Computational Science in Engineering
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Lecture – 03
Linear Algebra

So, let us continue the discussion on linear algebra, so what we have looked at it is the, how to look at the linear system like from row picture and then the column picture. And what we have concluded that when you go for large number of linear system or large number of unknowns, then your system becomes also large and that time it becomes like a row picture is not a feasible solution, so we need to look at the column picture.

And then column picture forms a different kind of column vector and how we can look at the solution. So, that is what we are in the discussion and in the meantime, we have looked at how to find out the inverse through the elimination process of the gauss elimination process and also parallelly we have seen how we can look at the elementary matrix.

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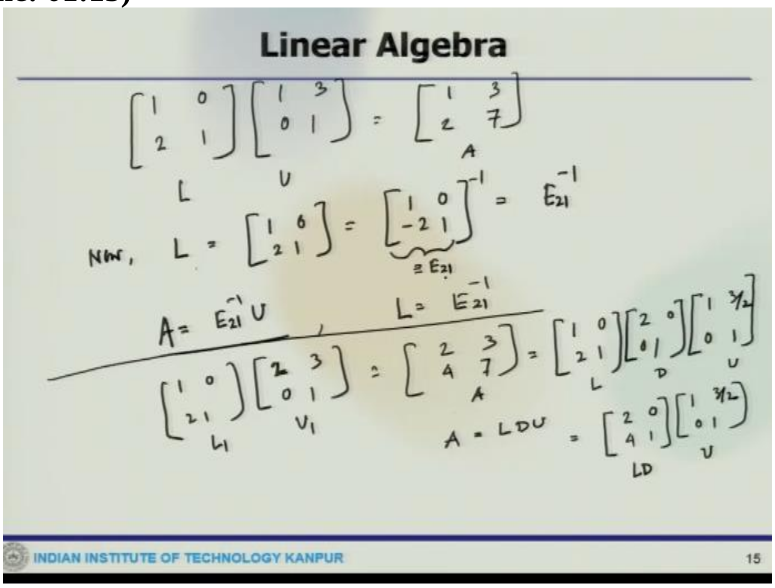

So, this is what we have actually started and like when you talk about this system and through this elementary operation actually you can finally write the system in terms of lower triangular matrix and upper triangular matrix. These lower triangular matrices this is a 2×2 system that we took an example. How the lower triangular matrix could be connected or associated with the elementary matrix. And elementary matrix is nothing but the matrix which is used to get this row operation.

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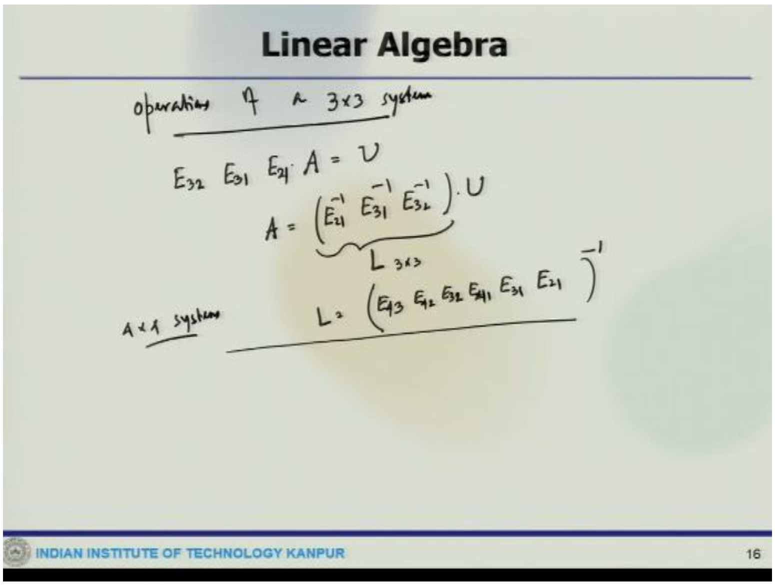

So, these essentially, these elementary matrices they actually represent the sort of row operations which are required for this elimination process.

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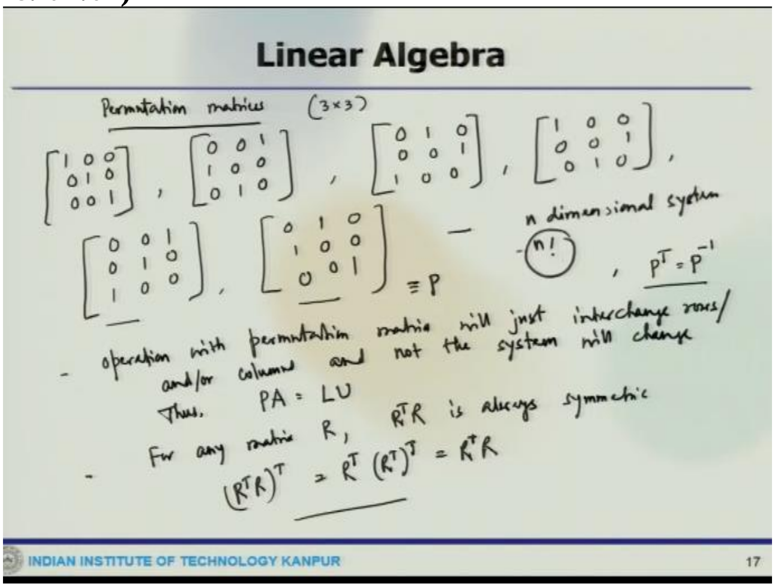

Now, we need to talk about one more particular system which is called that this also we have already looked at is the permutation matrices. Now, we have already looked at 2 × 2 system here what we are going to look at is the 3 × 3 system. So, now 3 × 3 system obviously, there will be different permutation matrix and for example, one can take some examples for 3 × 3 system which is $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

So, these are the different combinations which are possible. So, if you have a system like let us say if you have a n dimensional system, so, this is 3 × 3 system n dimensional system, then we can have a number of permutation matrix.

So, one of the important properties of this particular permutation matrix is that, if you look at if you say these are called permutation matrix, which are represented at as P then the property which is quite important is that P transpose would be P inverse and we can see the examples that we have already written down here that this should be satisfied. Now, what it does that is that any operation with permutation matrix will just interchange rows and or columns and not the system will change.

So, that is one thing, so thus one can say PA also can be written as lower triangular and upper triangular decompositions or this A permutated matrix can also be decomposed into a lower and upper triangular matrix that is what it means. Now, the second possibility is that for any matrix R, $(R^T R)^T = R^T (R^T)^T = R^T R$. So, that means, this is going to be always symmetric. So, these are some of the properties of a permutation matrix and already we have looked at the different permutation matrix for 2×2 system, how that change the system?

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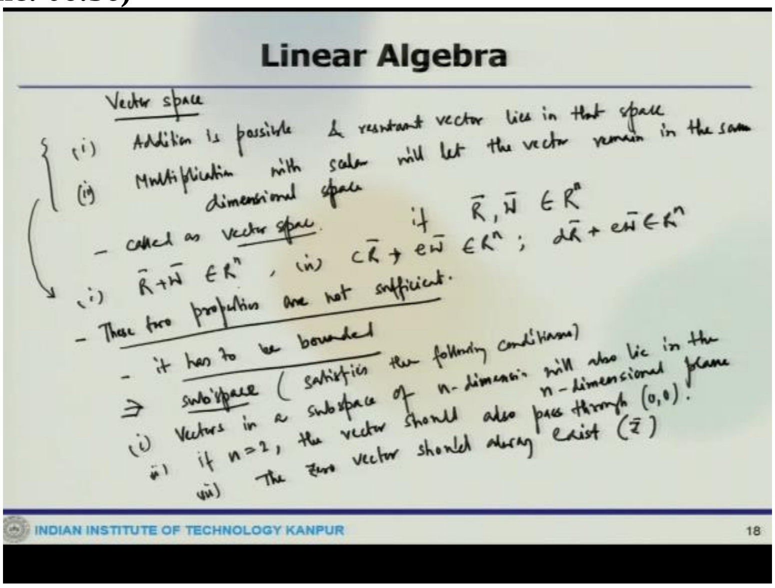

Now, with this sort of important understanding, we will look at the vector space or that space. So, what is important here is that obviously, to form a vector space, there would be certain important properties that need to be satisfied to form this space. So, one of the important properties is that addition is possible and so there are a set of vectors which are actually forming a space and to make that is a vector space this addition is possible and the resultant vector also lies in that space.

That means, when we do the addition amongst the vectors which are forming that space, it could be 2 dimensional space 1 dimensional space or 3 dimensional space or n dimensional space, they would also lie in that the resultant vector lie in that space. So, second is that any multiplication with scalar will let the vector remain in the same dimensional space. So, which means these are the two important properties that need to be satisfied then only the space which is sort of informed or produced by the vectors can be called as vector space.

So, now for example, let us say if there are two vectors $\vec{R}$ and $\vec{W}$ which belongs to R^n, then as per this $\vec{R} + \vec{W}$ also belongs to R^n or $c\vec{R} + e\vec{W}$ they also belong to R^n or $d\vec{R} + e\vec{W}$ they also belong to R^n. So, it actually satisfied these two important criteria that means, there are a couple of vectors which belong to n dimensional space and their addition is possible and also the resultant vector belongs to that space.

And the multiplication with scalar also produced some vector and they also lie in that space. Now, these are all possible linear combination of these R and W vector. However, one may note that these two properties are not sufficient, so that is a very important condition that these are different linear combinations which are possible or the possible linear combinations, but this does not mean that they are sufficient enough. Besides these thus space should be bounded in order to satisfy the definition of vector space that means it has to be bounded.

So, this is also very important, the vector space has to be bounded to satisfy the definition of vector space or the space would be to quality is that vector space, the space has to be bounded to satisfy the definition, so that is another condition. Now, a bounded domain always has to be predefined in order to form a vector space. So, now, if you have a subspace of an n dimensional space, they will satisfy certain condition that means if you have a subspace, so that satisfies the following conditions.

So, one is that vectors in a subspace of n dimension will also lie in the n dimensional plane. Second, if n = 2 the vectors should also pass through (0, 0) that means origin. The third one the zero vector should always exist. So, that means, if there is in subspace this has to satisfy these conditions, so these are the three conditions that they need to satisfy.

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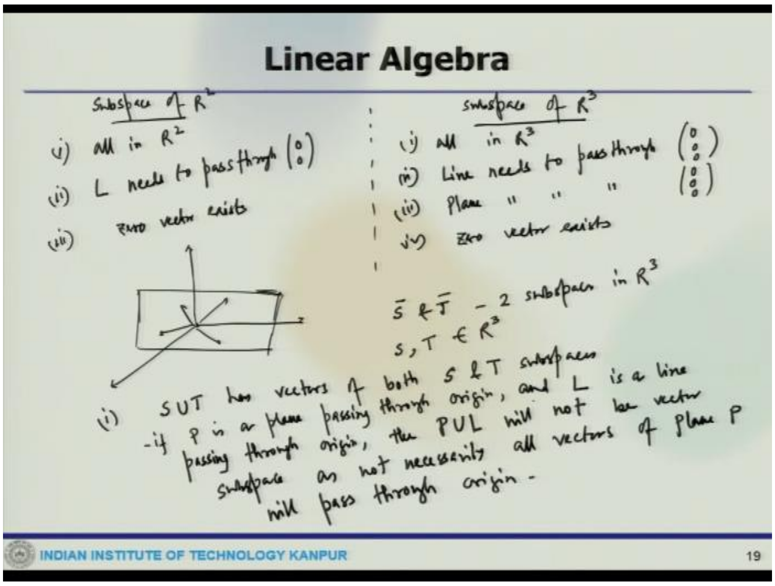

So, if we look at a 2 dimensional subspace or 3 dimensional subspace for example, let us say we have a subspace here of R^2 that means, 2 dimensional subspace we have a subspace here of R^3. So, which means they are all would be in R^2, here all in R^3. Second, L needs to pass through (0, 0) origin, here the line also needs to pass through (0, 0, 0). On top of that, since it is a 3 dimensional subspace, we should have a plane and planes also needs to pass through (0, 0, 0) origin.

Then the third condition here would be zero vector exists, here also the zero vector exists, so what can see kind of a typical scenario. So, there could be vector this direction, this direction, this direction and the plane could be formed like this. So, here the plane pass through the origin so, just like that it is not quite sort of an easy task to visualize that 3 dimensional space or plane in a 2 dimensional system or the pen and paper.

Let us see, let us say we consider two vectors S and T these are two subspaces in R^3. So, that means S, so this S and T belongs to R^3. Now, so what it will do that S union T has vectors of both S and T subspaces. Now, if P is a plane which is passing through origin and L is a line passing through origin then P union L will not be vector subspace as not necessarily all vectors of plane P will pass through origin. So, this is a very interesting statement here that S union T has the vectors of both S and T subspaces.

So, these are two subspaces so, they will have some different you can think about in terms of these are the 2 different matrix which has the column vectors which are forming the subspaces. But, the thing is that if there is a plane called P which is passing through origin and there is a

L which is also passing through origin, but the P union L will not be a vector subspace as not necessarily all the vectors in that will pass through the origin. So, this is very important.

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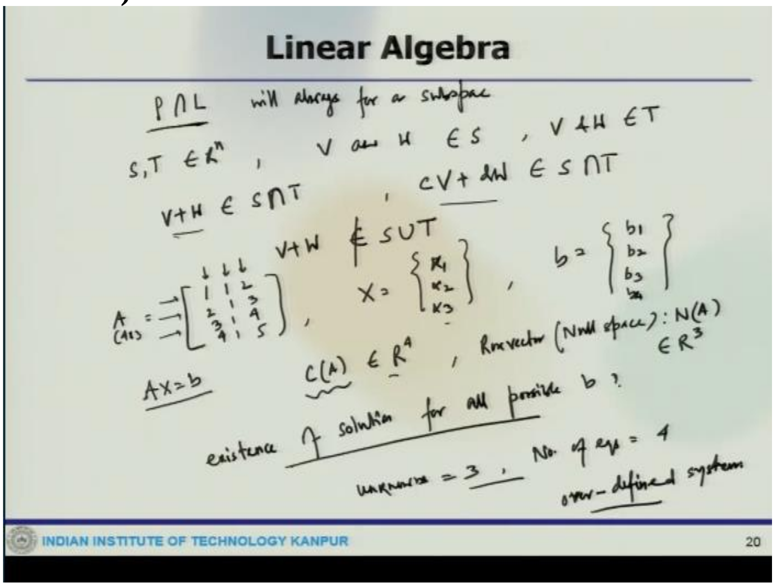

However, P intersection L will always form a subspace. So that is always true, so that union may not be but the intersection always, let us say S T belongs to R^n and V and W the belongs to S and V and W also belongs to T. So, V + W belongs to S intersection T and C V plus let us say dW they belongs to also S intersection T. So, these are the 2 conditions that is satisfied, but V + W does not belong to S union T. So, this is what it means, not necessarily always they are going to be like this.

So, let us see, let us take an example of matrix 4×3, so $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$, x as a vector $X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ and $b = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$. So, what do we are interested in ? We are interested to find out the solution of this Ax = b. Now, this is what we are interested in at the end of the day. Here these are the column vectors and the column vectors they belong to the space of R^4. So, this is what it is the column vectors belong to the R^4 and the row vector which is these vectors or one can say it is a null space.

So, the C(A) here is represented as column space and row vector is represented as null space which is like N(A). So, any matrix A, C(A) is column space in it is null space that belongs to

R^3. Here the important questions arise is the existence of the solution for all possible b. So, that is the important question that has to be sort of answered. Now, here if we look at this particular example, we have 3 vectors which mean to be solved for so, that means 3 unknown and number of equations that we have is 4. So, that means, this is an over defined system where we have less number of unknown, but much higher number of equations.

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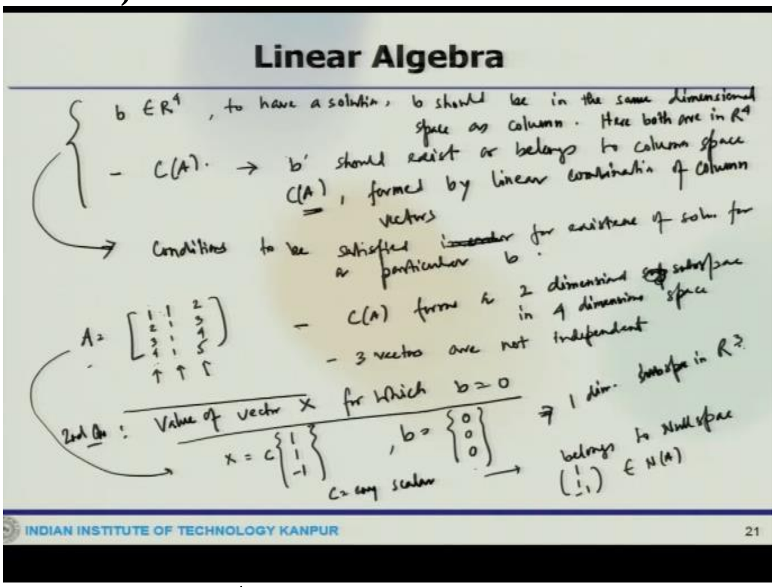

Now, if you see here b belongs to R^4 and to have a solution b should be in the same dimensional space as column here both are in R^4 so, that is fine. Now, again the column vectors are forming a column space which is called as C(A) and the b should exist or belongs to column space that is C(A). And this column space C(A), this is formed by linear combination of column vectors. So, that means for a given matrix, the column space is formed by the column vectors through some linear combination.

Now, to have a solution this b has to lie in this column space and that column space which has been formed by the linear combination of this. Now, so this means, if we put these things in one particular word that condition that needs to be satisfied in order so, these are some of the or you can think about the these are the conditions to be satisfied in order or for existence of solution for a particular b.

Now if you look at that particular example here, so here the example that we have taken, we have $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$, so this is column 1, this is column 2, this is column 3, so these are the column vectors. Now, they are only see, I mean if you see that carefully this column vectors, this column vector C_1 and C_2 if you so, this is the first column are the column vector 1, column vector 2, column vector 3 and if you look at that carefully, this linear combination of 1 and 2 is 3 or that means or rather 1 and 3 is 2 or 2 and 3 is 1.

So, that means the 3 column vectors any 2 of them are sort of independent, but the third one is dependent on the other 2 for any linear combination would work. So, they are forming only so, that means 2 vectors are forming the space not the third one because third one is the linear combination of the 2. So, which means here the C(A) forms a 2 dimensional subspace in 4 dimensional space, so, which means the all 3 vectors are not independent.

So, they are sort of kind of dependent on each other or the other 2, so any single one is dependent on the other 2, so that is first question. Second question which is that the value of vector x for which b is 0, so this is the second question, but what is the value of vector x for which b is 0. Now, here we have A here and we have $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ these are vectors and b could be anything but here we are trying to find out so, for this particular A, what could be this x vector that can give us b which is already $\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$.

And as we can see that one can just look at that mean to visualize and only one can see that, like here is the column 1, column 2 and column 3 they are connected. If we add 1 and 2, so that third one is getting there. And so that means if we add 2 and subtract the third one, this would be 0. So, we can have $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and with a multiplication of a scalar which is c is any scalar. So, this combination is possible a solution, this possible solution for having b 0.

So now, the particular vector which makes the matrix to get in b is zero, is also that also belong to null space. So, here the particular vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ belongs to a null space of A or other way one can say that the vectors which give back the 0 solution belong to null space. Here particularly this is an one dimensional subspace in R^3. So, you can see how these different issues are the questions that need to be answered even to get a solution for Ax = b. So, we will stop the discussion here and continue the discussion in the next session.