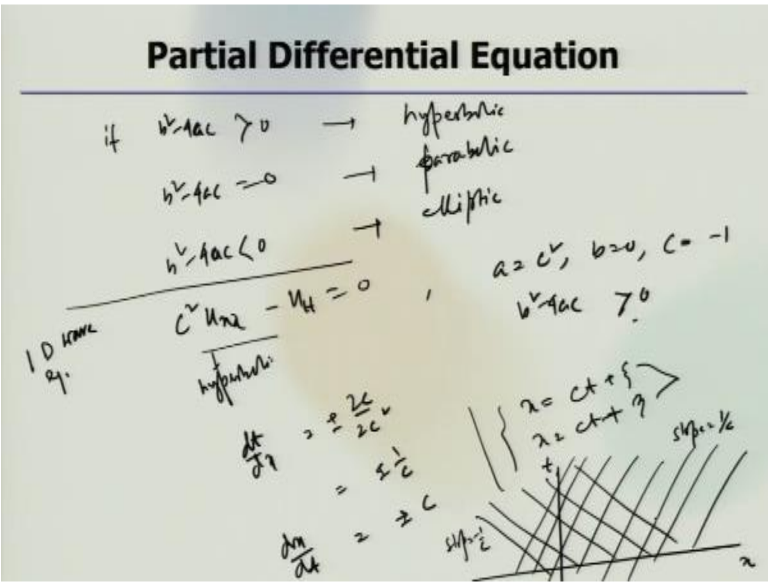

So, the possibilities are for this discriminant if $b^2 - 4ac$ greater than 0, then that PDE would be hyperbolic is nature if $b^2 - 4ac$ equals to 0, then this is a parabolic system. And if $b^2 - 4ac$ less than 0, this is an elliptic system, okay. So, we can see this taking the original wave equation, one dimensional wave equation, which is c square.

So 1D wave equation;

$$c^2 u_{xx} - u_{tt} = 0$$

So here the describing small amplitude vibration of a string now $a = c^2$ b is 0 and c is -1 in this case. So, you can see $b^2 - 4ac$ here greater than 0. So, this is hyperbolic system. So, in this PDE the equation for the characteristics curve would be

$$\frac{dt}{dx} = \pm\frac{2c}{2c^2} = \pm\frac{1}{c}$$

So, which one can write that

$$\frac{dx}{dt} = \pm c$$

So, the solution to this particular, there would be a solution, one solution would be

$$x = ct + \xi$$

Another would be

$$x = ct + \eta$$

ξ and η these two guys are here are the parameters for each of the families of characteristics curve, which is given through the solution.

So, if you look at that in a x-t plane like x and t plane, so this is how the, so characteristics curve of this particular hyperbolic system, they will look like in this kind of situation where the slope is always the slope is $1/c$. So, this family of characteristics curve described by this equation, they are parallel and the slope is $1/c$. So similarly, one can have now so this is slope is $1/c$.

Now one can have solution of the, so this is other side of the curve where the slope is negative then this would go like this, where the slope is $-1/c$. So, if you look at that, so that is how the system would look like.

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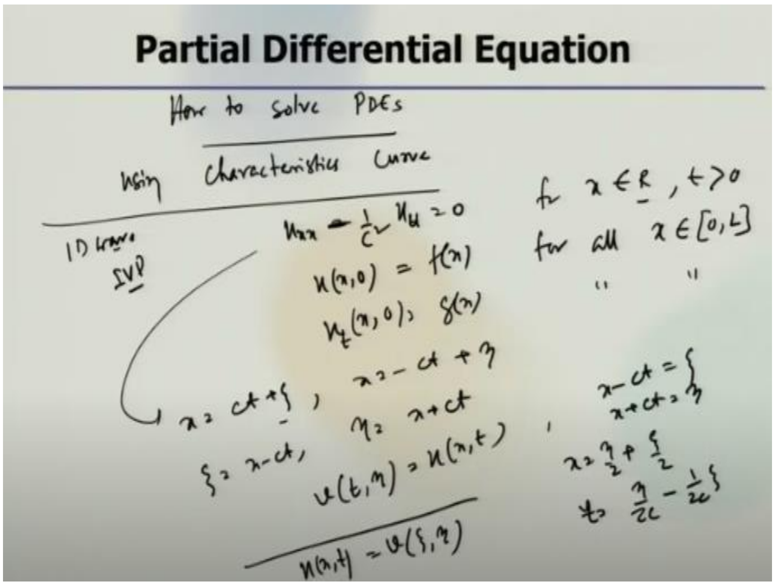

Now how would one solve the, so how these PDEs are solved? How to solve this PDEs? So, one best approach is that using characteristics curve like so that is probably the best way to get a solution for this kind of PDEs. Now we can look at this that only wave equation

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0$$

for x belongs to R and t greater than 0.

So, the initial value problem here, this is an 1D wave equation with initial value problem which is $f(x)$ for all x belongs to 0 to L. And $u_t(x, 0) = g(x)$ for all x belong to 0 to L. Now f and g are given continuous functions defined in R. Now already we have seen this kind of system has a solution. There would be two solutions. One is $ct + \xi$.

And another would be x is $-ct + \eta$. Now the families of curve consist of parallel line in the x-t plane, which will have a slope $1/c$ and $-1/c$. Now what we can do this by let us say considering this psi and eta parameter, now we set

$$\xi = x - ct$$

and

$$\eta = x + ct$$

Now you now given a solution u to the PDE here what you are there we can write, we can change the variables by using the set of variables like

$$v(t, \eta) = u(x, t)$$

and where x, t are obtained in terms of ξ and η. So, we will write

$$x = \frac{\eta}{2} + \frac{\xi}{2}$$

And

$$t = \frac{\eta}{2c} - \frac{\xi}{2c}$$

Or alternatively one can write like

$$u(x,t) = v(\xi, \eta)$$

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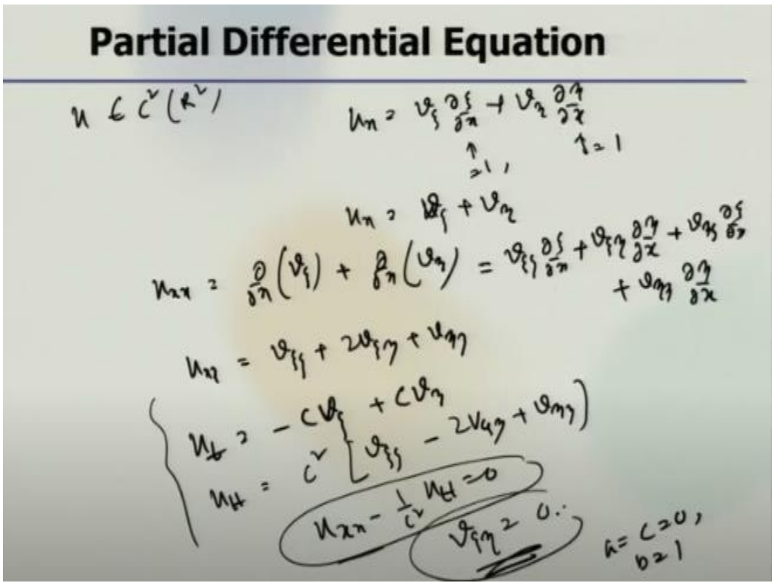

So now assuming u belongs to c square solve the PDE which is given here. We would like to derive the PDE, satisfy this v and v will satisfy this express term in terms of ξ *and* η. So, we will use the chain rule. So,

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x}$$

So, $\frac{\partial \xi}{\partial x}$ is 1 and $\frac{\partial \eta}{\partial x}$ is also 1. So, what we get

$$u_x = v_\xi + v_\eta$$

Now we take partial derivative both side with respect to x.

So, we get

$$u_{xx} = \frac{\partial}{\partial x}(v_\xi) + \frac{\partial}{\partial x}(v_\eta)$$

So, what we get

$$u_{xx} = v_{\xi\xi} \frac{\partial \xi}{\partial x} + v_{\xi\eta} \frac{\partial \eta}{\partial x} + v_{\eta\xi} \frac{\partial \xi}{\partial x} + v_{\eta\eta} \frac{\partial \eta}{\partial x}$$

Now already we have $\frac{\partial \xi}{\partial x}$ is 1 and $\frac{\partial \eta}{\partial x}$ is also 1. With that what we can write,

$$u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}$$

So similarly, we can find out for

$$u_t = -cv_\xi + cv_\eta$$

And

$$u_{tt} = c^2\{v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}\}$$

Now our original equation is

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0$$

So, when you replace back all this here, what we get $v_{\xi\eta} = 0$. So now this particular equation here, this is also hyperbolic second order linear PDE, in this case $a = c = 0$and b equals to 1. So, in contrast with the hyperbolic 3D here we can so this one instead of that this one can be directly solved.

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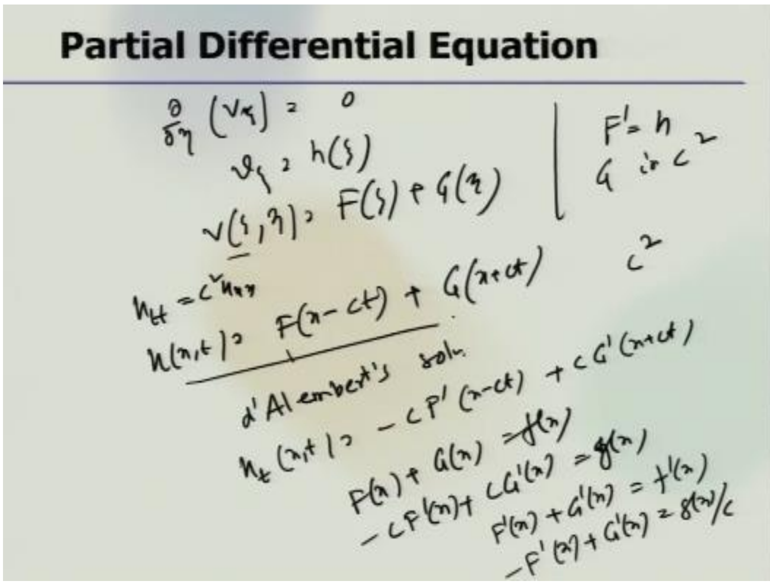

By writing like

$$\frac{\partial}{\partial \eta}(v_\xi) = 0$$

So, $v_\xi = h(\xi)$. So, what we get

$$v(\xi, \eta) = F(\xi) + G(\eta)$$

where f is antiderivative of h. So, $F' = h$ and g is an arbitrary function, arbitrary in C^2 function. Now the v is defined here. Now f and g are arbitrary in C^2 and so general solution we can write for one dimensional wave equation which is

$$u_{tt} = c^2 u_{xx}$$

$$u(x,t) = F(x - ct) + G(x + ct)$$

So, this F and G are the arbitrary C^2 functions of a single variable and this expression is known as d'Alembert's solution to 1D wave equation. Now we can use this general equation to 1D wave equations and now here we can take derivative like

$$u_t(x,t) = -cF'(x - ct) + cG'(x + ct)$$

and we can apply the general initial condition

$$F(x) + G(x) = f(x)$$

$$-\,cF'(x) + cG'(x) = g(x)$$

And we can take derivative again similarly, and write that

$$-\,F'(x) + G'(x) = \frac{g(x)}{c}$$

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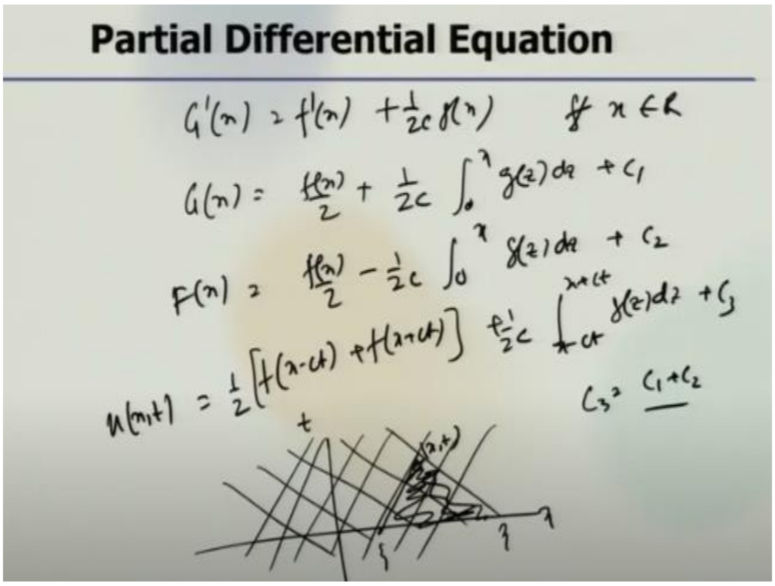

So now if we add these terms with the following, so we get

$$G'(x) = F'(x) + \frac{g(x)}{2c}$$

This is for all x belongs to R. So, integrating this equation what we get

$$G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(z)dz + c_1$$

Similarly, one can find

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(z)dz + c_2$$

So, what we will have the general solution for

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz + c_3$$

Now here the c_3 is another constant, which is taken as $c_3 = c_1 + c_2$. So, it follows the first initial conditions and so again if you see this curve in x-t plane, so there would be slope in these directions and the reverse slope in these directions. So, there would be a location which one can see. This would be ξ, η. Let us say this is η and this would be x-t solution.

So essentially, this is how the 1D wave equations gives a solution of in terms of two characteristics and the slope would go in two different directions. So, this is how one can look at this PDEs. So, we will continue the discussion on other type of PDEs in the next session. So, this is what we have looked at the hyperbolic system of the wave equation and where the characteristics go in two different directions with a different slope. So, we will continue the discussion for other PDEs in the next lecture.