Computational Science in Engineering

Prof. Ashoke De
Department of Aerospace Engineering
Indian Institute of Technology, Kanpur

Lecture – 33
Numerical Analysis

So, let us continue the discussion on this numerical analysis part. So, far we have looked at the finding of the roots for polynomials and the equations. And then we have also looked at the different methods that how we can solve, Ax = b system where direct method or iterative method. Now, we know the linear system that comes as an output of your physical system which is represented through some set of governing equations which are ODEs or PDEs and that lead to Ax = b system.

So, before we talk about the ODE and solution and such things like that, what we would like to do? We would like to talk something about interpolations and approximation. So, let us talk on that thing and then before we move to the ODE analysis.

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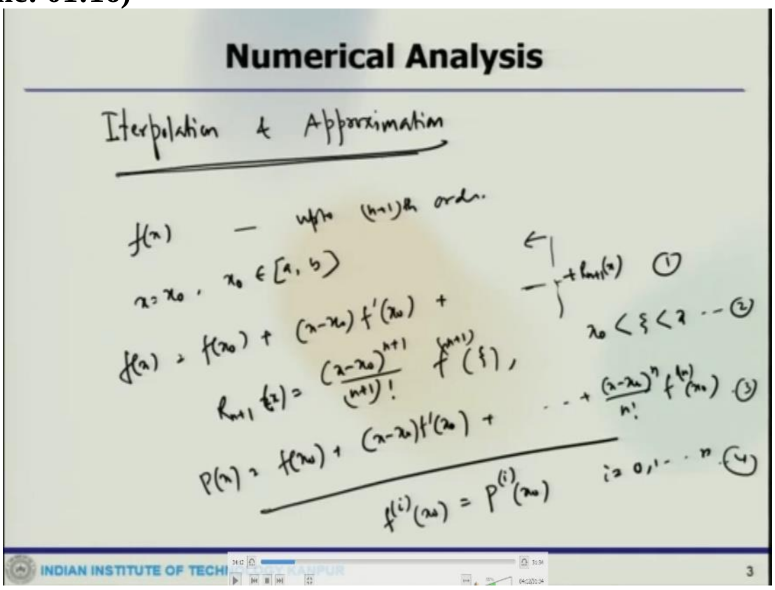

So, what we are going to look at now, we are going to look at some interpolation and approximation techniques which means that, the ways that, how we can interpolate a particular function and then what are the approximation that we use which will lead to some. So, essentially these are connected with some kind of error and other issues which would be kind of interesting to know, for someone who is doing a numerical analysis.

Now, let us say we have a function $f(x)$ which is the continuous derivative up to and including, let us say, up to (n + 1)th order. So, then we can write this Taylor series formula for a particular point around $x = x_0$ which belongs to an interval and which can be written which we have already seen, how we write a Taylor series expansion and like that and so on.

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots$$

And the remainder of that thing would be

$$R_{n+1}(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

where $x_0 < \xi < x$.

So, now, here this particular function, let us say one where neglecting the $R_{n+1}(x)$, so, here we can have $R_{n+1}(x)$. So, $R_{n+1}(x)$, if we neglect that, we can basically, the rest of the term up to this we get a polynomial of degree n which is nothing but

$$P(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^n(x_0)$$

so, this is a polynomial with degree n.

Now, the polynomial maybe interpolating polynomial which is satisfying (n + 1) conditions like

$$f^{(i)}(x_0) = P^{(i)}(x_0)$$

where i goes from 0, 1 to n which are called the interpolating conditions. Now, let us say these are, this could be equation 2 then these are polynomial could be equation 3 then this is equation 4. Now, this condition in 4 may be replaced by some more generic conditions as the values of $P(x)$ and certain other derivatives coincide with the corresponding values of a $f(x)$ within the some sort of a interval.

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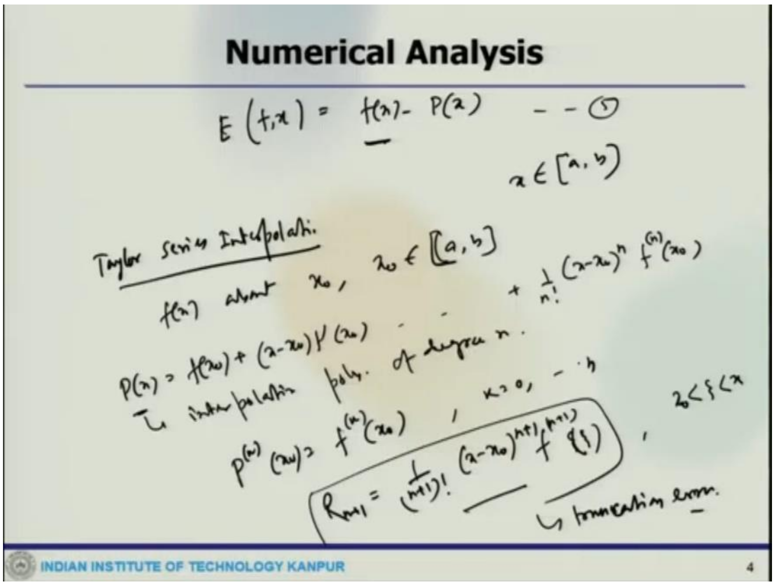

In general, the deviation of the remainder due to the replacement of function $f(x)$ by another function $P(x)$ may be written as, $E(f,x) = f(x) - P(x)$. So, this in approximation we measured the deviation of the given function, $f(x)$ and approximating the function $P(x)$ for all values of x which belongs to a and b in the interval. So, now, we can talk about some of the method for constructing the interpolating polynomials and approximating function for given function, $f(x)$.

Now, one thing is that which can be used as an obviously that Taylor series interpolation which is there. So, which again like the polynomial $P(x)$ is written in Taylor series expansion and for the function $f(x)$ about a point x_0 where x_0 belongs to a and b then, the polynomial can be written as

$$P(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0)$$

Now, the $P(x)$ may be regarded as an interpolating polynomial of degree n. So, this is the sort of interpolating polynomial of degree n and satisfying the condition that

$$P^{(k)}(x_0) = f^{(k)}(x_0)$$

where, K goes from 0 to n and the term which is remainder

$$R_{n+1}(x) = \frac{1}{(n+1)!}(x - x_0)^{n+1} f^{(n+1)}(\xi)$$

where $x_0 < \xi < x$. So, which usually has been neglected in the Taylor series is called the remainder of the truncation error.

So, this term actually lead to the truncation error. So, this already we have talked about while talking about the error of the numerical methods and all these. So, the number of the terms included in the Taylor series expression may be determined by the acceptable error.

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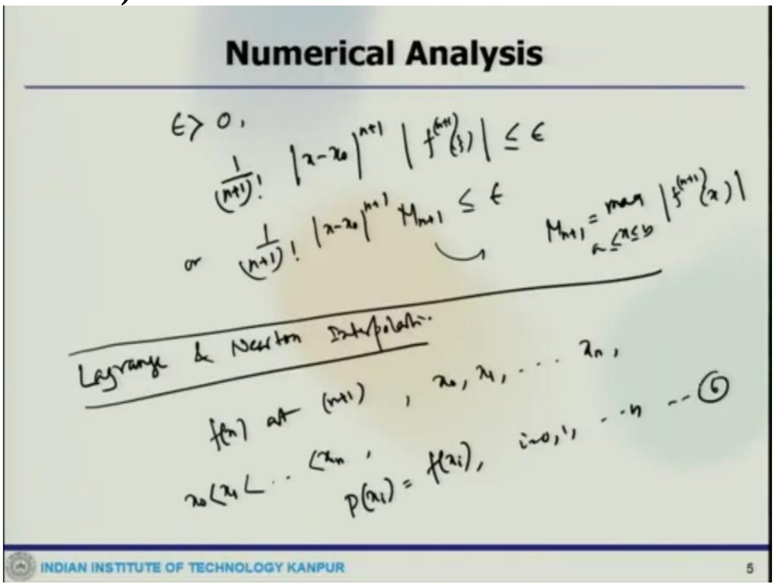

And in this error, epsilon which is a small number greater than 0 and the series is truncated at the time $f^{(n)}(x_0)$ then we can say that

$$\frac{1}{(n+1)!}|x-x_0|^{n+1}\left|f^{(n+1)}(\xi)\right| \leq \epsilon$$

or one can say that

$$\frac{1}{(n+1)!}|x-x_0|^{n+1}M_{n+1} \leq \epsilon$$

where this guy $M_{n+1} = \max_{a\leq x\leq b}\left|f^{(n+1)}(x)\right|$. So, assume that the value of in M_{n+1} or its estimate is available for a given epsilon. we can determine n and if n and x are prescribed, we can determine epsilon.

When both n and epsilon are given, we can find the upper bound on $(x - x_0)$ that is what it will give an interval about x_0, in which this Taylors polynomial approximate affects to the prescribed activities. So, depending on what is available we can do that. Now, the other one which would be interesting to look at is that Lagrange interpolation and Lagrange and Newton interpolations.

So, given a value of a function $f(x)$ at (n + 1) distinct point like, $x_0, x_1, x_2, \ldots x_n$, such that $x_0 < x_1 < x_2 < \cdots < x_n$. We can determine a unique polynomial $P(x)$ of degree n which satisfy a condition like

$$P(x_i) = f(x_i)$$

where i goes from 0, 1 to n.

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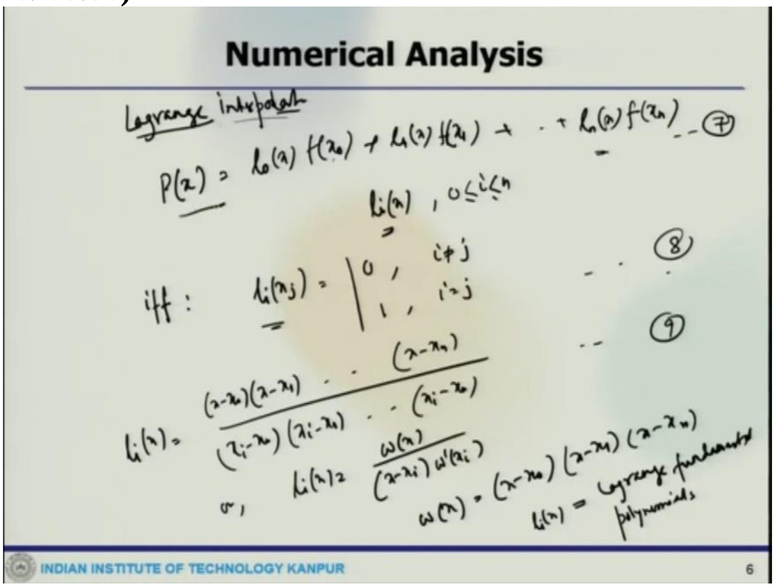

Now when we talk about Lagrange polynomial, a Lagrange interpolation, so what we are going to say that the maximum degree of the polynomial, satisfying the n plus, let us say, this is 6, satisfying the conditions in equation 6 will be n. So, we assume the polynomial $P(x)$ is in the form

$$P(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \cdots + l_n(x)f(x_n)$$

where $l_i(x)$, which lies $0 \le i \le n$, are polynomial of degree n, so, this is 7.

So, the polynomial which is given in equation 7 will satisfy the interpolating conditions of equation 6 here and so, if and only if $l_i(x_j)$ is 0 for $i \neq j$ and 1 for $i = j$, so this is the situation. Now, the polynomial $l_i(x)$ satisfying the conditions at equation 8, also can be written as

$$l_i(x) = \frac{(x - x_0)(x - x_1) \ldots (x - x_n)}{(x_i - x_0)(x_i - x_1) \ldots (x_i - x_n)}$$

which is 9 or one can say

$$l_i(x) = \frac{\omega(x)}{(x - x_i)\omega'(x_i)}$$

where, $\omega(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$.

Now, the function $l_i(x)$ where i goes from 0 to n are called the Lagrange fundamental polynomial. So, this $l_i(x)$ is called the Lagrange fundamental polynomial. So, what we get here, this one in the equation 7 this $P(x)$, this is called a Lagrange interpolation polynomial.

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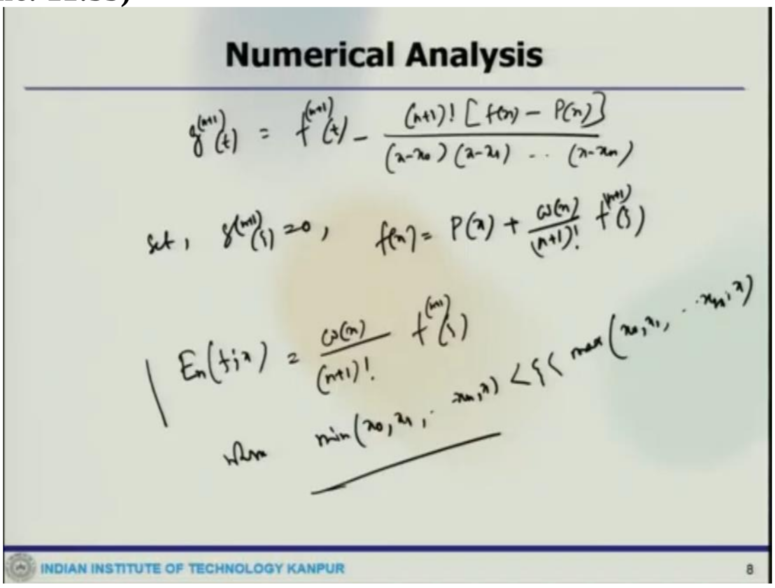

So, in equation 7 that is Lagrange interpolation polynomial, so, the truncation error in the Lagrange interpolation is given as $E_n(f, x) = f(x) - P(x)$, since $E_n(f, x) = 0$ at $x = \xi$ where i goes from 0 to n and x belongs to a and b. So, we can define a function $g(t)$.

Now when we apply the Rolle's theorem repeatedly, we can obtain that $g^{(n+1)}(\xi) = 0$ where ξ is some points as that,

$$\min(x_0, x_1, x_2, \ldots x_n, x) < \xi < \max(x_0, x_1, x_2, \ldots x_n, x)$$

Now, if we differentiate $g(t)$, (n + 1) times with respect to t, what we can write is that

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \frac{(n+1)!\,[f(x) - P(x)]}{(x - x_0)(x - x_1) \ldots (x - x_n)}$$

and one we can set

$$g^{(n+1)}(\xi) = 0$$

then we can solve for

$$f(x) = P(x) + \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi)$$

So, the truncation error in Lagrange interpolation is given as

$$E_n(f, x) = \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\xi)$$

for

$$\min(x_0, x_1, x_2, \dots x_n, x) < \xi < \max(x_0, x_1, x_2, \dots x_n, x)$$

So, this is what we get as a truncation error in the Lagrange interpolation. Now, similarly, what we can get in that?

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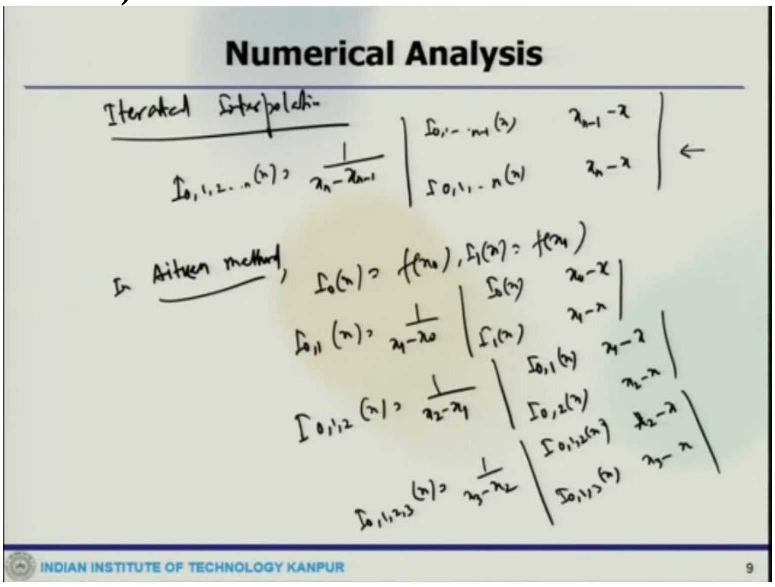

Like iterated interpolation also we can see what is that, iterated interpolation what is essentially the iterated form of the Lagrange polynomial and one can write

$$I_{0,1,2,\dots,n}(x) = \frac{1}{x_n - x_{n-1}} \begin{vmatrix} I_{0,1,2,\dots,n-1}(x) & x_{n-1} - x_n \\ I_{0,1,2,\dots,n}(x) & x_n - x \end{vmatrix}$$

The interpolating polynomials appearing on this equation, particular equation here are any 2 independent (n – 1) degree polynomial which could be constructed in number of ways.

So that is where we can get, I mean like in Aitken method, we construct the successive iterated polynomials like in Aitken method. So, interpolation is identical with the Lagrange interpolation polynomial, but it is much simpler to construct.

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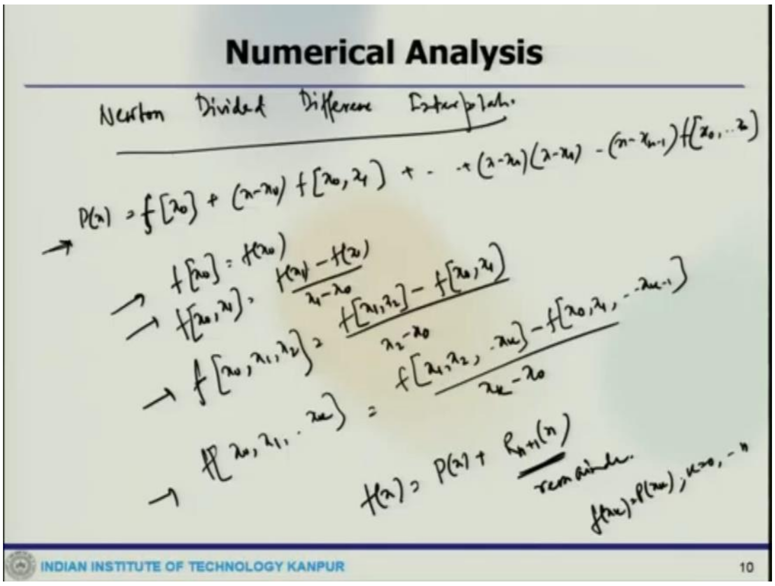

Now, we can also write or look at like, Newton divided difference interpolation. So, an interpolation polynomial which is satisfying the condition which is given in equation 6 here, one can write like

$$P(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})f[x_0, x_1, \ldots x_n]$$

where

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f[x_0, x_1, x_2, \ldots x_k] = \frac{f(x_1, x_2, \ldots x_k) - f(x_0, x_1, x_2, \ldots x_k)}{x_k - x_0}$$

So, these are the 0th, first, second and sort of Kth order divided differences which are written in this. The polynomial given here is called the Newton divided difference interpolation polynomial and the function effects can be written as

$$f(x) = P(x) + R_{n+1}(x)$$

and $R_{n+1}(x)$ is the remainder which is there. Since $P(x)$ is a polynomial of degree n which satisfy the condition like

$$f(x_k) = P(x_k)$$

for K = 0 to n, the reminder R_{n+1} vanishes at x = K.

So, now, one can note that the interpolation polynomial which is satisfying the condition given in equation 6 earlier, is unique and the polynomial. Therefore, what we can do the given this polynomial px here, must be identical with the Lagrange interpolation polynomial.

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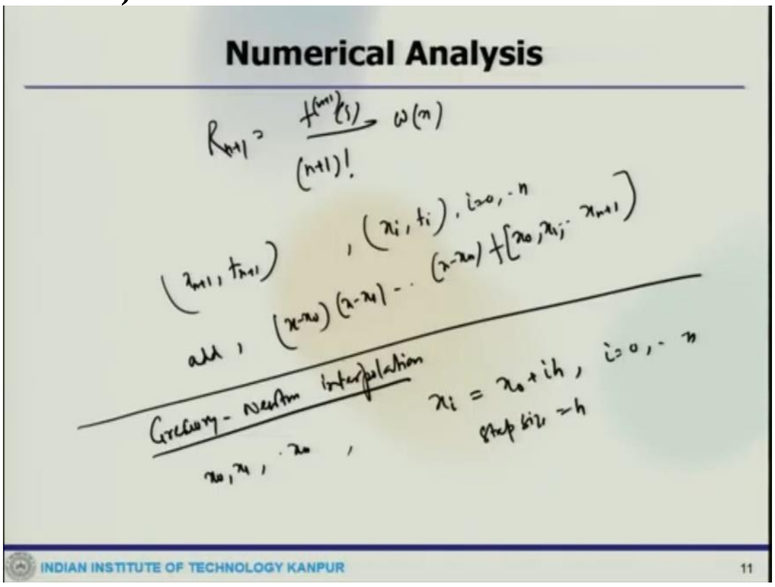

So, we can write this reminder as

$$R_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega(x)$$

So, when a data item is added at the beginning and the end of the tabular data and if it is possible to derive an interpolating polynomial by adding one more term to the previously calculated interpolating polynomial then, such an interpolating polynomial is a process performance property. Obviously, Lagrange interpolating polynomial does not possess this property. Interpolating polynomial based on divided differences has the performance property.

If one more data item, let us say (x_{n+1}, f_{n+1}) is added to the given data of (x_i, f_i) where i goes from 0 to n then in case of Newton's divided difference formula, we need to add a term $(x - x_0)(x - x_1) \dots (x - x_n) f[x_0, x_1, x_2, \dots x_{n+1}]$. So, this will be added to the previously nth degree interpolation polynomial. Now, we can move to some like, Gregory Newton interpolation, so assume that the tabular points, let us say, $x_0, x_1, x_2, \dots x_n$ are equally spaced then anytime we can write $x_i = x_0 + ih$ where i goes from 0 to n. The step size is h. So, given that step size, so, we can have these finite difference operators.

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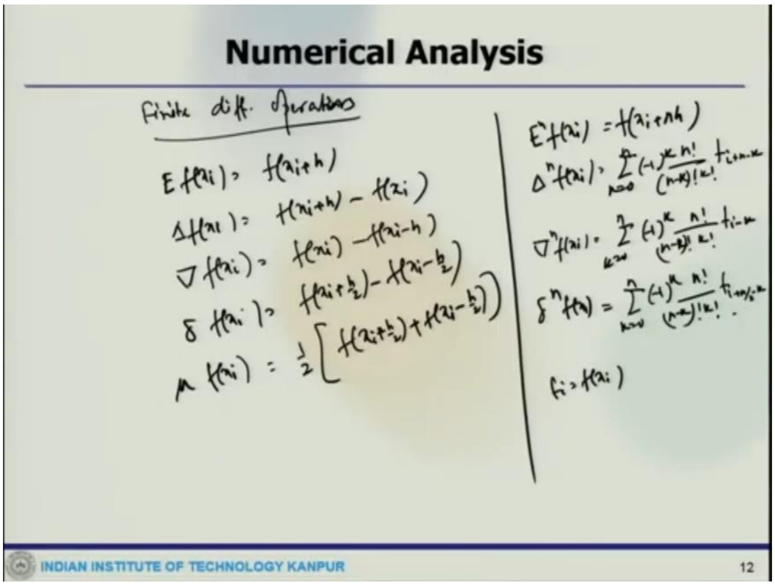

So, we can define this finite difference operators like

$$Ef(x_i) = f(x_i + h)$$

this is the shift operator then

$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$

this is the forward difference operator

$$\nabla f(x_i) = f(x_i) - f(x_i - h)$$

this is a backward difference operator. Then we can write

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

this is a central difference operator and

$$\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)\right]$$

this is the averaging operator.

So, the repeated application of the difference operator if you can get the higher order differences like, similarly one can write

$$E^n f(x_i) = f(x_i + nh)$$

$$\Delta^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!\,k!} f_{i+n-k}$$

then you have

$$\nabla^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!\,k!} f_{i-k}$$

And

$$\delta^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!\,k!} f_{i+\frac{n}{2}-k}$$

where $f_i = f(x_i)$. Now, you can also write some other intermediate these things also.

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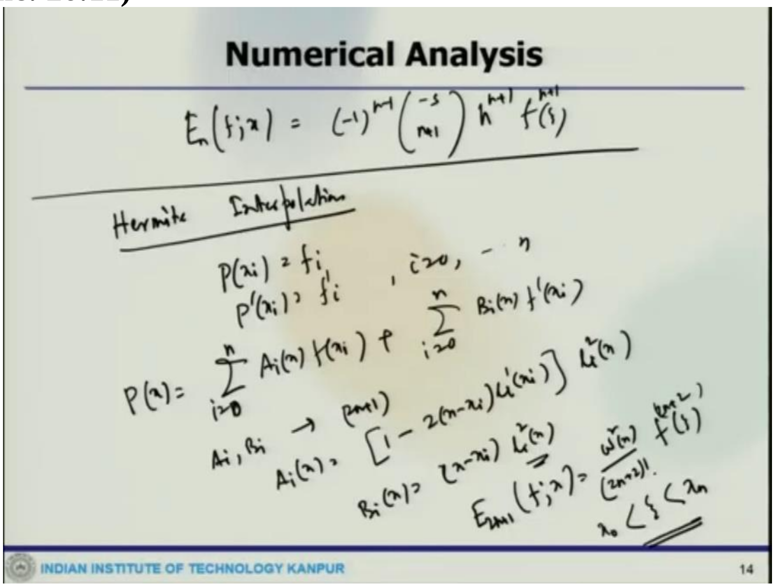

Now, write Gregory Newton forward difference interpolation, so the repeated divided interpolation that we had, we can replace that with the forward differences. So, similarly, this is forward and if you write the same thing with the backward then one can write for backward differencing.

And that case the error would be in the backward case the error would be,

$$E_n(f;x) = (-1)^{n-1} \binom{-s}{n+1} h^{n+1} f^{(n+1)}(\xi)$$

So, this would be the error for the backward case. Now, we can also look at Hermite interpolation.

We can determine the unique polynomial of degree which is less than (2 n + 1) which satisfied that $P(x_i) = f_i$ and $P'(x_i) = f'_i$ where i goes from 0 to n. So, the required polynomial is written as

$$P(x) = \sum_{i=0}^{n} A_i(x)\ f(x_i) + \sum_{i=0}^{n} B_i(x)\ f'(x_i)$$

Where A_i, B_i are polynomial of degree, so, these are polynomial of degree and (2 n + 1) which are given that

$$A_i(x) = [1 - 2(x - x_i)l'_i(x_i)]{l_i}^2(x)$$

So, $l_i(x)$ is the Lagrange polynomial, fundamental polynomial and this case the error would be

$$E_{2n+1}(f; x) = \frac{\omega^2(x)}{(2n+2)!} f^{(2n+2)}(\xi)$$

This is the function ξ where $x_0 < \xi < x_n$. So, this is how you do the Hermite polynomial. Now, there are other interpolation techniques also and we will stop here today and discuss the other interpretation technique in the next session.