Computational Science in Engineering
Prof. Ashoke De
Department of Aerospace Engineering
Indian Institute of Technology-Kanpur

Lecture - 24

Okay, so now we will continue the discussion on the PDE. Now what we are going to look at now how to apply that kind of theory like separation of variable or then the theoretical discussion on these things that we have done so far.

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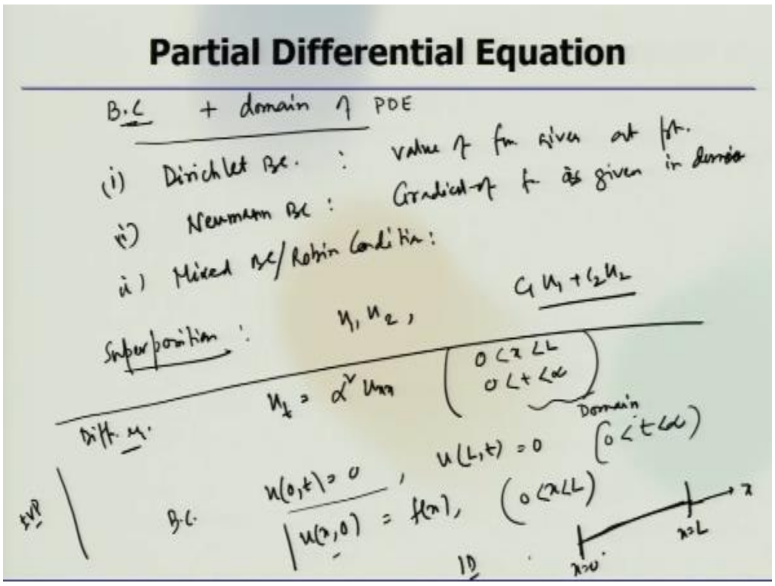

So let us look at those things. And but before we again refresh the, just to give an, whenever we solve this PDEs we require some sort of a boundary conditions. And these are along so that is one important aspect plus the domain of PDE.

So, these are the two things that one must have it to get a solution of a PDE, because this will give you an idea about what is the domain where it is defined, and what are the boundary conditions that you have. So, first kind of boundary condition that is important is that called the Dirichlet boundary condition okay. So that means which assume that value of function given at point.

And the second would be Neumann boundary condition, which is the gradient of the function, gradient of function is given in domain. So, it so third would be mixed boundary condition. Mixed boundary condition or sometime it is called Robin condition. So which means is that you will have both the Dirichlet condition.

So, it is just like when we see this example, then you can see that how these Dirichlet conditions are defined, and then the mixed condition would have both the fixed values and the gradient. And just to solution pattern, we have already talked about the superposition, which is that if u_1 and u_2 are solution of a linear homogeneous PDE, then let us say u_1 and u_2 are solution, then any combination of $c_1u_1 + c_2u_2$ this also going to be solution.

And obviously, the other thing is that when you have a homogeneous solution, then that solution is going to be solution of the non-homogeneous equation. Now we have already also seen the systems which are parabolic, elliptic or hyperbolic in nature. Now we will see some let us see example of diffusion equation, which is given as

$$u_t = \alpha^2 u_{xx}$$

So, this is obviously the domain, okay. So, when you define a PDE the domain is important and then obviously the boundary condition. So, the what are the boundary conditions here? That

$$u(0, t) = 0$$
$$u(L, t) = 0$$

for t greater than 0 to infinity. And

$$u(x, 0) = f(x)$$

for $0 < x < L$. So now we see this is a so this sort of an initial value problem one can sees.

This is an initial value problem where at the t=0 condition, the initial value of the function is given. And then the domain information or the boundary conditions given for other t at x=0 and so since this is a 1D diffusion equation, so in the one direction the boundary conditions are provided. Now one it is clear that this system is 1D and it is bounded between let us say if it is direction of x, the bound is x equals to 0 and x equal to L.

So in between that only we are trying to find out the solution. And $f(x)$ is given as an initial condition. But another interesting thing one has to note that this initial condition is not defined at x=0 or x equal to L. This is given between x equal is greater than 0 and x less than 0. So, this is not defined at these conditions. The reason is that at x=0 and x=L these are already defined.

So, we do not want to have a system which is over defined. So that is important. So, we have this all initial conditions between this boundedness and these two points are the. Now this is one dimensional system but this will give you a very clear idea when you are dealing with multi-dimensional problem. So, at this corner points there is a chance of over defining the system. Or sometimes it is a corner point, sometimes it is an intersection of these things.

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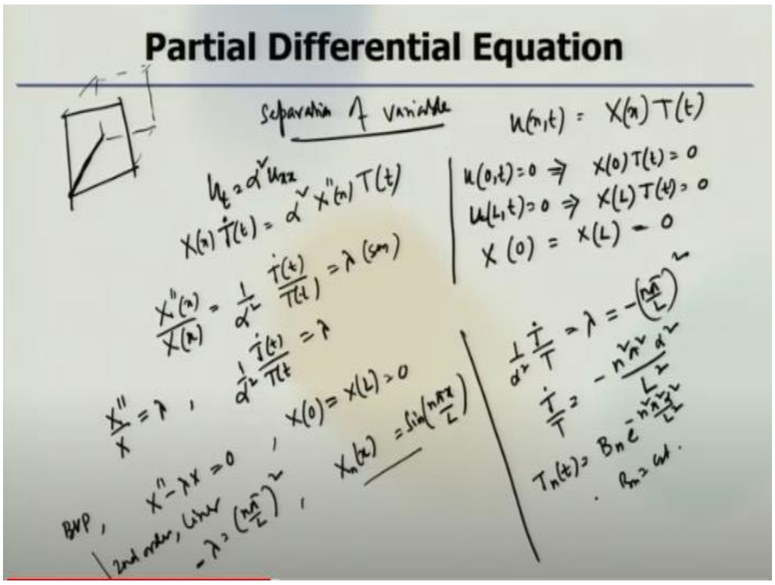

So, in a multi-dimensional problem for example if you take 2D example then this is again a corner point. Or if you take for example 3D, then this would be intersection surface. So that is another issue is there. So, in that time whenever you have this multi-dimensional problem so this corner or intersection of surface, we have to choose one of which is defined. Now let us come back to this original problem of this 1D diffusion.

So here we will use separation of variable. So let us say we define like

$$u(x,t) = X(x)T(t)$$

So that is what we have defined. So, this is what we have already seen in the theoretical calculation. So, when you do that our

$$u_t = \alpha^2 u_{xx}$$

So now if we put it back, this would be

$$X(x)\dot{T}(t) = \alpha^2 X''(x)T(t)$$

And all the boundary conditions correspond to this transformed system would become like for example

$$u(0,t) = 0$$

This would go

$$X(0)T(t) = 0$$

$$u(L,t) = 0$$

that would get to

$$X(L)T(t) = 0$$

$$X(0) = X(L) = 0$$

So now what we can write this guy is that

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{\dot{T}(t)}{T(t)} = \lambda$$

So, this then it will separate to 2 ODEs, which is

$$\frac{X''(x)}{X(x)} = \lambda$$

And

$$\frac{1}{\alpha^2}\frac{\dot{T}(t)}{T(t)} = \lambda$$

Now this boundary value problem becomes minus

$$X'' - \lambda X = 0$$

where we have $X(0) = X(L) = 0$. So now the PDE is due to the separation of variable. Now we come down to the ODEs and these are boundary value problem.

So, this is a second order linear ODE with the eigenvalue problem. So, this will have some non-trivial solution for

$$-\lambda = \left(\frac{n\pi}{L}\right)^2$$

Now what we get the solution for this is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

So that is from the first part of the system. Now the second part which we have is the

$$\frac{1}{\alpha^2}\frac{\dot{T}(t)}{T(t)} = \lambda = -\left(\frac{n\pi}{L}\right)^2$$

Now what we can write that

$$\frac{\dot{T}}{T} = -\frac{n^2\pi^2\alpha^2}{L^2}$$

So, the after doing the integration this will get like

$$T_n(t) = B_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}}$$

where B_n is constant.

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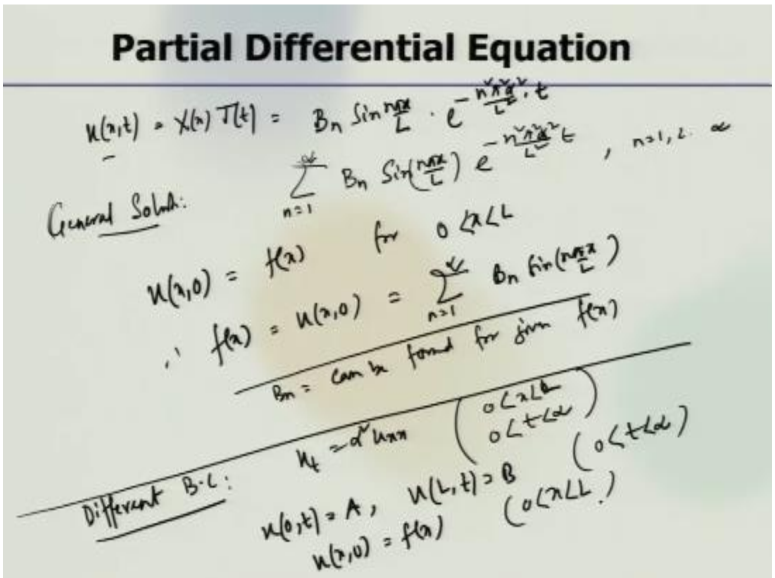

Now if we combine this and this the general solution would look like

$$u(x,t) = X(x)T(t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}}$$

and general solution would be

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}}$$

for n is equals to 1 to infinity.

Now we have got now this like initial condition

$$u(x,0) = f(x)$$

for $0 < x < L$. So what we get

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

So, and from here B_n would be, so you can find out, B_n can be found for given $f(x)$. And the final solution can be found out.

So, this is where $f(x)$ we kept it beta generic so that any given $f(x)$ one can find out the B_n from here and the final solution. Now the same 1D problem if we have a different boundary condition. Let us say the boundary conditions are different, equation is same like we have the same

$$u_t = \alpha^2 u_{xx}$$

and define $0 < x < L \ and \ 0 < t < \infty$. Here the boundary condition is given

$$u(0,t) = A \ and \ u(L,t) = B$$

This is for any t. So these are the Dirichlet conditions where the x=0 and x=L at these two points things are defined and $u(x,0) = f(x)$. Now if you compare with the previous boundary conditions where we have given this is 0. Now we are moving to a situation where they are given A and B. These are more generic. So, A and B if they are 0 then it comes down to the system that we have already looked at.

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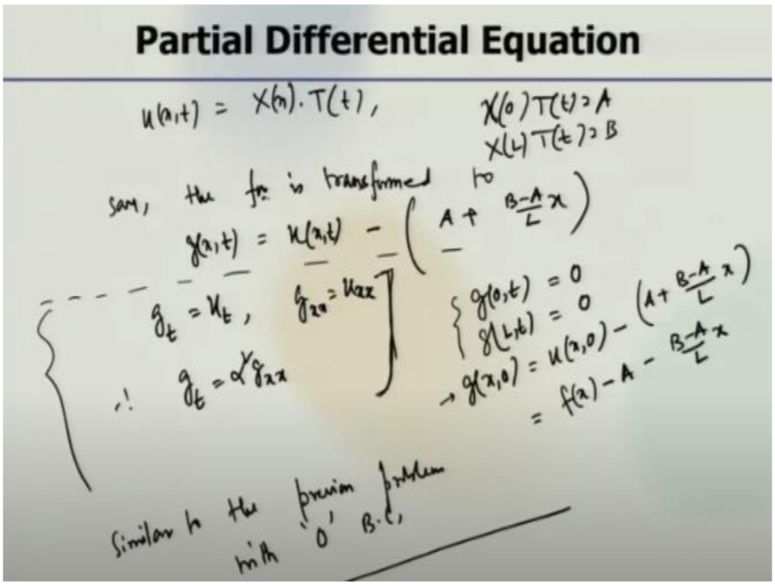

Now same way we can do the transformation like $u(x,t)$ would be the separation of variable. Then

$$u(x,t) = X(x)T(t)$$

$X(0)T(t) = A, X(L)T(t) = B$. So, the let us say the function is transformed to

$$g(x,t) = u(x,t) - \left(A + \frac{B-A}{L}x\right)$$

And where $g_t = u_t$, $g_{xx} = u_{xx}$. So, our equation system becomes

$$g_t = \alpha^2 g_{xx}$$

And then for this set of equation what we have like the boundary conditions which are going to have

$$g(0,t) = 0$$
$$g(L,t) = 0$$

And

$$g(x,0) = u(x,0) - \left(A + \frac{B-A}{L}x\right) = f(x) - A - \frac{B-A}{L}x$$

So, these are two boundary conditions. This is initial conditions on everything gets transformed to this g.

Now once this happens, so this portion of the system which has been defined this is exactly to or similar to the previous problem with zero boundary conditions. So now what it gives you an idea that this A and B is more generic and if A and B is 0, then that becomes a special case or alternatively what one can do, transform this system.

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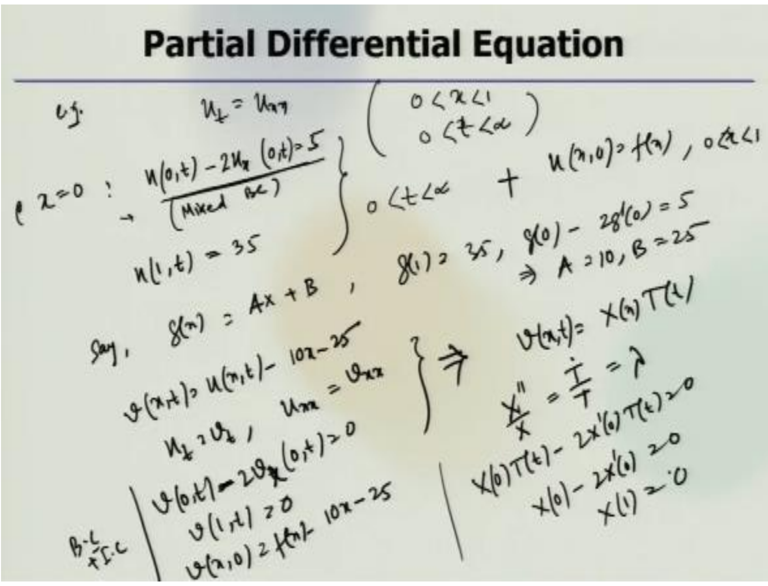

So let us look at some examples like $u_t = u_{xx}$. The domain is defined between, so these are some numerical values here; t is infinity to 0 and the boundary conditions are given like

$$u(0,t) - 2u_x(0,t) = 5$$

So, this is at x equals to 0. And this particular condition is known as mixed boundary condition where you have the both Dirichlet and the derivative.

Now at

$$u(1,t) = 35$$

and these are for t less than infinity greater than 0. And plus, we are given the initial condition is

$$u(x,0) = f(x)$$

for x between 0 to 1. So, as I said this is a mixed boundary condition. So let us say we defined

$$g(x) = Ax + B$$

and $g(1) = 35$, $g(0) - 2g'(0) = 5$, which gets us A equals to 10 and B equals to 25. So, we set let us say

$$v(x,t) = u(x,t) - 10x - 25$$

So,

$$u_t = v_t$$

And

$$u_{xx} = v_{xx}$$

And the conditions would be

$$v(0,t) - 2v_x(0,t) = 0$$

$$v(1,t) = 0$$

$$v(x,0) = f(x) - 10x - 25$$

So, these are all boundary condition plus initial conditions for which it has to be solved. Now how do we do that? Now it is straightforward. We can see that

$$v(x,t) = X(x)T(t)$$

So, the equation becomes

$$\frac{X''}{X} = \frac{\dot{T}}{T} = \lambda$$

So,

$$X(0)T(t) - 2X'(0)T(t) = 0$$

$$X(0) - 2X'(0) = 0$$

$$X(1) = 0$$

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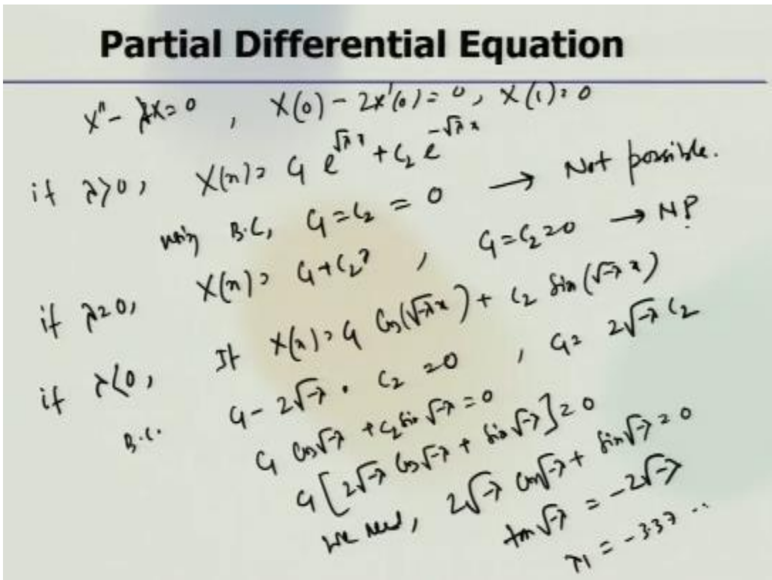

So, you got two ODEs. One is

$$X'' - \lambda X = 0$$

with boundary conditions $X(0) - 2X'(0) = 0$ and $X(1) = 0$. So, we get if $\lambda > 0$, we get this one as

$$X(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$$

Now if we using the boundary conditions what we get $C_1 = C_2$, they would be 0 which is not possible.

Or if $\lambda = 0$ then

$$X(x) = C_1 + C_2 x$$

where $C_1 = C_2 = 0$, that is also not possible. So, we can see what are the values of lambda or if lambda less than 0 then if we get now

$$X(x) = C_1 \cos(\sqrt{-\lambda}x) + C_2 \sin(\sqrt{-\lambda}x)$$

So, the boundary condition if we put

$$C_1 - 2\sqrt{-\lambda}C_2 = 0$$

Where,

$$C_1 = 2\sqrt{-\lambda}C_2$$

So,

$$C_1 \cos(\sqrt{-\lambda}) + C_2 \sin(\sqrt{-\lambda}) = 0$$

where

$$C_1\left[2\sqrt{-\lambda}\cos(\sqrt{-\lambda}) + \sin(\sqrt{-\lambda})\right] = 0$$

So, what we need here is that

$$2\sqrt{-\lambda}\cos(\sqrt{-\lambda}) + \sin(\sqrt{-\lambda}) = 0$$

So, which is

$$\tan\sqrt{-\lambda} = -2\sqrt{-\lambda}$$

So that means,

$$\lambda_1 = -3.37$$

something like that.

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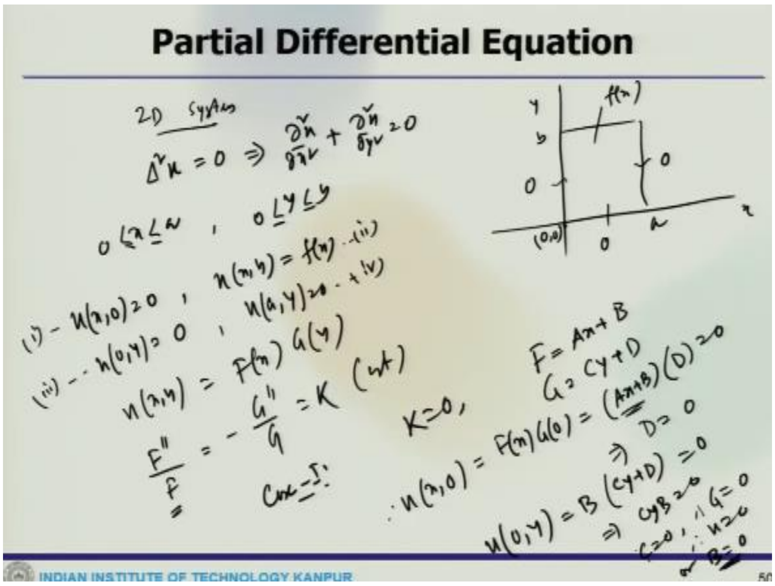

So now let us look at a 2D system where you have

$$\Delta^2 u = 0$$

which is corresponding to an equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is also defined on a plane x and y and these are the plane A and B. And where this is within the domain, x is 0 to a and y is 0 to b. And the boundary conditions are x 0 is 0. So, this is origin.

And x, b, which is also $f(x)$, so this is $f(x)$.

$$u(x,0) = 0$$

That means this guy is 0. And

$$u(a,y) = 0$$

So, this is also 0. So, these are the conditions. So let us say this is boundary condition i, this could be ii, this could be iii, this could be iv, and we will use that. And the other condition then we can write the solution as

$$u(x, y) = F(x)G(y)$$

And the equation becomes

$$\frac{F''}{F} = -\frac{G''}{G} = K$$

Now there would be different cases. For example, let us start with the case I where K is 0. So, the solution for the first ODE would be

$$F = Ax + B$$

And the second one would be

$$G = Cy + D$$

Now we apply the boundary condition. So,

$$u(x, 0) = F(x)G(0) = (Ax + B)(D) = 0$$

So, which means if this guy is not 0, D is 0. Now the second condition

$$u(0, y) = F(0)G(y) = (B)(Cy + D) = 0$$

Now D is already 0, so this gives us B is 0. Now we can have $C = 0$. If it is $C = 0$, then G is 0 and u is 0. Or we can have B is 0, okay, because C cannot be 0, then G would be 0 and then u would be 0. So, you can have B is 0.

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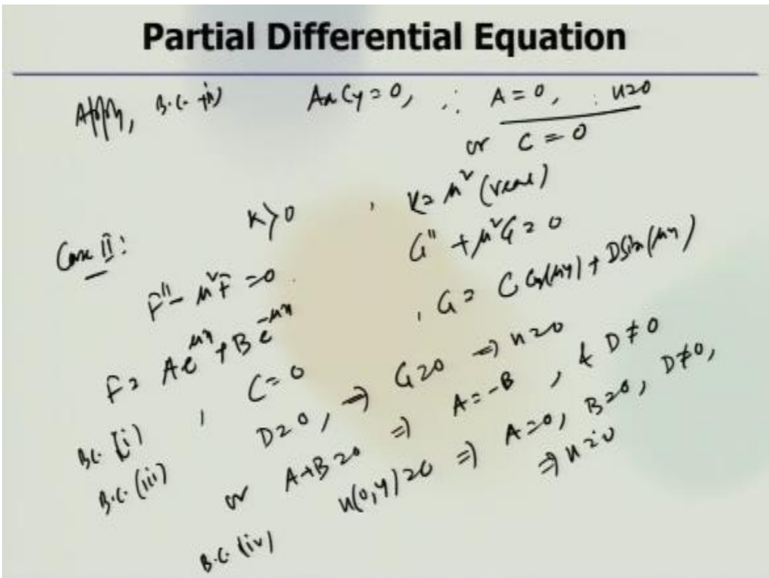

Now if we apply the boundary condition iv, what we get

$$AxCy = 0$$

So now A would be 0. So, u would be 0. So, this is not possible or the C is 0, okay. So, this is how we can do the coefficient. Now this case K is positive. Let us say $K = \mu^2$, which is real. Then our system becomes

$$F'' - \mu^2 F = 0$$

and

$$G'' + \mu^2 G = 0$$

So, this gives us a solution

$$F = Ae^{\mu x} + Be^{-\mu x}$$

Here

$$G = C\cos\mu y + D\sin\mu y$$

Now we use boundary condition ii, sorry i. Then you get C is 0. If we use boundary condition iii, D would be 0, which means G is 0, which means u is 0. Or what we can have A + B = 0, which means A = -B and D should be not equals to 0.

Now we use boundary condition iv, which is $u(0, y) = 0$. That gives us A equals to 0, B equals to 0 and D not 0. So, which means u = 0.

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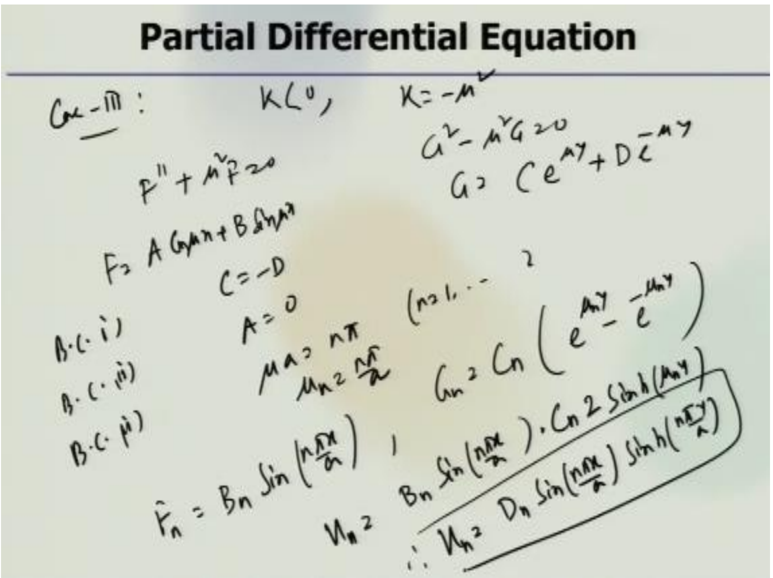

So, this is second case or case iii, the possibility is that K is less than 0 and let us say it is

$$K = -\mu^2$$

Then our system becomes

$$F'' + \mu^2 F = 0$$

$$G'' - \mu^2 G = 0$$

So, this case the solution would be

$$F = A\cos\mu x + B\sin\mu x$$

This case

$$G = Ce^{\mu y} + De^{-\mu y}$$

okay. So now what we get now boundary condition i we get $C = -D$.

If we use boundary condition ii, A would be 0. Boundary condition iii which gives

$$\mu a = n\pi$$

where n equals to 1, 2, 3. So

$$\mu_n = \frac{n\pi}{a}$$

So, our

$$F_n = B_n \sin\left(\frac{n\pi x}{a}\right)$$

and

$$G_n = C_n(e^{\mu_n y} - e^{-\mu_n y})$$

So, this is what you get. So, our

$$u_n = B_n \sin\left(\frac{n\pi x}{a}\right) C_n 2 \sinh \mu_n y$$

$$u_n = D_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

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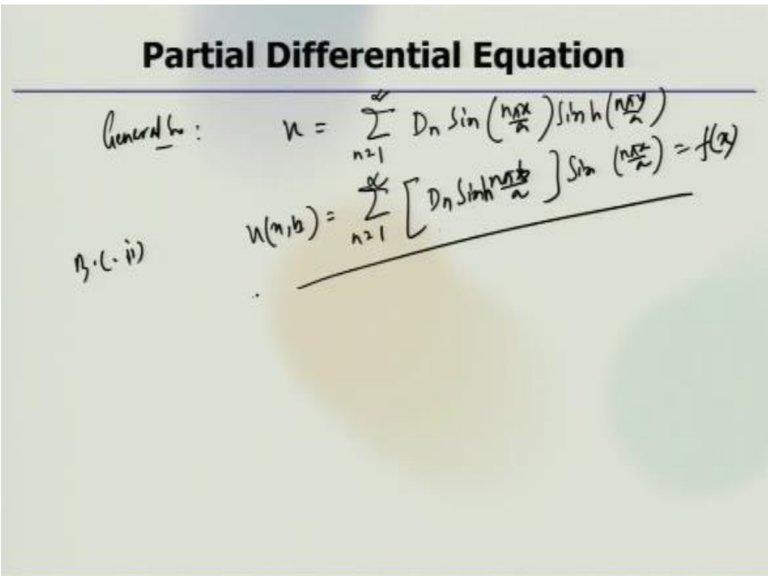

So, if you write the general solution for this, we have

$$u = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

Now we are using boundary condition ii where

$$u(x, b) = \sum_{n=1}^{\infty} \left[D_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right) = f(x)$$

So, we can get this is how you can find out the D_n from there. So, this is how one has to tackle this kind of problem and then find out this coefficient like C and D from this kind of given conditions. So that is pretty much give you an idea about the different kind of solution approach.

But primarily you do separation of variable, then you get to the second order ODEs I mean separated ODEs where you can solve and find out for different boundary conditions. So that is pretty much we conclude the discussion on PDE. Now we will continue other discussion in the next session.