Computational Science in Engineering

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Lecture - 30
Numerical Analysis

So, now we have discussed how to find out roots whether simple root, multiple roots or complex roots and like that. Now, with that discussion we are moving to looking at the different situation of linear system of equation or the linear algebra how you can compute those things.

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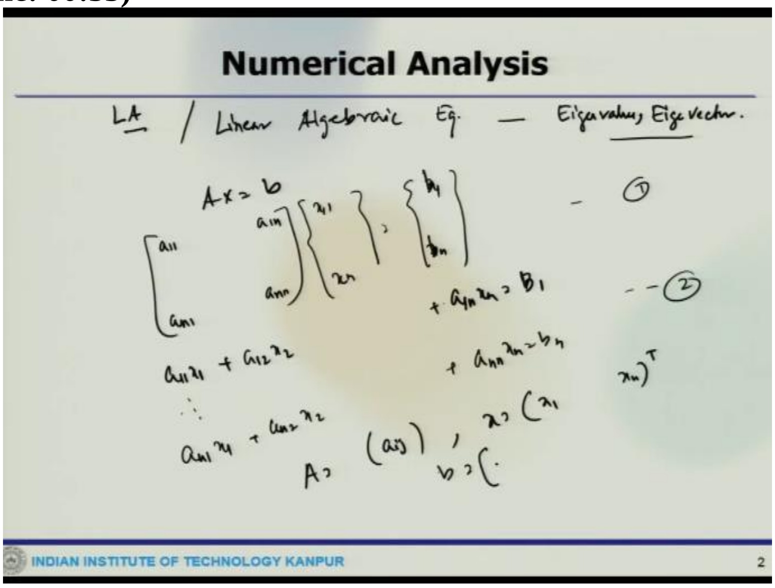

So, let us start with the discussion on linear algebra or linear algebraic equations whatever you call it or linear algebraic equations and then we can have Eigen values, Eigen vectors. Now, if you recall from the discussion on linear algebra, these are the things which we have already talked upon quite a bit of in details like how to find out the Eigen values and Eigen systems and all these things.

And also, we have talked about so many different kinds of matrices, but just to go about it quickly, so what we will do? We will just start with the let us take a system like you have $Ax = b$ here the system of equations, which is going to be like

$$\begin{bmatrix} a_{11} & & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ \\ b_n \end{Bmatrix}$$

so, this will form a linear system. So, just like you can think about like it is writing

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

so, something like that.

So, essentially what you can say, A has all a_{ij}, x has all $(x_1, x_2, \ldots, x_n)^T$ b has all $(b_1, b_2, \ldots, b_n)^T$.

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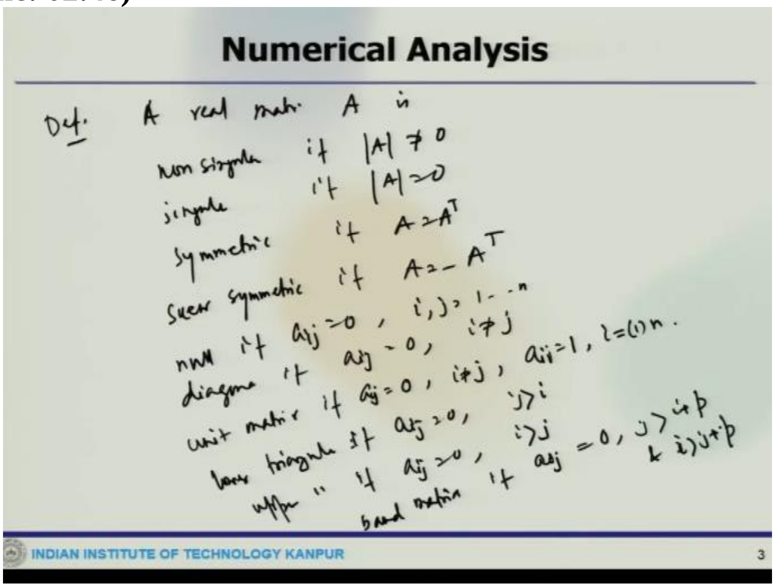

Now, there are some definitions quickly we will say the A real matrix A is non-singular so, this is non-singular if $|A| \neq 0$. This is singular if $|A| = 0$, this is symmetric if $A = A^T$. So, we are just quickly recalling the definitions that we have discussed earlier, this is skew symmetric if $A = -A^T$ this would be null if $a_{ij} = 0$, ij is goes from 1 to n, it is diagonal if $a_{ij} = 0$ for $i \neq j$, it is unit matrix if $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$ for i = 1 to n.

Now, this would be lower triangular so, like if $a_{ij} = 0$ $j > i$ this is upper triangular if $a_{ij} = 0$ for $i > j$. Now, this could be a band matrix if $a_{ij} = 0$ for $j > i + p$ and $i > j + p$ with band width p + q + 1.

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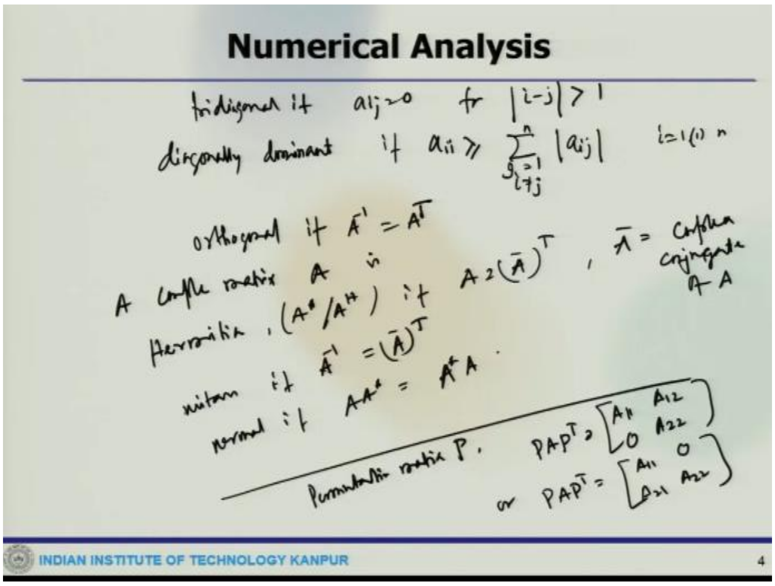

Now, it could be like a tridiagonal if $a_{ij} = 0$ for $|i - j| > 1$, so that is tridiagonal, then it could be diagonally dominant. So, it is if

$$a_{ii} \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$$

where i goes from 1 to n, it is orthogonal if $A^{-1} = A^T$. Now, a complex matrix A is Hermitian which is denoted by A^* or A^H if A is where this is the complex conjugate of A so, that is what it is going to happen.

Now, this is a unitary if $A^{-1} = (\overline{A})^T$, this is normal if $AA^* = A^*A$. Now, the matrix is said to be permutation matrix if it has exactly 1 in each row and column and other matrices are 0. So, that is one permutation matrix, now the other one the matrix A is reducible if there exists a like permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

or

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

are square some matrices.

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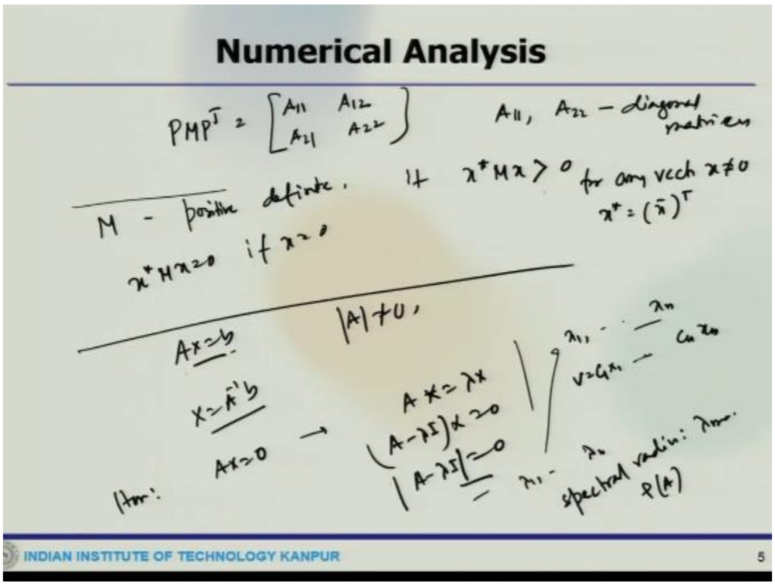

Now, the real matrix M is said to have property A if there exists a permutation matrix P says that

$$PMP^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11}, A_{22} these are diagonal matrix or matrices. Now, a matrix M is positive definite if $x^*Mx > 0$ for any vector x which is not equals to 0 and $x^* = (\bar{x})^T$ further, we can say that $x^*Mx = 0$ if x = 0. Now here A is Hermitian strictly diagonal dominant matrix with positive real diagonal entries when A is positive definite.

So, positive definite matrix has some of the other properties which we have already discussed that it should have certain properties and then if we come back to this $Ax + b$ and the solution exists, if determinant of A is not equals to 0 and also, we have already talked that b has to lie in the column space of A and the solution would can be obtained like

$$x = A^{-1}b$$

like this.

Now, if the system is homogeneous, then the b would be 0 so, then we can find out all the null space vector and all these things. And now, when you consider the homogeneous system which is A = 0. So, we can find out from there the Eigen values lambda like a saying that there is

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

So, system will have non trivial solution for the determinant to be 0. And we can get a polynomial of degree n lambda, which is called the characteristics equation.

And then we find out the roots like $\lambda_1, \lambda_2, \ldots \lambda_n$ and then the largest Eigen value would be the spectral radius. Spectral radius is the lambda max that we have got and which is denoted by $\rho(A)$. And then corresponding to each λ we will get $x_1, x_2, \ldots x_n$ these are called the corresponding Eigen vectors and Eigen values are distinct then if this Eigen values are distinct then these Eigen vectors which we obtain there would be also independent Eigen vectors and they are forming n dimensional space where they can be returned.

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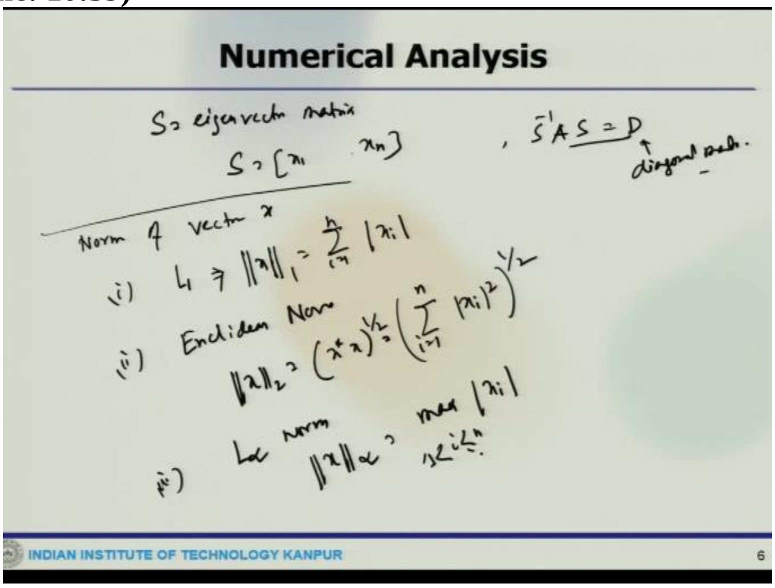

Now, if you have n Eigen values and S denote the matrix of the corresponding Eigen vectors, so, this is an Eigen vector matrix then you can write is

$$S = [x_1 \ \ldots \ x_n]$$

then we can write

$$S^{-1}AS = D$$

where D is the diagonal matrix and Eigen values A are located on the diagonal of D further if S is an orthogonal matrix so, this will result in a true even if the Eigen values are not distinct, but the problem complex system of Eigen vectors.

Now, we can find the norm of vector x like one can find out absolute norm or L_1 norm which is like

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

you can have Euclidean norm like

$$\|x\|_2 = (x^*x)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

and one can have L_∞ norm so, one is a L_1 norm, Euclidean norm or this is called L_2 norm and then we can have L_∞ norm so, which is written as

$$\|x\|_\infty = \max|x_i|$$

where $1 < i < n$.

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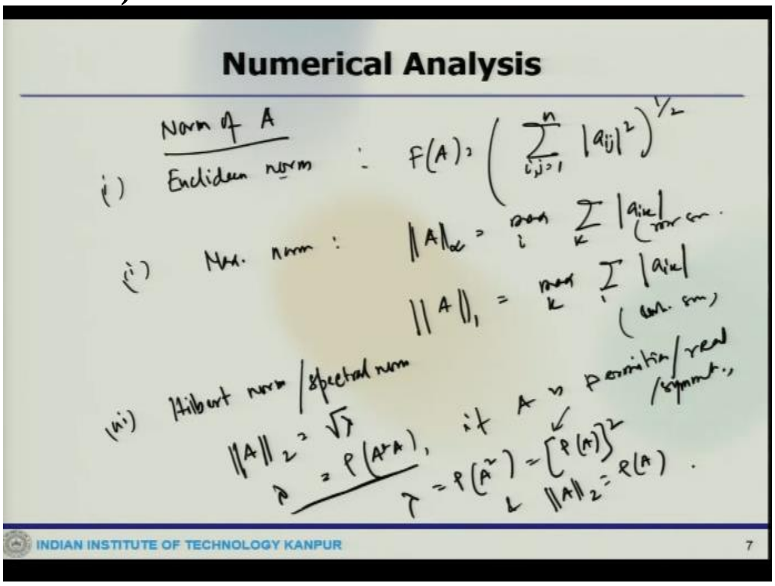

Now, similarly, we can write the norm of A of the matrix A.

i) the Euclidean norm so, we can write that all this is called subvenious norm also, so we write

$$F(A) = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

ii) Then we can have maximum norm that is written as

$$\|A\|_\infty = \max_i \sum_k |a_{ik}|$$

So, this is maximum absolute row sum or one can have

$$\|A\|_1 = \max_k \sum_i |a_{ik}|$$

this is maximum absolute column sum so; this is column sum this is row sum.

iii) Now then we can have Hilbert norm or one can say spectral norm so, that case this would be

$$\|A\|_2 = \sqrt{\lambda}$$

where λ is given at spectral radius of A^*A where if A is Hermitian or real or symmetric. Then we can write

$$\lambda = \rho(A^2) = \rho[A]^2$$

and this is

$$\|A\|_2 = \rho(A)$$

then this falls down to finding the system like this.

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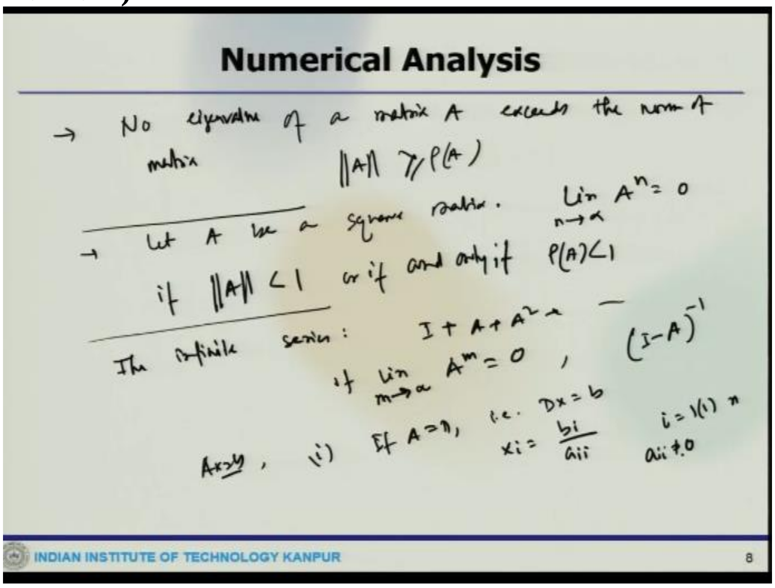

Now, the other thing that is that if the no Eigen value of a matrix A exceeds the norm of the matrix, then what do we get $\|A\| \geq \rho(A)$. So, that is the other thing is that let A be a square matrix so, then what we can write that limit

$$\lim_{n\to\infty} A^n = 0$$

if this $\|A\| < 1$ or if and only if the spectral radius is less than 1. So, that is another criterion the other one is that let us say the infinite series we have $I + A + A^2 + \cdots$ and so on.

These converges if limit

$$\lim_{m\to\infty} A^m = 0$$

the series converges to $(I - A)^{-1}$. Consider now the system which has been already given an equation 1 and 2 that Ax = b so we have like Ax = b so, what we can write, if A is diagonal, then this becomes Dx = b, then the solution would be given like

$$x_i = \frac{b_i}{a_{ii}}$$

where i = 1 to n and obviously, $a_{ii} \neq 0$.

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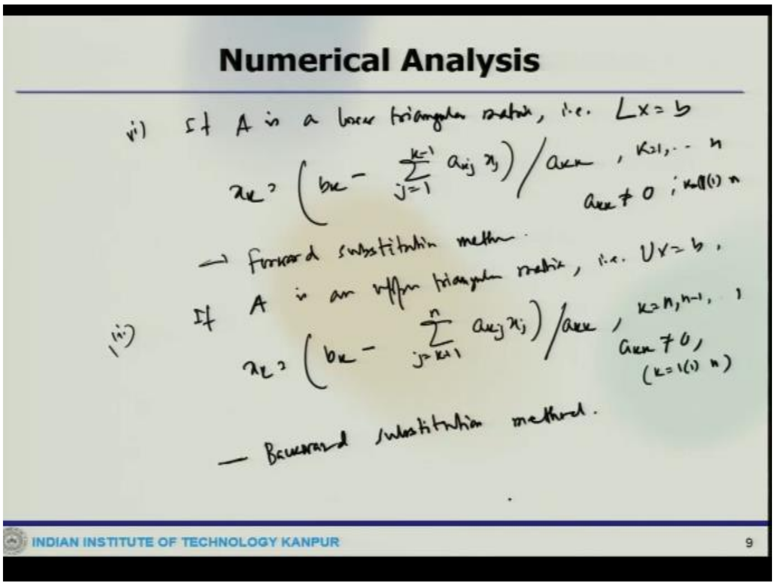

Now, similarly, if A is a lower triangular matrix so, that is one can write Lx = b then the solution one can obtain

$$x_k = \frac{\left(b_k - \sum_{j=k-1}^{n} a_{kj}x_j\right)}{a_{kk}}$$

where k goes from 1 to n obviously, here $a_{kk} \neq 0$ this method is known as the forward substitution method. Now, if A is an upper triangular matrix, then we can write Ux = b and one can find out the solution like

$$x_k = \frac{\left(b_k - \sum_{j=k+1}^{n} a_{kj}x_j\right)}{a_{kk}}$$

Obviously, k goes from n, (n – 1) to 1 and $a_{kk} \neq 0$ where k goes from 1 to n here k goes from also 1 to n. So, this is known as like previously that was forward substitution method. So, this is backward substitution method. So, this is what you get like without the forward substitution method or backward substitution method. Now, these are some of the important definitions that already we have also discussed while talking about linear algebra in details.

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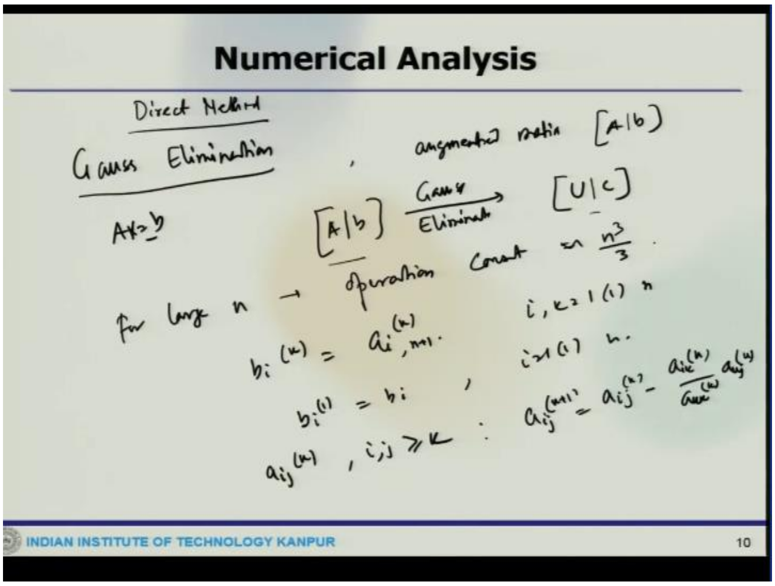

And now we look at essentially how we solve Ax = b so, first we start with some of the, let us say direct methods like first one is the Gauss elimination. So, we have already seen that this is first thing that we have discussed how to do this Gauss elimination and find out the pivot columns and all this to define. So, we have in let us say an augmented matrix like we have A like this so, and the system that we are trying to solve is Ax = b.

Now, we do elementary row operation, so, the gauss elimination method reduces to so, this augmented matrix reduces due to gauss elimination we get upper triangular into C. So, now, we can use these back substitution that already we have talked about, so that find out the solution vector x for large n so, the matrix if the for large n the operation count becomes quite heavy, which is of order of roughly $\frac{n^3}{3}$.

Now, during this process the successive elements after each elimination procedure, which are obtained like

$$b_i^{(k)} = a_{i,(n+1)}^{(k)}$$

where i and k goes 1 to n, which would be b_i where i goes from 1 to n so, this is how you find out all these different matrices, I mean the different elements of that.

So, we can see how these properties that we have discussed and we can use but this is I mean the start with this augmented system and you get this upper triangular system and where you can use back substitution. So, we will complete this discussion so, we will stop it here and complete the discussion in the next session.