Computational Science in Engineering

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Lecture - 22

So we have continued the discussion on the PDEs and now we are going to look at the solution of the first order PDE.

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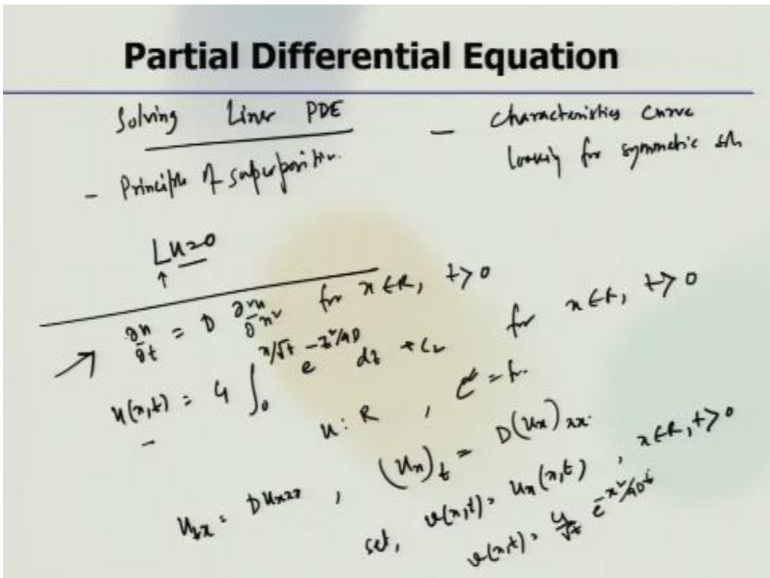

So, what we do now we do the solving linear PDE, okay. And we can see how we can. So, all the previous discussion where we have seen that there are two general approaches for finding solution to the first or the second order PDEs either using the characteristics curve that is one or looking for symmetric solution, okay. But in theory, these methods could be applied to nonlinear or linear equation.

So here we will try to explore methods that exploit the special structure provided by the linear PDEs, okay. So, like we will look at here principle of superposition. So, which is the linear combination of the system like is a linear differential operator, L is a linear differential operator. So, in principle, we can use super position to construct the solution of this linear PDEs.

So, I will see how we can do that, and then we can find out this solution to that kind of, fundamental solution to this. So let us, to start with let us again start with the diffusion equation, where you have an equation of this kind

$$\frac{\partial u}{\partial t} = D\ \frac{\partial^2 u}{\partial x^2}$$

for x belongs to R and t greater than 0.

So, this is a very special solution to this and the solution that we have already dilation invariant solution that we have already obtained, which is

$$u(x,t) = C_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{z^2}{4D}}\, dz + C_2$$

for x belongs to R and t greater than 0; C_1 and C_2 is constant. Now we can observe that the function u is in R. So, which is essentially a C^∞ function.

So, it then follows the fundamental theorem of calculus, the chain rule and then what we get that

$$u_{tx} = Du_{xxx}$$

By the equality of the mixed derivative, we can write

$$(u_x)_t = D(u_x)_{xx}$$

which shows that u_x is also a solution to that one dimensional diffusion equation that is given here. Now taking the partial derivative with respect to x we obtain another solution to the one dimensional diffusion equation.

So let us set

$$v(x,t) = u_x(x,t)$$

where x belongs to R and t is positive. So, u is already given. So using the fundamental theorem of calculus and chain rule, what we can write that

$$v(x,t) = \frac{C_1}{\sqrt{t}} e^{-\frac{x^2}{4Dt}}$$

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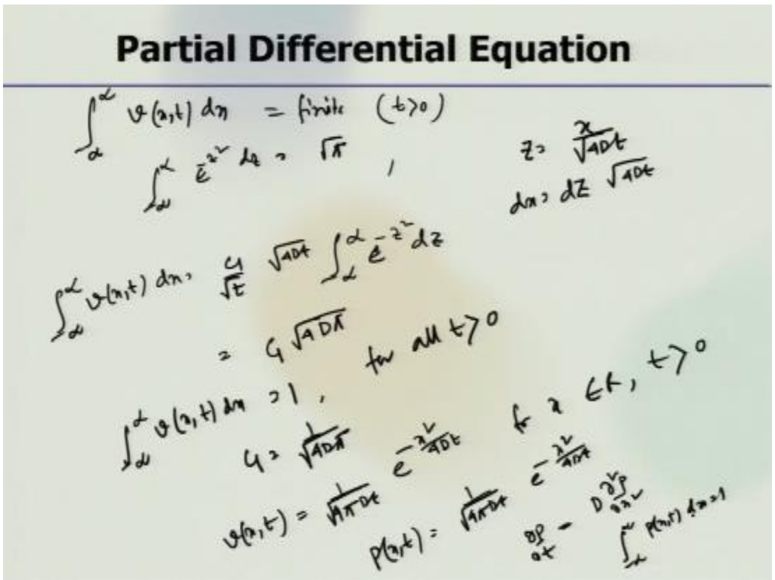

Now and some interesting property of the function, which is given here that $v(x,t)$ is that this integral of this like

$$\int_{-\infty}^{\infty} v(x,t)\,dx = finite$$

okay. And for obviously t greater than 0. And indeed, this integration that

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

So now here we make the change of variable like

$$z = \frac{x}{\sqrt{4Dt}}$$

So

$$dx = dz\sqrt{4Dt}$$

So, what we get that

$$\int_{-\infty}^{\infty} v(x,t)\,dx = \frac{C_1}{\sqrt{t}}\sqrt{4Dt}\int_{-\infty}^{\infty} e^{-z^2} dz = C_1\sqrt{4D\pi}$$

So, we can choose the constants C_1 so that in such a way that this integration becomes $\int_{-\infty}^{\infty} v(x,t)\,dx = 1$ for all t greater than 0. So then or

$$C_1 = \frac{1}{\sqrt{4D\pi}}$$

Now we substitute this value in the $v(x,t)$ expression and what we get

$$v(x,t) = \frac{1}{\sqrt{4D\pi}} e^{-\frac{x^2}{4Dt}}$$

which is for x belongs to R and t greater than 0.

Now we can express this

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

So here p also defined as a C^{∞} function. So, what we get

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}$$

And it follows that

$$\int_{-\infty}^{\infty} p(x,t)\, dx = 1$$

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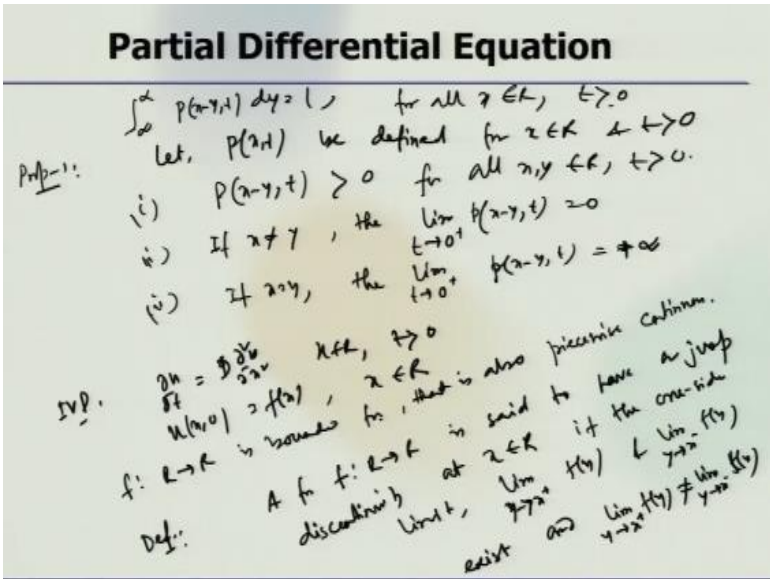

And also,

$$\int_{-\infty}^{\infty} p(x-y,t)\, dy = 1$$

for all x belongs to R and t greater than 0. So, in addition to this, P will have some other properties like we can have some propositions let us say one, so this is the properties of P. So let $p(x,t)$ be defined like this where for x belongs to R t greater than 0. So $p(x,t)$ defined for x R and t greater than 0, then $p(x-y,t)$ is greater than 0 for all x, y belongs to R and t greater than 0.

Second if $x \neq y$ then the

$$\lim_{t\to 0^+} p(x-y,t) = 0$$

If $x = y$ then

$$\lim_{t\to 0^-} p(x-y,t) = +\infty$$

So now here we will show how to use this properties or this proposition to get the solution to the initial value problem of

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

where x belongs to R and t greater than 0. And

$$u(x,0) = f(x)$$

where x belongs to R.

And f is a bounded function that is also piecewise continuous, okay. So, there must be some definition which are associated with that for piecewise a function f which is said to have a jump discontinuity at x belongs to R if the one sided limits like

$$\lim_{y \to x^+} f(y)$$

and

$$\lim_{y \to x^-} f(y)$$

one sided derivative exists and this limit

$$\lim_{y \to x^+} f(y) \neq \lim_{y \to x^-} f(y)$$

So, with that f is a piecewise continuous function, if it is a continuous except at an most constable number if has jump discontinuities.

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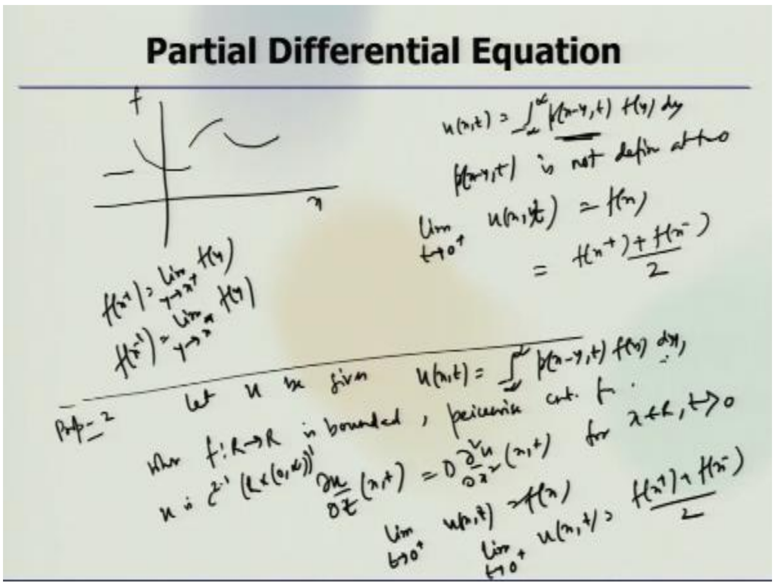

So, like one can see some of this like if an x like this guy can come like this, then like this, like this, this like this. So, this is a sketch of

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)\, f(y) dy$$

is a candidate for a solution of the initial value problem that is given. So we have to note that $p(x-y,t)$ is not defined at t equals to 0. So, we have that limit

$$\lim_{t\to 0^+} u(x,t) = f(x) = \frac{f(x^+) + f(x^-)}{2}$$

for f has a jump discontinuity, where $f(x^+)$ is given that $\lim_{y\to x^+} f(y)$. And $f(x^-)$ is given that $\lim_{y\to x^-} f(y)$. So, this is an important definition that we will have. Now that would also follow some subsequent propositions, let us say proposition number 2. Let u be given like this

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)\, f(y) dy$$

u is given like that where f is bounded and piecewise continuous function. Then u is $C^{2.1}(R \times (0,\infty))$.

$$\frac{\partial u}{\partial t}(x,t) = D\, \frac{\partial^2 u}{\partial x^2}(x,t)$$

for x belongs to R, t greater than 0. Furthermore, we will have the limit

$$\lim_{t\to 0^+} u(x,t) = \frac{f(x^+) + f(x^-)}{2}$$

f is continuous at x so that f has the jump discontinuity at x.

So, once we prove this proposition, then we can have a solution for u and all this. Now and we can have a solution in this nature like $u(x,t)$ given to the initial value problem.

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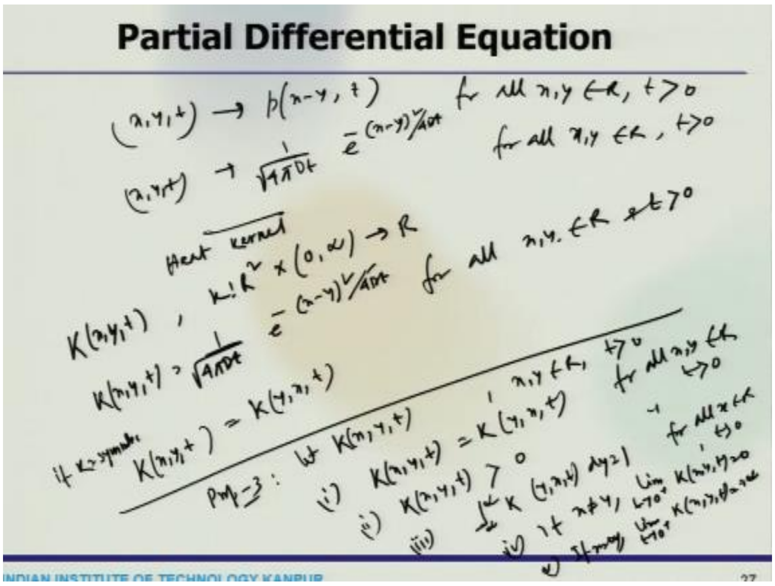

And the solution would be in kind like, so the map would be $(x, y, t) \rightarrow p(x-y, t)$ for all x, y belongs to R and t greater than 0. So,

$$(x, y, t) \rightarrow \frac{1}{\sqrt{4\pi Dt}} e^{\frac{-(x-y)^2}{4Dt}}$$

for all x, y R. So, this is called the usually the called the Heat Kernel. So, and we can also call it the fundamental solution to the one dimensional diffusion equation.

Now if we denote this kernel as a $K(x, y, t)$ where K belongs to R^2, then this

$$K(x, y, t) = \frac{1}{\sqrt{4\pi Dt}} e^{\frac{-(x-y)^2}{4Dt}}$$

for all x, y belongs to R, t greater than 0. So we can reiterate the properties of the heat kernel so that we have already said here, so that $K(x, y, t)$ if K is symmetric, if K is symmetric then $K(x, y, t) = K(y, x, t)$. Now there are some associated propositions.

So, like for this heat kernel if let $K(x, y, t)$ be defined for x, y belongs to R and t greater than 0, then we have $K(x, y, t) = K(y, x, t)$. This is for all x, y belongs to R and t greater than 0. We have $K(x, y, t) > 0$ for all x, y belongs to R. Then we have

$$\int_{-\infty}^{\infty} K(y, x, t)\, dy = 1$$

That is also for all x belongs to R and t greater than 0.

If $x \neq y$, then the limit

$$\lim_{t \to 0^+} K(x, y, t) = 0$$

If $x = y$ then at the limit

$$\lim_{t \to 0^+} K(x, y, t) = +\infty$$

So now this is what is going to be the property for the.

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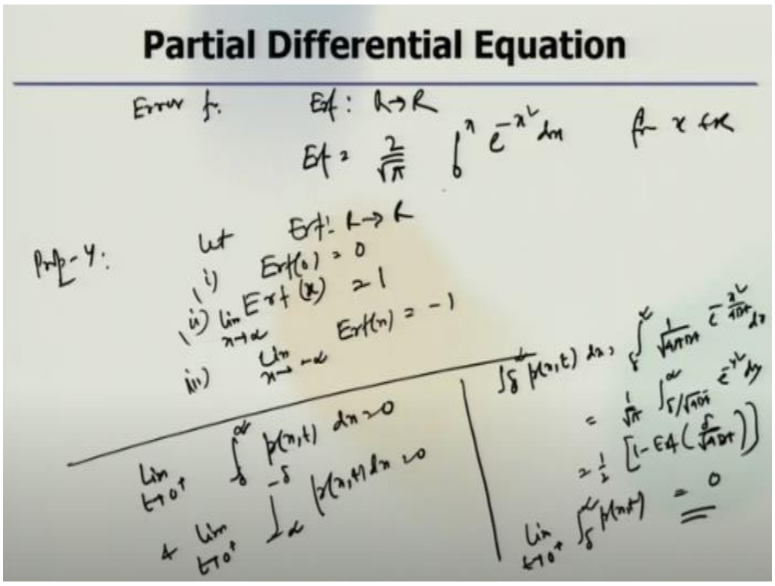

Now, there are another thing which so which would be that also I mean involve some of the function called error function which is Erf which is also defined in R and

$$Erf = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

where x belongs to R. So, error function also had certain properties and like let Erf is given like this, then it has $Erf(0) = 0$.

$$\lim_{x \to \infty} Erf(x) = 1$$

$$\lim_{x \to -\infty} Erf(x) = -1$$

So, these are some of the properties of the error function. Now also like the definition of the $p(x, t)$ that we have used. So, we can have some properties for that too. Like for limit

$$\lim_{t \to 0^+} \int_\delta^\infty p(x, t)\, dx = 0$$

$$\lim_{t \to 0^+} \int_\infty^\delta p(x, t)\, dx = 0$$

So, $p(x, t)$ is already defined.

So now one can see this group which would be kind of like straight forward I mean

$$\int_{\delta}^{\infty} p(x,t)\,dx = \int_{\delta}^{\infty} \frac{1}{\sqrt{4\pi Dt}} e^{\frac{-x^2}{4Dt}} dx = \frac{1}{\sqrt{\pi}} \int_{\delta/\sqrt{4Dt}}^{\infty} e^{-y^2} dy = \frac{1}{2}\left[1 - Erf\left(\frac{\delta}{\sqrt{4Dt}}\right)\right]$$

And so, we have used the error function definition here.

So, this is what we get. So now if we take the limit

$$\lim_{t\to 0^+} \int_{\delta}^{\infty} p(x,t)\,dx = 0$$

So similarly, the other one, one can show that how we can prove these things.

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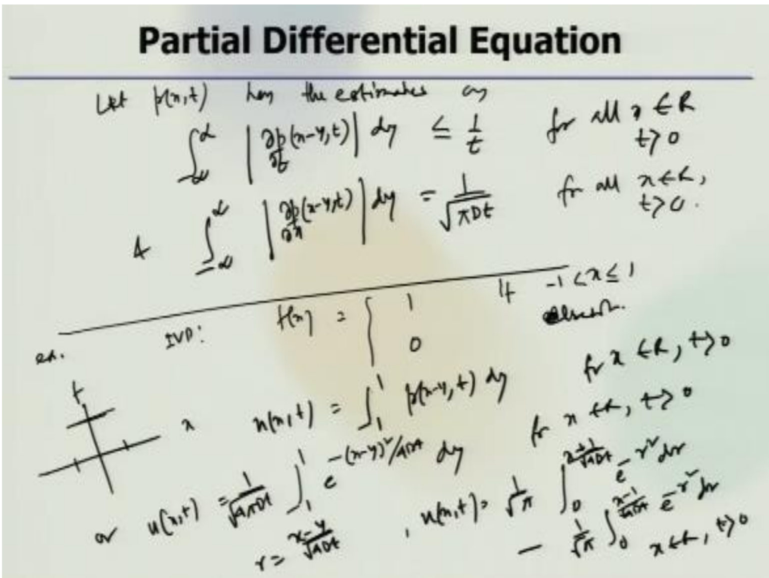

Now there is another situation where let $p(x,t)$ has the estimate as

$$\int_{-\infty}^{\infty} \left|\frac{\partial p}{\partial t}(x-y,t)\right| dy \leq \frac{1}{t}$$

for all x belongs to R t greater than 0. So, this also and second one is that

$$\int_{-\infty}^{\infty} \left|\frac{\partial p}{\partial x}(x-y,t)\right| dy = \frac{1}{\sqrt{\pi Dt}}$$

So that also for all x belongs to R and t greater than 0. So these are other two estimate of $p(x,t)$ which again one can prove these things.

So now all this proofs actually this requires some sort of an involved mathematics. So, but anyway since we can have this from textbook also and so we are not going to carry out this proof or the detailed proof here. Now one here and there with the small ones we can do but the other ones we can have it from the.

So, we will finish this discussion with a small example like let us say we have an initial value problem of $f(x)$ which is given as 1 on 0 if x less than -1 or elsewhere. So, the function if we plot, so this is the function looks like, this is f.

$$u(x,t) = \int_{-1}^{1} p(x-y,t)\,dy$$

for x belongs to R and t greater than 0. Or

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-1}^{1} e^{\frac{-(x-y)^2}{4Dt}} dy$$

for x belongs to R, t greater than 0.

So, change of variables we use,

$$r = \frac{x-y}{\sqrt{4Dt}}$$

So, what it becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+1}{\sqrt{4Dt}}} e^{-r^2} dr - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-1}{\sqrt{4Dt}}} e^{-r^2} dr$$

where x belongs to R and t greater than 0.

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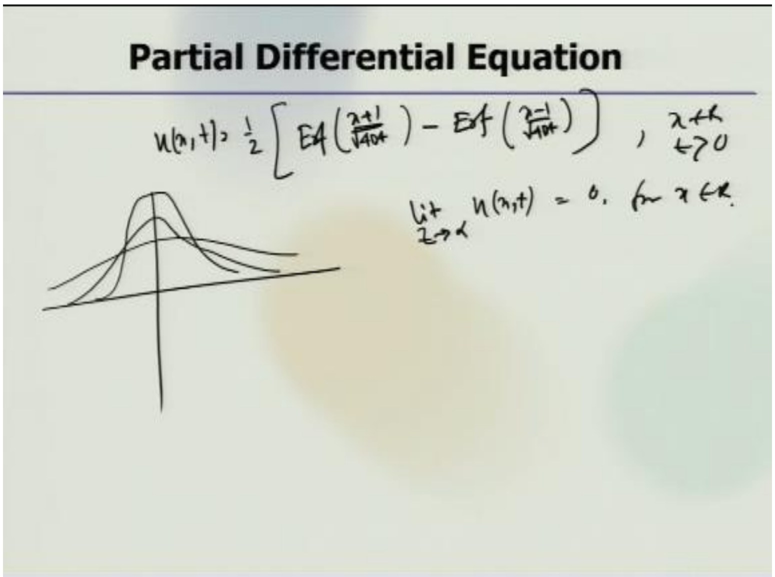

So, if we use the error function here then our solution would be

$$u(x,t) = \frac{1}{2}\left[Erf\left(\frac{x+1}{\sqrt{4Dt}}\right) - Erf\left(\frac{x-1}{\sqrt{4Dt}}\right)\right]$$

So, for x belongs to R and t greater than 0. So, the solution if you look at, the solution would look like this. This would be now this will go like this. Then like this. So this would be the solution and for the limit

$$\lim_{t\to\infty} u(x,t) = 0$$

for all x belongs to R.

So, this is what you can find out and you can see how this error function and other things become handy in defining this thing. So, we will stop the discussion here and look at the things like uniqueness and all this in the next session.