Computational Science in Engineering
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Lecture - 23

Okay, so let us continue the discussion on the PDE. So now we have looked at the different ways to solve this PDEs. Now we are looking at, going to look at the uniqueness of the diffusion equation and the other thing.

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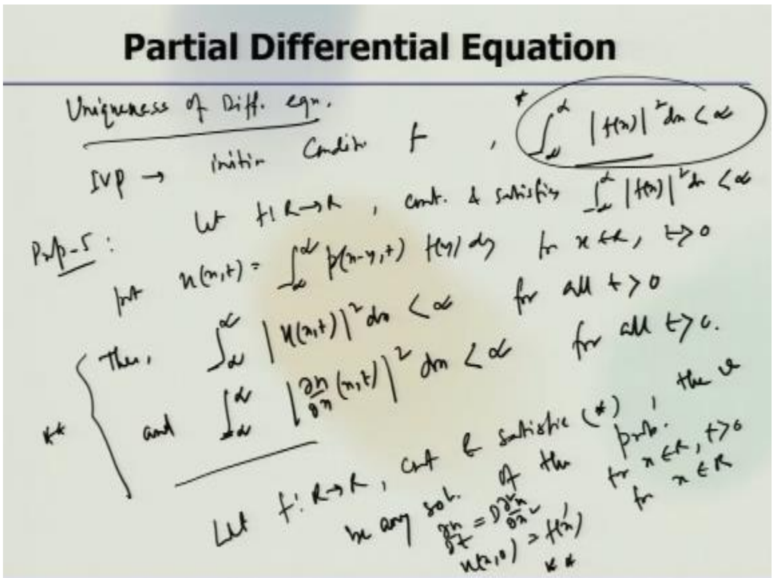

So now when you go to the uniqueness, so what we have already seen the solution and all these, like the solution that we have obtained for this kind of initial value problem. So, these are I mean there are general solution for the equation in which let us say for this initial value problem, there is an initial condition which is f, so f it is square and integrable then this should satisfy that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

So that is what it is supposed to do. Now before we establish that any solution in the initial value problem satisfy these things so we need to have certain information or properties or propositions on this. So, like propositions let us say we had 4, now 5. We say let f is a function continuous and satisfying this f is continuous and satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

That is, so we can put you

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)\, f(y) dy$$

for x belongs to R and t greater than 0. Then what we have

$$\int_{-\infty}^{\infty} |u(x,t)|^2\, dx < \infty$$

for all t greater than 0 and

$$\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}(x,t)\right|^2 dx < \infty$$

for all t greater than 0.

So again, one can look at this proof in a textbook, but we will continue with the some of the other properties like let f is continuous and satisfies this and this particular equation. Then v be any solution of the problem where we define

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

for x belongs to R the initial values are

$$u(x,0) = f(x)$$

for x belongs to R. And we have other two condition also here.

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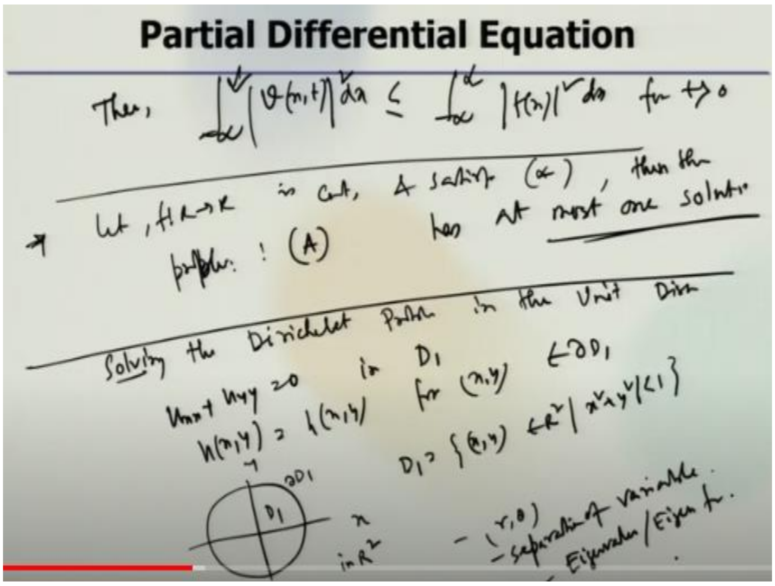

Then what we can have, then we have

$$\int_{-\infty}^{\infty} |v(x,t)|^2\, dx \le \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

So, this is again like so if you have a solution and then like this, so this again one can prove this. So similarly, we can say that let f is continuous and satisfying that equation then the problem that which is defined here, let us say A. Then the problem which is defined in A has at most one solution.

So, this is another important information and again this is another theory which one can prove it. Like now we can solve or solving the Dirichlet problem in the unit disk, okay. So, like let us say we have an

$$u_{xx} + u_{yy} = 0$$

in D_1, this is a two dimensional Laplacian and

$$u(x, y) = h(x, y)$$

for x and y belongs to ∂D_1. Now what is that? This is x, this is y, this is ∂D_1 and this is D_1.

So, this is a unique disk in R^2. So,

$$D_1 = \{(x, y)\epsilon R^2, |x^2 + y^2| < 1\}$$

So, this is called the unit disk in R^2. And h is given a function that is continuous in a neighborhood at the unit circle ∂D_1. Thus, we would like to find a solution u that is harmonic in D_1 that takes the values given in this.

So now first by looking at this radial symmetry of the domain we can probably express this in polar coordinates (r, θ). Then once we define that $r\ and\ \ \theta$ then we can use the separation of variable and then we can look at the eigenvalue or eigenfunction and then try to find out the solution.

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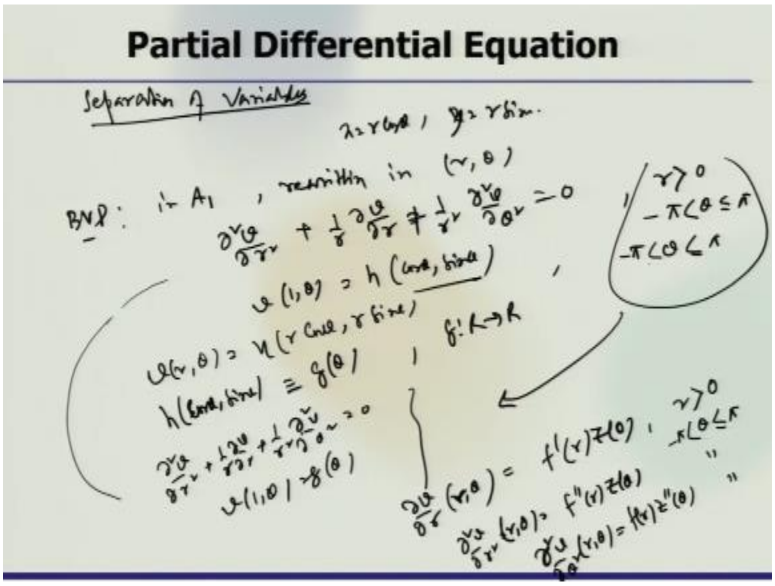

Now first look at the separation of variables. So, by looking at the symmetry of the domain, so we said that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

in polar coordinate. So now we exploit the linearity of the PDE and the boundary conditions given in this particular problem, let us say A_1 given in the problem A_1 we superposing simple solution of the problem.

The strategy here is to first find a special class of function $r\ and\ \theta$ and then find the solution. So, we start with the boundary value problem which is given in A_1 and rewriting in $r\ and\ \theta$, what we get is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Where $r > 0$, $-\pi < \theta < \pi$.

And

$$v(1,\theta) = h(\cos\theta, \sin\theta)$$

where θ again going. So, where we have set

$$v(r,\theta) = h(r\cos\theta, r\sin\theta)$$

Now here this

$$h(\cos\theta, \sin\theta) \equiv g(\theta)$$

So, then what we will write this equation is that

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

And

$$v(1, \theta) = g(\theta)$$

And all the conditions here they will remain there. Now f is a continuous function, then how we can partial derivative that we will write;

$$\frac{\partial v}{\partial r}(r, \theta) = f'(r)z(\theta)$$

where r greater than 0 and $-\pi < \theta < \pi$. So,

$$\frac{\partial^2 v}{\partial r^2} = f''(r)z(\theta)$$

for the same condition. And

$$\frac{\partial^2 v}{\partial \theta^2} = f(r)z''(\theta)$$

in this given condition.

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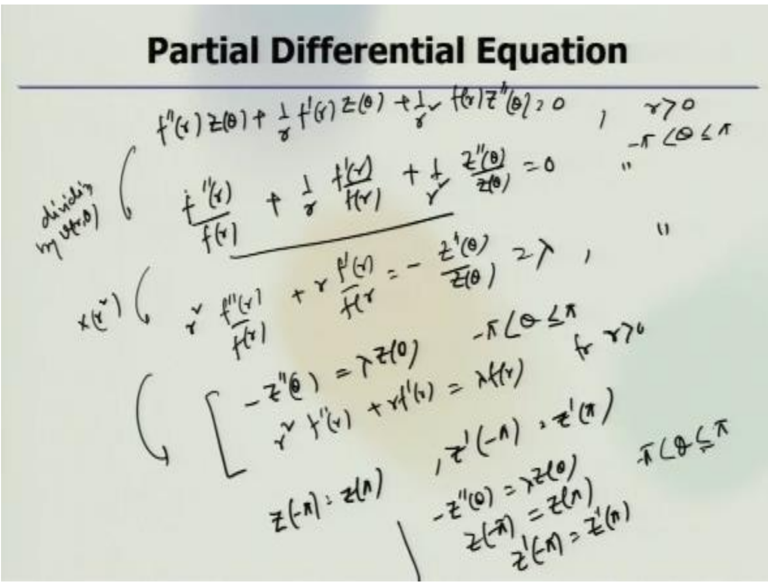

Now once we replace back in this r, θ system what we get

$$f''(r)z(\theta) + \frac{1}{r}\, f'(r)z(\theta) + \frac{1}{r^2} f(r)z''(\theta) = 0$$

where r is greater than 0 and $-\pi < \theta < \pi$. So now assuming $v(r, \theta)$ is a nonzero for all values of $r\ and\ \theta$. So, then we divide by both side by that and what we get that

$$\frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + \frac{1}{r^2}\frac{z''(\theta)}{z(\theta)} = 0$$

Now this is the condition for which this is valid. Now here if we multiply both side by r^2 we notice that the equation can be written in such a way that the functions that depends only r on one side and the other side on theta. For example, if we multiply this by r^2 we get

$$r^2 \frac{f''(r)}{f(r)} + r \frac{f'(r)}{f(r)} = -\frac{z''(\theta)}{z(\theta)} = \lambda$$

So here we have got everything separated.

Now that we got two differential equations, which lead to

$$-z''(\theta) = \lambda\, z(\theta)$$

for theta between $-\pi < \theta < \pi$ and we got

$$r^2 f''(r) + r f'(r) = \lambda\, f(r)$$

where r greater than 0. Now we can have a requirement for the function g which is periodic of with the period 2π yields that

$$z(-\pi) = z(\pi)$$

and

$$z'(-\pi) = z'(\pi)$$

So, in other words, now if I put everything this two-point boundary value problem, which becomes like

$$-z''(\theta) = \lambda\, z(\theta)$$

$$z(-\pi) = z(\pi)$$

$$z'(-\pi) = z'(\pi)$$

where theta goes between $-\pi < \theta < \pi$. So, this is what we get.

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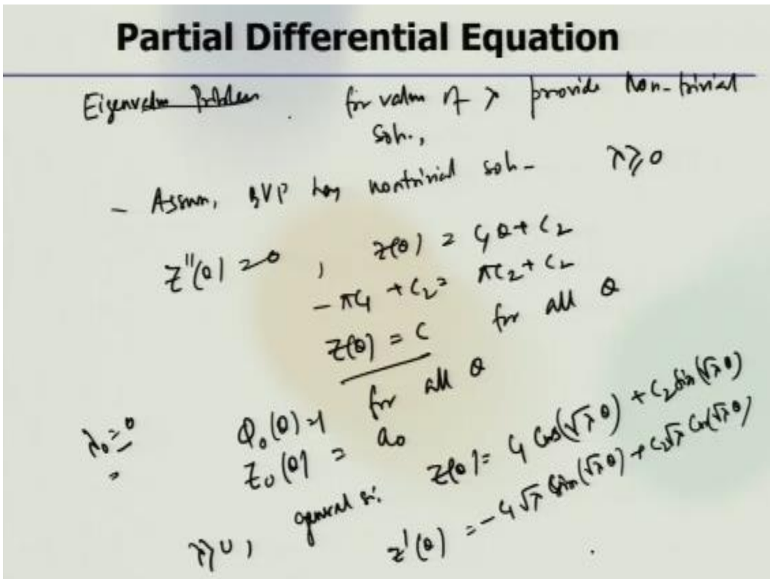

Now we have an, we can have an eigenvalue problem. So, like we can see that function $z(\theta) = 0$ for all values of θ. So, and when it solves, we get and refer to the solution is a trivial solution. But we are interested for the non-trivial solution and the special solution for this boundary value problem. So, we can see that this non-trivial solution depends on the value of lambda of the ODE.

So, there is certain values of lambda for which has a non-trivial solution but the rest this boundary value problem will have trivial solution. So now the value of lambda for which this two-point boundary value problem which has non-trivial solutions, so that would call the so that say for values of lambda provide non-trivial solution are called the eigenvalues and corresponding function will be eigenfunction.

So now this again one can see this proof and all these that this is possible and like if we let us say, assuming that two points has a non-trivial solution, then lambda has to be so, if we assume this boundary value problem has non trivial solution, then lambda has to be 0. So that is an interesting. Now how to find out the solution basically, that lambda 0 now we have this equation.

So, we get $z''(\theta) = 0$, which is a general solution is that

$$z(\theta) = C_1\theta + C_2$$

Now if we apply the boundary condition, So, this is a solution that we have for all theta which is non-trivial solution.

And that is $\lambda = 0$ is an eigenvalue. Now this one we can say that $\lambda_0 = 0$. So, we pick the special eigenfunction that is $\phi_0(\theta) = 1$ for all theta. So,

$$z_0(\theta) = a_0$$

for all theta. Now since $\lambda_0 = 0$. Now we can look for the positive eigenvalues of this solution where $\lambda > 0$, then the general solution would be in the form of

$$z(\theta) = C_1 \cos\sqrt{\lambda}\theta + C_2 \sin\sqrt{\lambda}\theta$$

So that is also for all theta that we have. So, we have

$$z'^{(\theta)} = -C_1\sqrt{\lambda}\cos\sqrt{\lambda}\theta + C_2\sqrt{\lambda}\sin\sqrt{\lambda}\theta$$

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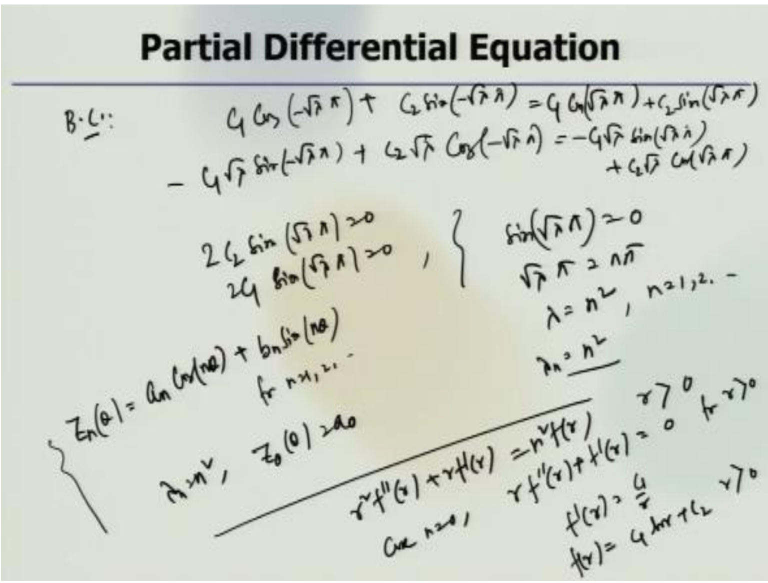

Now once we put this with the boundary condition, so if we use the boundary conditions, we get

$$C_1\cos(-\sqrt{\lambda}\pi) + C_2\sin(-\sqrt{\lambda}\pi) = C_1\cos(\sqrt{\lambda}\pi) + C_2\sin(\sqrt{\lambda}\pi)$$

And second one

$$-C_1\sqrt{\lambda}\cos(-\sqrt{\lambda}\pi) + C_2\sqrt{\lambda}\sin(-\sqrt{\lambda}\pi) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)$$

So here we can divide. So, what we get from here that

$$2C_2\sin(\sqrt{\lambda}\pi) = 0$$

and

$$2C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) = 0$$

So, essentially $\sin(\sqrt{\lambda}\pi) = 0$. So,

$$\sqrt{\lambda}\pi = n\pi$$

So,

$$\lambda = n^2$$

for n equals to 1, 2, 3 and so on. So, the eigenvalues would be given as this. So, the positive eigenvalues are. So essentially, we can say that $\lambda_n = n^2$.

So,

$$z_n(\theta) = a_n cos\,(n\theta) + b_n sin(n\theta)$$

for n equals to 1, 2, 3 and so on. Now if we put these things together in the, then we say that all these eigenvalues and eigenfunction for this two-point boundary value problems are given by like $\lambda_n = n^2$.

And the corresponding eigenfunction $z_0(\theta) = a_0$ and $z_n(\theta)$ is given by like this

$$z_n(\theta) = a_n cos\,(n\theta) + b_n sin(n\theta)$$

where $a_n\ and\ b_n$ for corresponding. So now we have another radial component of that boundary value problem which is like

$$r^2 f''(r) + rf'(r) = n^2\, f(r)$$

for r greater than 0. So, for case n equals to 0, the equation becomes

$$rf''(r) + f'(r) = 0$$

for r greater than 0.

So, from here what we get

$$f'(r) = \frac{C_1}{r}$$

And

$$f(r) = C_1 \ln r + C_2$$

where r greater than 0. Now C_1 and C_2 are the constants and $C_1 \neq 0$.

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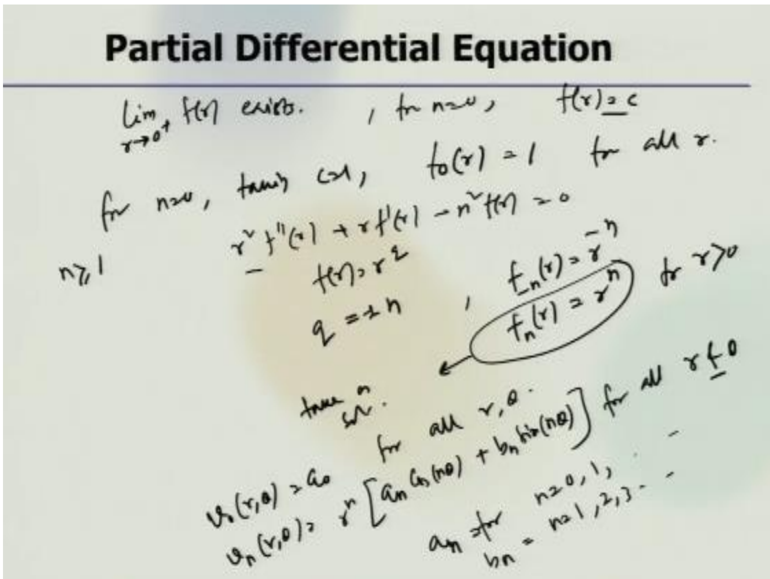

So, we are looking for a C^2 function defined on the closer unit disk D^1, so we must set $C_1 = 0$. So, the equivalent proposition is that $\lim_{r \to 0^+} f(r)$ exist and for $n = 0$, the $f(r)$ constant. So now for some constant $c = 1$ if you take for $n = 0$, taking $c = 1$, which will give

$$f_0(r) = 1$$

for all r. Now we can consider other cases where n greater than equals to 1.

So that differential equation becomes

$$r^2 f''(r) + rf'(r) - n^2 f(r) = 0$$

So here we can use

$$f(r) = r^q$$

So, this is an again second order ODE. So, after doing all these, what we will get is that $q = \pm n$. So, it will have that

$$f_{-n}(r) = r^{-n}$$

And

$$f_n(r) = r^n$$

for r greater than 0. So, the boundary condition if we apply, so we take this is the physical solution that we will have.

So, we will take this as the take as solution. So finally, what we get that

$$v_0(r, \theta) = a_0$$

for all $r \; and \; \theta$;

$$v_n(r,\theta) = r^n[a_n cos\ (n\theta) + b_n sin(n\theta)]$$

for all r and θ. And a_n is for n equals to 0, 1, 2 and b_n for n equals to 1, 2, 3 are the constants. So, this is what you get. And now surely these things can be expanded in terms of eigenfunction and all this.

So, the other thing which would be interesting to see is that like some sort of a Poisson integral kernel or like so now the point here one important another thing which could be interesting to see is the Poisson integral representation.

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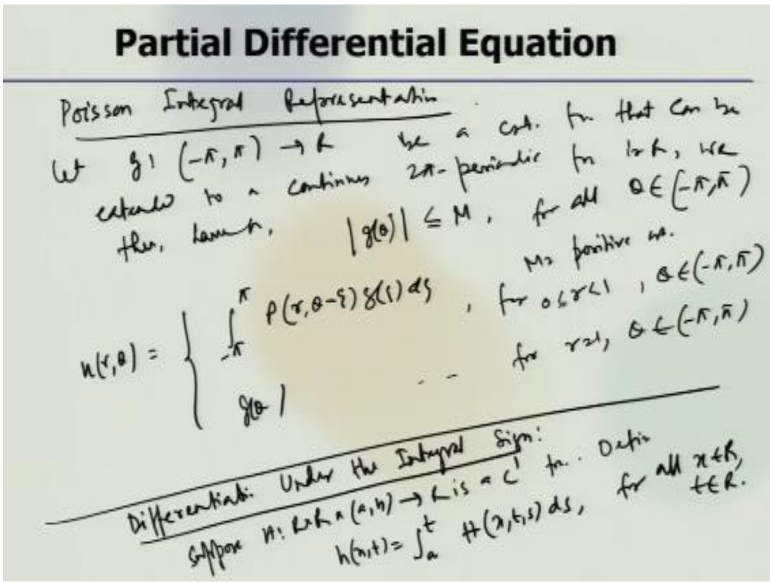

Like Poisson integral representation. So, what it says that let g, which is defined in $-\pi\ to\ \pi$ be a continuous function that can be extended to a continuous 2π periodic function in R. We then have that

$$|g(\theta)| \le M$$

for all theta belongs to $-\pi\ to\ \pi$, where M is some positive constant.

So here the idea here is that we can use some of these properties of the Poisson kernel and extend this one some of these solutions with the unit disk and we can show that $u(r,\theta)$ would be $\int_{-\pi}^{\pi} p(r, \theta - \xi)\ g(\xi) d\xi$ or $g(\theta)$. So, this is for r greater than 0, theta belongs to $-\pi\ to\ \pi$. And this is for r equals to 1 and theta belongs to $-\pi\ to\ \pi$. So, this is another thing, which could be also bit of handy.

But we will see some examples and where the separation of variable or these things we can use. So, another last point which is important to note here is that we can have differentiation or differentiating under the integral sign. So how do we do that? Actually, so this is an suppose, let us say suppose H which is given like is a C^1 function and the define is that

$$h(x,t) = \int_a^t H(x,t,s)\,ds$$

for all x belongs to R, t belongs to R.

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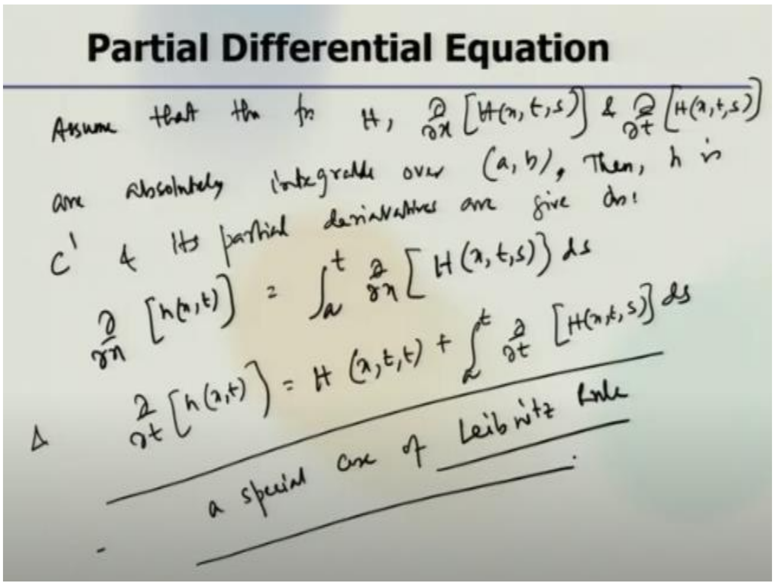

So here we assume that the function $H, \frac{\partial}{\partial x}[H(x,t,s)]$ & $\frac{\partial}{\partial t}[H(x,t,s)]$ are absolutely integrable over a and b. Then h is C^1 and its partial derivative are given as

$$\frac{\partial}{\partial x}[H(x,t)] = \int_a^t \frac{\partial}{\partial x}[H(x,t,s)]\,ds$$

And what we can write

$$\frac{\partial}{\partial t}[H(x,t)] = H(x,t,t) + \int_a^t \frac{\partial}{\partial t}[H(x,t,s)]\,ds$$

So, this is a, this is an again a fundamental theorem of calculus.

And this is a sort of a special case of, this is a special case of Leibnitz rule where you can have differentiation under integration. So that is pretty much sort of an conclude the theoretical aspect of this PDEs. Now we can see some practical examples and then that would give you an idea how to use all this theoretical discussion that we had so far. And that we will do in the next session.