Computational Science in Engineering
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Lecture - 21

Okay, so let us continue the discussion on partial differential equation. We have started just discussing about the PDEs and we have looked at the classification and then in the last session, we have looked at the 1D wave equation.

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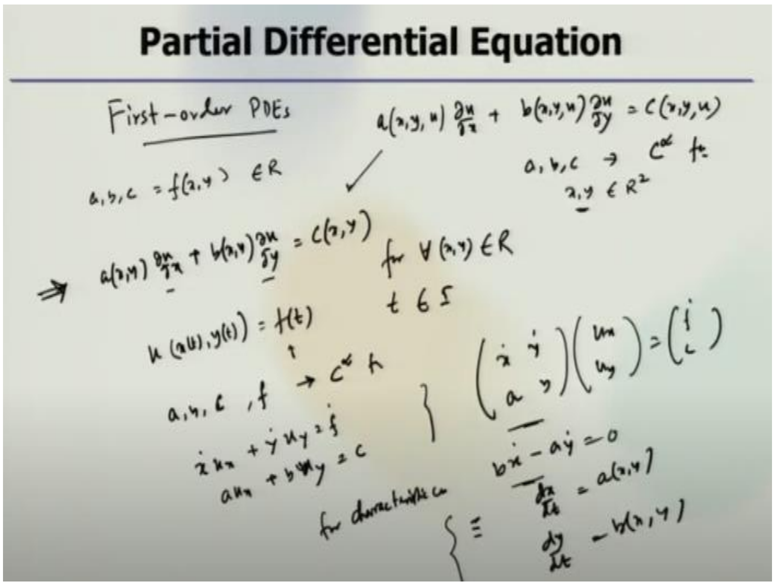

So now we are going to look at the first order PDE. So essentially how to solve that first order PDEs. Now in so the characteristics curve of this kind of PDE would look like

$$a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c(x,y,u)$$

So, this is the and here $a, b\ and\ c$ these are all C^{∞} function of three variable x, y, z, where x y lies in an open region. So, x, y belongs to R^2 in which coefficient function $a, b\ and\ c$ depend only on x, y.

So let us say if this guy a, b, c they become function of x and y which belongs to R then this particular ODE will bring down to

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y)$$

for all x, y which belongs to R. So first we define the concept of characteristics curve for this PDE what we have written here and then the discussion would be similar to the second order equation.

So let us say suppose we try to solve this PDE here and to an initial condition and they are given like

$$u\big(x(t), y(t)\big) = f(t)$$

where t belongs to I. Now so f is a known smooth function defined on I interval. So here the idea is that the given information, we can use that information together with the PDE to obtain the values of the derivative like u_x and u_y on gamma.

So now we have already defined these a, b, c functions, since we are assuming this a, b, c and the initial data f, these are all C infinity function. So, then what we can write that

$$\dot{x}u_x + \dot{y}u_y = \dot{f}$$

$$au_x + bu_y = c$$

So, if you put it that into a matrix form, so

$$\begin{pmatrix} \dot{x} & \dot{y} \\ a & b \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \dot{f} \\ c \end{pmatrix}$$

So, the determinant of this matrix would be

$$b\dot{x} - a\dot{y} = 0$$

to get the characteristics curve, for characteristics curve. Now so if you see this guy here, so these are ODEs here, which is equivalent to writing like

$$\frac{dx}{dt} = a(x, y)$$

and

$$\frac{dy}{dt} = b(x, y)$$

So, these are sort of a system of ODE that you can think about from here.

The system ODE is defined the characteristics curve for the first order linear PDE. So, these are going to define the system.

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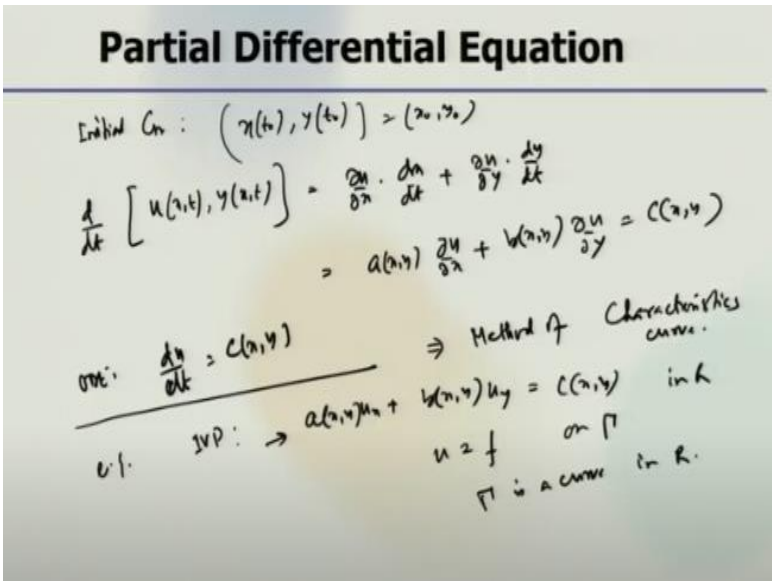

Now also we have the initial condition like

$$\big(x(t_0), y(t_0)\big) = (x_0, y_0)$$

So, the characteristics curve always can be found out from this. Now what we do? Let us say suppose we have computed the characteristics curve for this PDE that is given here and according to the system of this ODE system of ODEs which is represented by this set of curves.

Now what we can do we can write down the chain rule and expand this like

$$\frac{d}{dt}\,[u(x,t), y(x,t)] = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

So now what we get? So, using the definition of characteristics curve this we can write

$$\frac{d}{dt}\,[u(x,t), y(x,t)] = a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)$$

along the characteristics curve. Now this suggests a way to construct a solution for the initial value problem.

So, and this is known as method of characteristics curve, okay. So, this is how using method of characteristics you can solve. For example, let us say you want to solve an initial value problem where

$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$

in R and $u = f$. Then gamma Γ is a curve in R. That is not a characteristic curve.

The method of characteristic curve consists of first finding the characteristics curve of the PDE given here by solving the system of ODEs that we get here. So essentially, when you have a first order PDE with initial value problem, we can bring down to the system of ODEs and then that would form the characteristics curve. So, what we can see and quick let us say an example.

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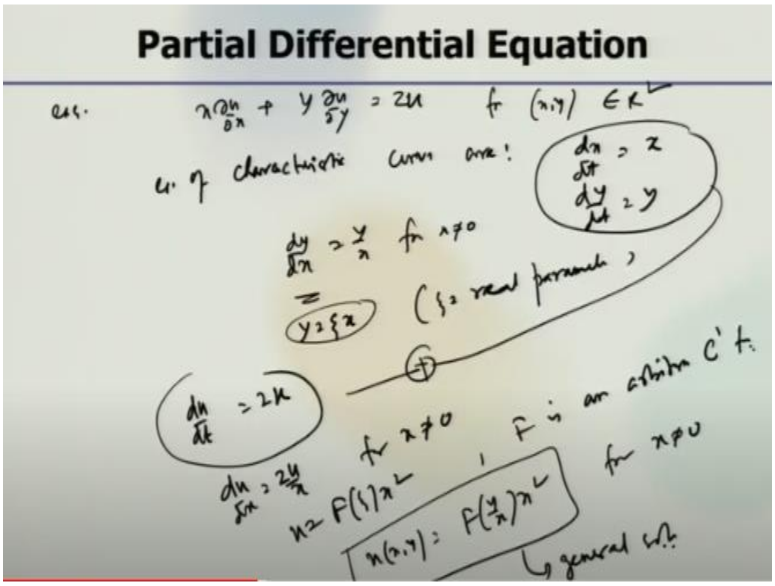

Let us say numerical example, we can find the general solution of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

for x and y which belongs to R^2. So, the equations of characteristics curves are

$$\frac{dx}{dt} = x$$

and

$$\frac{dy}{dt} = y$$

So, using chain rule, what we obtain that

$$\frac{dy}{dx} = \frac{y}{x}$$

for $x \neq 0$. So, this can be solved by separation of variables. So let us say

$$y = \xi x$$

and where psi ξ is a real parameter.

The characteristic curve for the PDE is a pencil of straight lines through the origin in R^2. So, along the characteristics curve of the given PDE here you solve the ODE like

$$\frac{du}{dt} = 2u$$

So now we combine this ODE with the first order ODE given here. So, these two if we combine then what we get that

$$\frac{du}{dx} = \frac{2u}{x}$$

for $x \neq 0$. So now this one if we solve, we get

$$u = F(\xi)x^2$$

So, F is an arbitrary function or arbitrary C^1 function and ξ is already given. So, solving for ξ if we solve for ξ from here and substitute here what we will,

$$u(x, y) = F\left(\frac{y}{x}\right)x^2$$

for $x \neq 0$. So, this is the general solution of this particular PDE.

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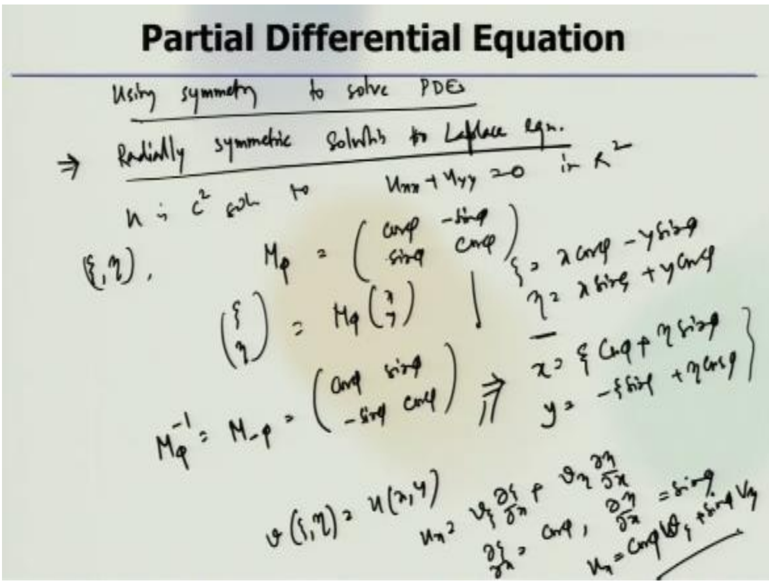

Now we move to the, we can use using symmetry to solve PDEs. So partial differential equation is said to be invariant under a group of transformation and it form does not change after changing variables according to the transformation in the group. So, we will look at the first kind of example of radially symmetric solution to Laplace equation.

So, suppose that u is an C^2 function C^2 solution to Laplace equation of

$$u_{xx} + u_{yy} = 0$$

in R^2. So, we consider that what happens to this particular equation when we change a new variable to ξ and η. So, like given by one parametric group of rotation the matrix is given

$$M_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

So, this is a sort of a rotation matrix. So, the rotations are counterclockwise by angle ϕ.

So, what we set

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = M_\phi \begin{pmatrix} x \\ y \end{pmatrix}$$

where we get

$$\xi = x\cos\phi - y\sin\phi$$

$$\eta = x\sin\phi + y\cos\phi$$

So now this particular equation, this can be now solved for x and y in terms of ξ and η by inverting the matrix or we can write

$$M_\phi{}^{-1} = M_{-\phi} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

So that what we get

$$x = \xi\cos\phi + \eta\sin\phi$$

$$y = -\xi\sin\phi + \eta\cos\phi$$

Now we think of u as a function of ξ and η by looking at this and which we will denote by let us say,

$$v(\xi,\eta) = u(x,y)$$

Now we can apply chain rule. So, what we get

$$u_x = v_\xi \frac{\partial\xi}{\partial x} + v_\eta \frac{\partial\eta}{\partial x}$$

And

$$\frac{\partial\xi}{\partial x} = \cos\phi$$

$$\frac{\partial\eta}{\partial x} = \sin\phi$$

So, what we get

$$u_x = \cos\phi\, v_\xi + \sin\phi\, v_\eta$$

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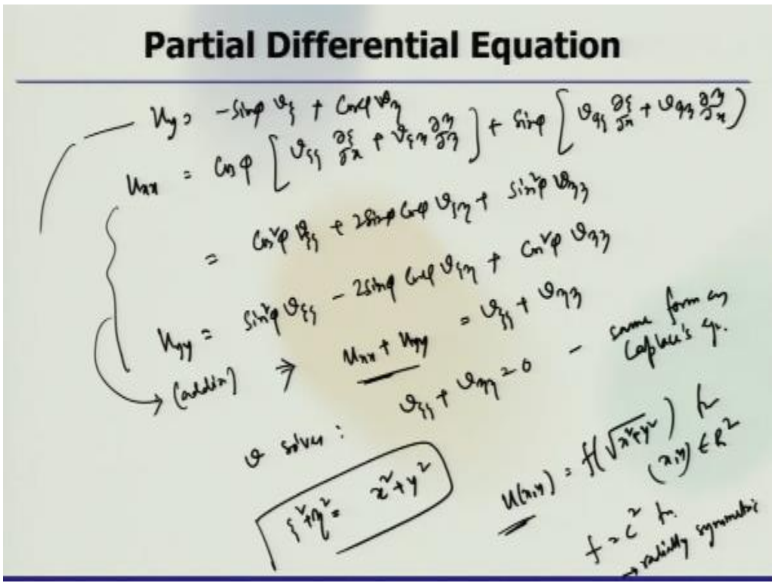

So similarly, we can calculate and we get

$$u_x = -\sin\phi\, v_\xi + \cos\phi\, v_\eta$$

So now we differentiate both side of this equation and what we get

$$u_{xx} = \cos\phi\left[v_{\xi\xi}\,\frac{\partial\xi}{\partial x} + v_{\xi\eta}\,\frac{\partial\eta}{\partial x}\right] + \sin\phi\left[v_{\eta\xi}\,\frac{\partial\xi}{\partial x} + v_{\eta\eta}\,\frac{\partial\eta}{\partial x}\right]$$

So, this will get us. So also, the mix second partial derivatives are C^2 functions.

So, this will give us

$$u_{xx} = \cos^2\phi\, v_{\xi\xi} + 2\sin\phi\cos\phi\, v_{\xi\eta} + \sin^2\phi\, v_{\eta\eta}$$

Now similarly we can take the partial derivative of this guy. And what do we get,

$$u_{yy} = \sin^2\phi\, v_{\xi\xi} - 2\sin\phi\cos\phi\, v_{\xi\eta} + \cos^2\phi\, v_{\eta\eta}$$

Now if we add these two guys together so what we get

$$u_{xx} + u_{yy} = v_{\xi\xi} + v_{\eta\eta}$$

So hence if you solve the Laplace equations here the original Laplace equation

$$u_{xx} + u_{yy} = 0$$

then

$$v_{\xi\xi} + v_{\eta\eta} = 0$$

which has the same form same form as Laplace equation. And is invariant under rotation, okay.

So, this suggests that we look for solution of the original Laplace equation

$$u_{xx} + u_{yy} = 0$$

that functions of a combination of the independent variable that is independent of the rotation parameter p. So, to obtain such combination we can write like

$$\xi^2 + \eta^2 = x^2 + y^2$$

So, the rotation parameter that is they are rotationally invariant. So, we therefore look for a solution of the Laplace equation in a form like

$$u(x, y) = f\left(\sqrt{x^2 + y^2}\right)$$

for x and y belongs to R^2 where f is a C^2 function of a single variable. And this particular solution is radially, so this is radially symmetric, okay.

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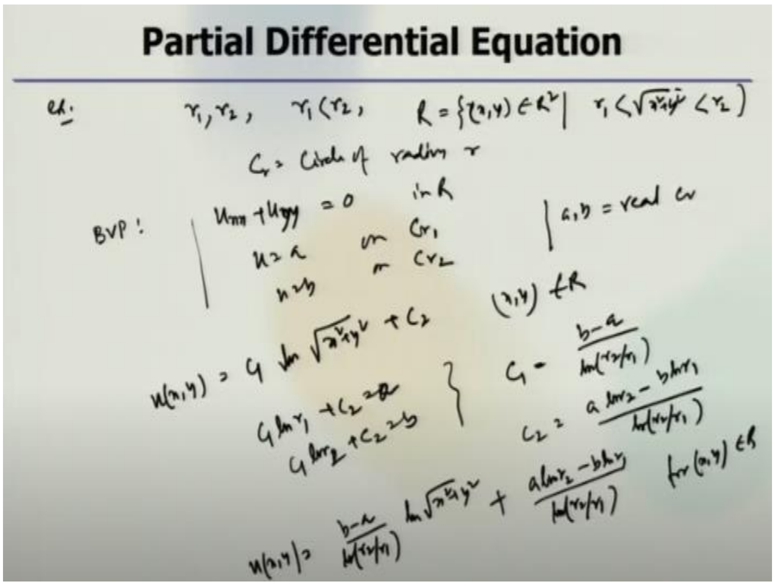

So now we can look at a quick example of that. Like let us say we have a problem, Dirichlet problem on annulus. So, we have some positive numbers r_1 *and* r_2 where $r_1 < r_2$. So, we define R which is x and y belongs to R^2 where $r_1 < \sqrt{x^2 + y^2} < r_2$. So, if C_r is the radius of the circle, the center at the origin. So, C_r is the circle of radius r.

Then we want to solve this boundary value problem of

$$u_{xx} + u_{yy} = 0$$

in R where $u = a$ on C_{r_1}, $u = b$ on C_{r_2}C r 2; a, b are real constant. So how we go about it? We can have a solution in this form

$$u(x, y) = C_1 \ln \sqrt{x^2 + y^2} + C_2$$

for x, y belongs to R. So, C_1 and C_2 some constant. So, the boundary conditions imply that

$$C_1 \ln r_1 + C_2 = a$$
$$C_1 \ln r_2 + C_2 = b$$

So, solving this what we get;

$$C_1 = \frac{b - a}{\ln \frac{r_2}{r_1}}$$

$$C_1 = \frac{a \ln r_2 - b \ \ln r_1}{\ln \frac{r_2}{r_1}}$$

So, once we substitute these values so our solution would be in the form of

$$u(x, y) = \frac{b - a}{\ln \frac{r_2}{r_1}} \ln \sqrt{x^2 + y^2} + \frac{a \ln r_2 - b \ \ln r_1}{\ln \frac{r_2}{r_1}}$$

where x and y belongs to R. So, this is the solution for the problem given here, the boundary value problem, okay.

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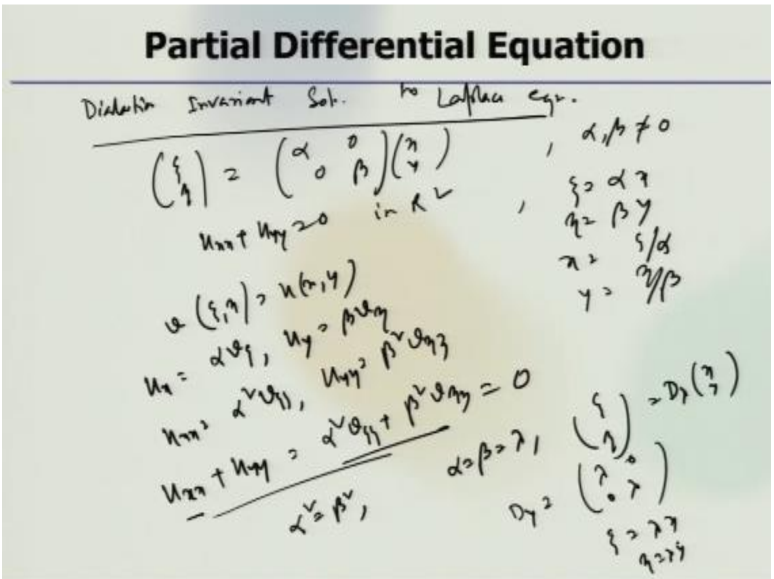

So now look at the second like the dilation invariant solution, dilation invariant solution to Laplace equation. So, like so you can define the so you can see the change of variable like

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So, for nonzero coefficients of α, β which are nonzero the two dimensional Laplace equation is given as

$$u_{xx} + u_{yy} = 0$$

in R^2. So, the change of variable where,

$$\xi = \alpha x$$
$$\eta = \beta y$$

So,

$$x = \frac{\xi}{\alpha}$$
$$y = \frac{\eta}{\beta}$$

So, we set

$$v(\xi, \eta) = u(x, y)$$

So then again, we can write like we can take the derivative and we can write that

$$u_x = \alpha v_\xi$$
$$u_y = \beta v_\eta$$

and

$$u_{xx} = \alpha^2 v_{\xi\xi}$$
$$u_{yy} = \beta^2 v_{\eta\eta}$$

So,

$$u_{xx} + u_{yy} = \alpha^2 v_{\xi\xi} + \beta^2 v_{\eta\eta} = 0$$

If you solve the Laplace equation this then we also solve this. So now that also is in R^2 and invariant under the scaling transformation provided $\alpha^2 = \beta^2$ with therefore set $\alpha = \beta = \lambda$. So,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = D_\lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

And

$$D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

So, what we get that

$$\xi = \lambda x$$
$$\eta = \lambda y$$

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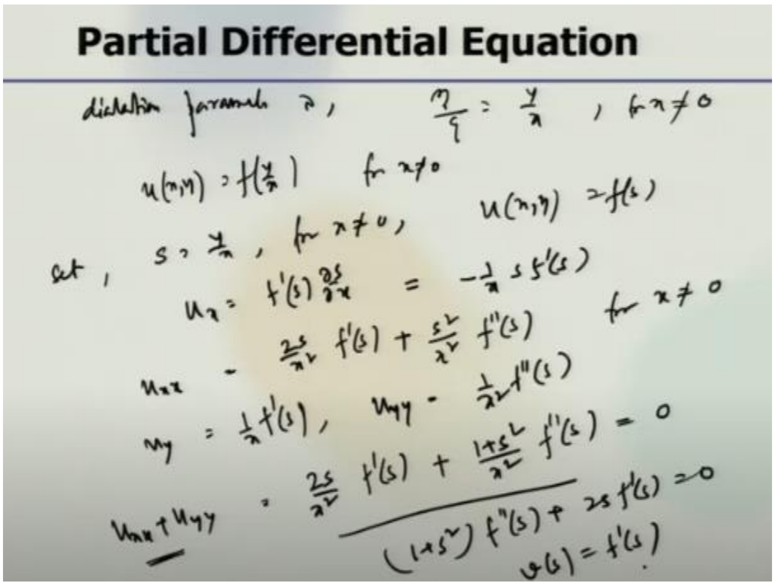

If a combination of this variable is independent of the dilation parameter lambda, this is a dilation parameter lambda, then we get

$$\frac{\eta}{\xi} = \frac{y}{x}$$

for $x \neq 0$. So, this suggests that we look for a solution to the Laplace equation in R^2 of the form

$$u(x, y) = f\left(\frac{y}{x}\right)$$

for $x \neq 0$. So let us set

$$s = \frac{y}{x}$$

for $x \neq 0$. So, we get

$$u(x, y) = f(s)$$

and where s is given like $\frac{y}{x}$.

Now we look for the solution to the Laplace equation in terms of this

$$u(x, y) = f(s)$$

So, what we get

$$u_x = f'(s)\,\frac{\partial s}{\partial x}$$

which would be

$$u_x = f'(s)\,\frac{\partial s}{\partial x} = -\frac{1}{x}\,s\,f'(s)$$

Similarly, we can get

$$u_{xx} = \frac{2s}{x^2}\ f'(s) + \frac{s^2}{x^2}\ f''(s)$$

for $x \neq 0$. Now similarly we get for

$$u_y = \frac{1}{x}\ f'(s)$$

and

$$u_{yy} = \frac{1}{x^2}\ f''(s)$$

So, this would give us

$$\frac{2s}{x^2}\ f'(s) + \frac{1+s^2}{x^2}\ f''(s) = 0$$

Now if you solve the Laplace equation, this equals to 0 in R^2, then f also follows this equation and this should be then 0. So, which one can write

$$(1+s^2)f''(s) + 2s\, f'(s) = 0$$

So, this is an ODE, second order ODE and we can solve this by setting that

$$v(s) = f'(s)$$

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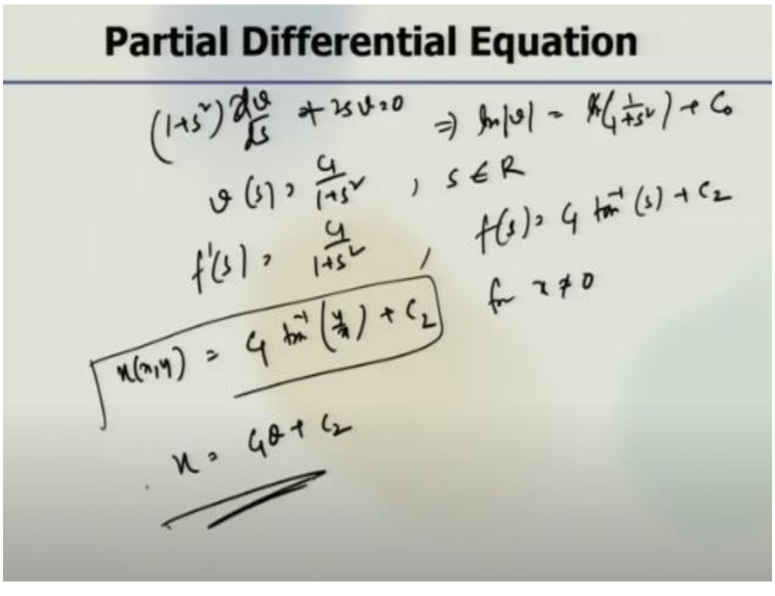

So, we get

$$(1+s^2)\frac{dv}{ds} + 2s\, v = 0$$

So, which gives us a solution

$$\ln|v| = \ln\left(\frac{1}{1+s^2}\right) + C_0$$

So that is a constant. So, our

$$v(s) = \frac{C_1}{1+s^2}$$

for s belongs to R. So,

$$f'(s) = \frac{C_1}{1+s^2}$$

So, we can integrate this to get a solution for

$$f(s) = C_1 \tan^{-1}(s) + C_2$$

Now all these things if we couple then the solution of the Laplace equation would be

$$u(x,y) = C_1 \tan^{-1}(s) + C_2$$

for $x \neq 0$. So, C_1 and C_2 are constant. So, this is this shows that the dilation invariant harmonic function in R^2 is linear function of angle theta. The point x and y mix with the positive axis so which is essentially like

$$u = C_1\theta + C_2$$

This is what you can see. Now similarly, one can find out the dilation invariant solution.

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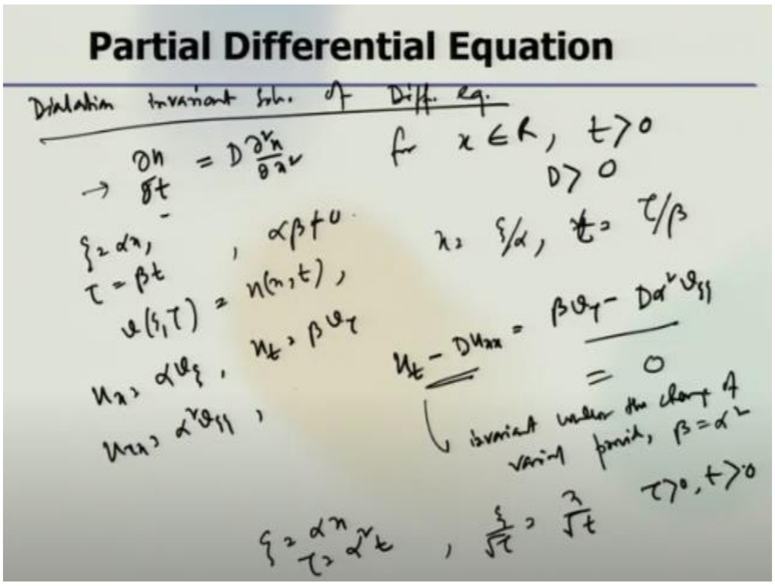

Dilation invariant solution of diffusion equation. So, the diffusion equation is given by

$$\frac{\partial u}{\partial t} = D\ \frac{\partial^2 u}{\partial x^2}$$

for x belongs to R, t greater than 0 where D is also greater than zero which is a diffusive coefficient. So now we can find the conditions for parameter $\alpha\ and\ \beta$ such that

$$\xi = \alpha x$$

and we can say

$$\tau = \beta t$$

and alpha beta not equals to 0.

So, what we can write that

$$v(\xi, \tau) = u(x, t)$$

So,

$$x = \frac{\xi}{\alpha}$$

and

$$t = \frac{\tau}{\beta}$$

So, what we can get that we can take the derivative and we get

$$u_x = \alpha v_\xi$$

and

$$u_t = \beta v_\tau$$

And

$$u_{xx} = \alpha^2 v_{\xi\xi}$$

And so, what we can write

$$u_t - Du_{xx} = \beta v_\tau - D\alpha^2 v_{\xi\xi}$$

So, if u solves this diffusion equation, then we solve this equation, so this would be 0. So here so this diffusion equation given here or this one invariant under the change of variable provided. So, this is invariant under the change of variable provided $\beta = \alpha^2$. So now what we have that following this we have

$$\xi = \alpha x$$

$$\tau = \alpha^2 t$$

So that means

$$\frac{\xi}{\sqrt{\tau}} = \frac{x}{\sqrt{t}}$$

for τ greater than 0 and t greater than 0.

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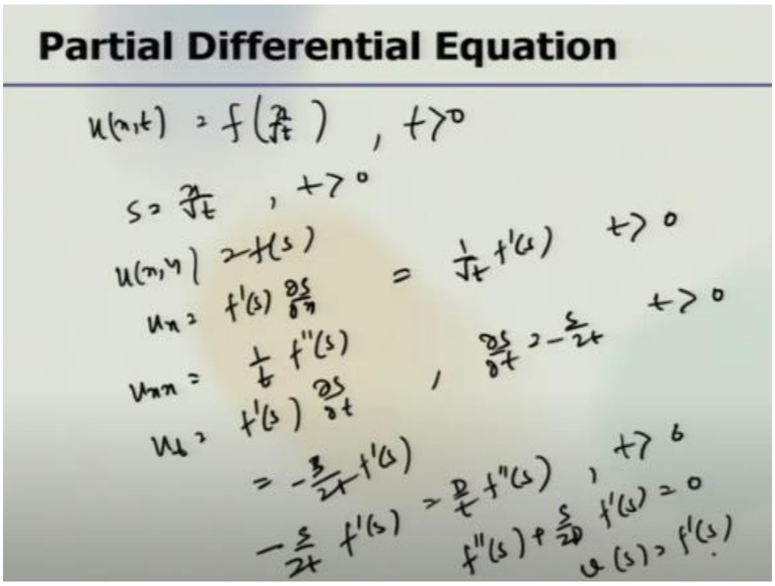

So, in order to find the dilation invariant solution of this one diffusion equation, our solution we look for a solution would be

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right)$$

where t greater than 0 and f is a C^2 function of single variable, so

$$s = \frac{x}{\sqrt{t}}$$

for t greater than 0. So,

$$u(x,t) = f(s)$$

Now what we can take derivative like previously we have done on both sides. We get

$$u_x = f'(s)\frac{\partial s}{\partial x} = \frac{1}{\sqrt{t}}f'(s)$$

for t greater than 0. And

$$u_{xx} = \frac{1}{t}f''(s)$$

And so, and

$$u_t = f'(s)\frac{\partial s}{\partial t}$$

Where,

$$\frac{\partial s}{\partial t} = -\frac{s}{2t}$$

where t greater than 0. So, this would become

$$u_t = -\frac{s}{2t}f'(s)$$

So now if we put back everything together then we get

$$-\frac{s}{2t}f'(s) = \frac{D}{t}f''(s)$$

for all t greater than 0. So, this will bring down to a second order again ODE like this.

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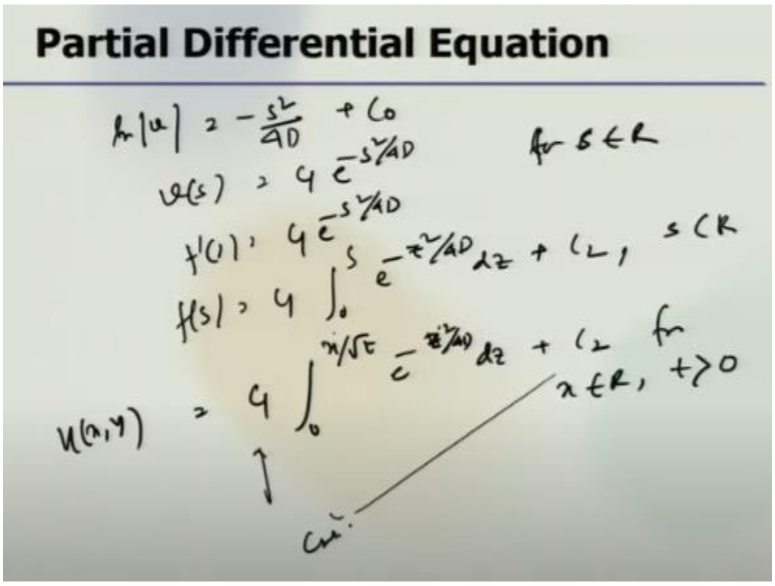

And when we do the solution of this system, so finally we get

$$\ln|v| = -\frac{s^2}{4D} + C_0$$

C_0 is constant. Now exponentiate both sides, what we get is that

$$v(s) = C_1 e^{-\frac{s^2}{4D}}$$

for s belongs to R and

$$f'(s) = C_1 e^{-\frac{s^2}{4D}}$$

So, our

$$f(s) = C_1 \int_0^s e^{-\frac{s^2}{4D}} dz + C_2$$

for s belongs to R. So, where C_1, C_2 are constant.

Now so final solution if we put everything together back this would be

$$u(x,y) = C_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4D}} dz + C_2$$

for x belongs to R, t greater than 0. And C_1 and C_2 are constant. So, this is what you get as a dilation invariant solution. So, we stop the discussion here and continue to look at the other things in the next session.