Computational Science in Engineering

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Lecture – 34
Numerical Analysis

So, let us continue the discussion on interpolation. So, we are talking about different kinds of interpolation for function. And we have looked at a few different ways. Now we are going to talk about the other one now.

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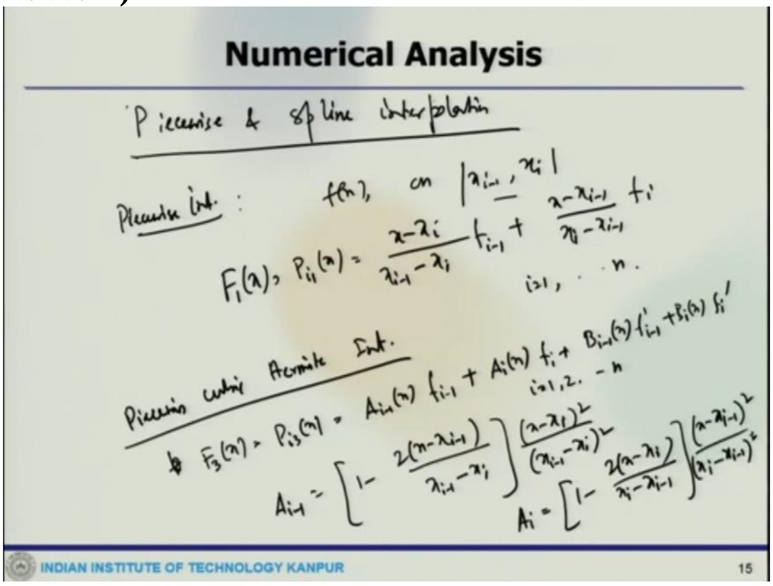

So, what we are going to talk about now the Piecewise and Spline interpolation, so which we are going to like the other way of doing things. So, for example, piecewise and spline interpolation, so, now in order to keep the degree of the interpolating polynomial small and also to obtain accurate results, we use piecewise interpolation. So, now for piecewise interpolation, what we do? Let us say, we replace $f(x)$ on $|x_{i-1}, x_i|$ with the Lagrange linear polynomial, so that we can write

$$F_1(x) = P_{il}(x) = \frac{x - x_i}{x_{i-1} - x_i} f_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} f_i$$

i goes from 1 to n.

Now similarly, the Piecewise Cubic Hermite, so, if somebody write piecewise cubic Hermite interpolation, so, in that case, we have the values of $f(x)$ and $f'(x)$ which are given at point $x_0, x_1, x_2, \dots x_n$ then we can replace the function $f(x)$ within like this interval, like $F_3(x)$ is a cubic Hermite interpolation polynomial.

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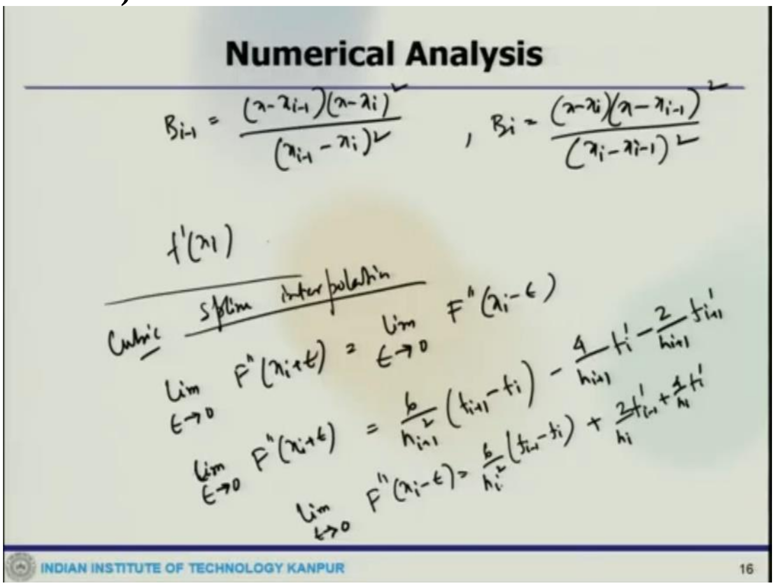

And we get

$$B_{i-1} = \frac{(x - x_{i-1})(x - x_i)^2}{(x_{i-1} - x_i)^2}$$

and

$$B_i = \frac{(x - x_i)(x - x_{i-1})^2}{(x_i - x_{i-1})^2}$$

So, we note that this piecewise cubic Hermite interpolation requires prior knowledge of $f'(x_i)$ for i 1 to n and also if we only use a f_i the resulting piecewise cubic polynomial still interpolate $f(x)$ at x_0 and all this. Since $P_3(x)$ is twice continuously differentiable on a and b we determine M_i is using this condition. Such an interpolation is called the spline interpolation.

So, now we will look at this cubic spline interpolation, so we assume that continuity of second derivative which is already given here. So, what we can write then that

$$\lim_{\epsilon \to 0} F''(x_i + \epsilon) = \lim_{\epsilon \to 0} F''(x_i - \epsilon)$$

and what we can get that

$$\lim_{\epsilon \to 0} F''(x_i + \epsilon) = \frac{6}{{h_{i+1}}^2}(f_{i+1} - f_i) - \frac{4}{h_{i+1}} f_i' - \frac{2}{h_{i+1}} f_{i+1}'$$

So, we can say that,

$$\lim_{\epsilon \to 0} F''(x_i + \epsilon) = \frac{6}{{h_i}^2}(f_{i-1} - f_i) + \frac{2}{h_i} f_{i-1}' + \frac{4}{h_i} f_i'$$

So, if we equate the right hand side what we get?

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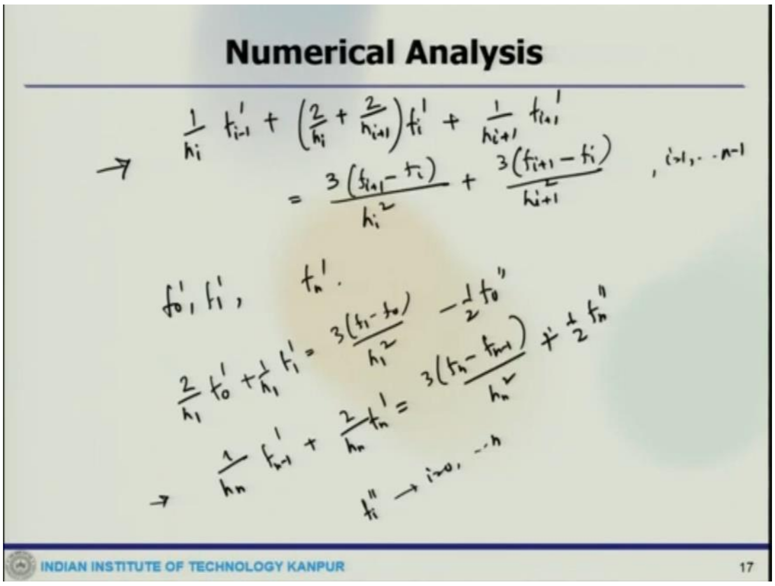

That,

$$\frac{1}{h_i}f'_{i-1} + \left(\frac{2}{h_i} + \frac{2}{h_{i+1}}\right)f'_i + \frac{1}{h_{i+1}}f'_{i+1} = \frac{3(f_{i-1} - f_i)}{{h_i}^2} + \frac{3(f_{i+1} - f_i)}{{h_{i+1}}^2}$$

where i goes from 1 to (n – 1). So, these are (n – 1) equation and (n + 1) unknowns. So, $f'_0, f'_1, f'_2, \dots . f'_n$, so, these are the unknowns. But if not double prime or f''_n are prescribed then from these equations what we get is that

$$\frac{2}{h_1}f'_0 + \frac{1}{h_1}f'_1 = \frac{3(f_1 - f_0)}{{h_1}^2} - \frac{1}{2}f''_0$$

Similarly,

$$\frac{1}{h_n}f'_{n-1} + \frac{2}{h_n}f'_n = \frac{3(f_n - f_{n-1})}{{h_n}^2} + \frac{1}{2}f''_n$$

Here, the derivative of f'_1 which goes from, i goes from 0 to n are determined by solving these equations and then we can determine other things. Now, for equi-spaced points, so this particular equation here and this one, they will now, we can replace that thing.

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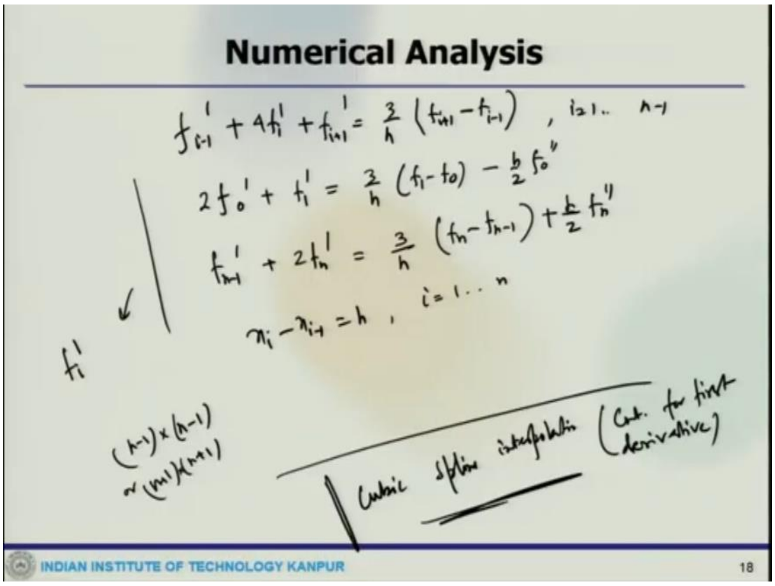

With the equi-spaced system, so we can write,

$$f'_{i-1} + 4f'_i + f'_{i+1} = \frac{3}{h}(f_{i+1} - f_{i-1})$$

where i goes from 1 to (n – 1) and

$$2f'_0 + f'_1 = \frac{3}{h}(f_1 - f_0) - \frac{h}{2}f''_0$$

$$f'_{n-1} + 2f'_n = \frac{3}{h}(f_n - f_{n-1}) + \frac{h}{2}f''_n$$

where we have, $x_i - x_{i-1} = h$ where, i goes from 1 to n. So, this procedure here will give us the values of a f'_i.

So, this actually gives us f'_i. Now, this one we substitute in the piecewise cubic interpolating polynomial, we obtain the required cubic spline polynomial. Also, one has to note that here, one has to solve only $(n-1) \times (n-1)$ or $(n+1) \times (n+1)$, tri diagonal system for equation f'_i. So, this method is computationally much less expensive than the direct method.

So, this is one of the biggest advantages of this particular method. So, similarly, here what we have talked about here is that, this is for continuity of second derivative. Now similarly, one can find out this cubic spline interpolation for fast derivative also. So, that one can look at some of the textbook and find out that thing. So, essentially the point here is that you can also get like cubic spline interval interpolation for continuity for first derivative, so that is what is there. So, this can be, one can find that out too that, what would be that thing?

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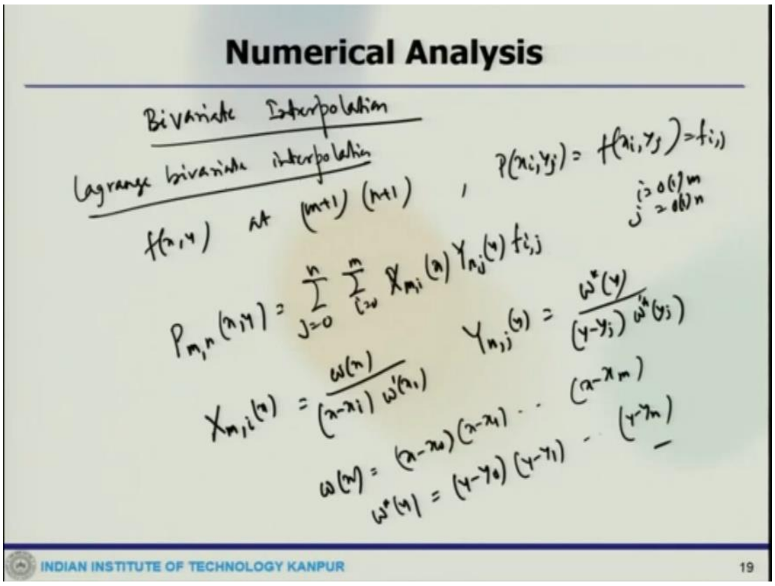

So, now, we look at some other thing like bivariate interpolation. So, here first thing that we will look at is the Lagrange bivariate interpolation. So, if the values of the function $f(x, y)$ at $(m+1)$ and $(n+1)$ distinct point are given then the polynomial

$$P(x_i, y_j) = f(x_i, y_j) = f_{i,j}$$

where, i goes from 0 to m, j goes from 0 to n, so which could be given as the

$$P_{m,n}(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{m} X_{m,i}(x) Y_{n,j}(x)\, f_{i,j}$$

Now, here we have

$$X_{m,i}(x) = \frac{\omega(x)}{(x - x_i)\omega'(x_i)}$$

And

$$Y_{n,j}(y) = \frac{\omega^*(y)}{(y - y_j)\omega^{*\prime}(y_j)}$$

So, where

$$\omega(x) = (x - x_0)(x - x_1) \dots (x - x_m)$$

And

$$\omega^*(y) = (y - y_0)(y - y_1) \dots (y - y_n)$$

so this is how one can. Similarly, I mean, there are Newton's bivariate interpolations also one can find out. Now, there are other, I mean, these are different kinds of interpolation that one can do and different techniques are used for different kinds of system depending on what is user requirement.

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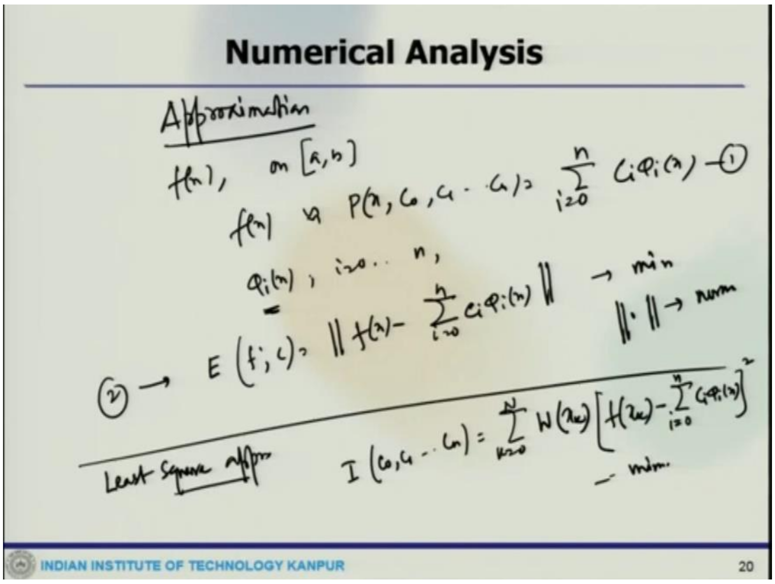

So, we will move to a little bit of talking about approximation. So, in that will, let us say we approximate a given function $f(x)$ on a and b by which has the form like

$$f(x) = P(x, c_0, c_1, \dots, c_n) = \sum_{i=0}^{n} c_i \phi_i(x)$$

and this $\phi_i(x)$ goes from i = 0 to n or (n + 1) approximately chosen, linearly independent function. So, these are (n + 1) linearly independent function and $c_0, c_1, ..$, these are the parameters to be defined such that E which is defined

$$E(f; c) = \left\| f(x) - \sum_{i=0}^{n} c_i \phi_i(x) \right\|$$

So, this is minimum where, this guy is the norm which is well defined. By using these different norms, we can find the different type of actual approximation like once a particular norm is chosen, there is a function chosen, the function which minimizes the error here, I mean, let us say, this is function 1 and this is 2, called the best approximation and the function $\phi_i(x)$ is called a coordinate function.

Now, one such thing is that one can say is that, you have like, least square approximation, so what you do here? We determine the parameters c_0 to c_n, such that

$$I(c_0, c_1, \dots, c_n) = \sum_{k=0}^{N} W(x_k) \left[f(x_k) - \sum_{i=0}^{n} c_i \phi_i(x) \right]^2$$

so that is minimum. So that is what we get. Here the values $f(x)$ are given at (n + 1) distinct point.

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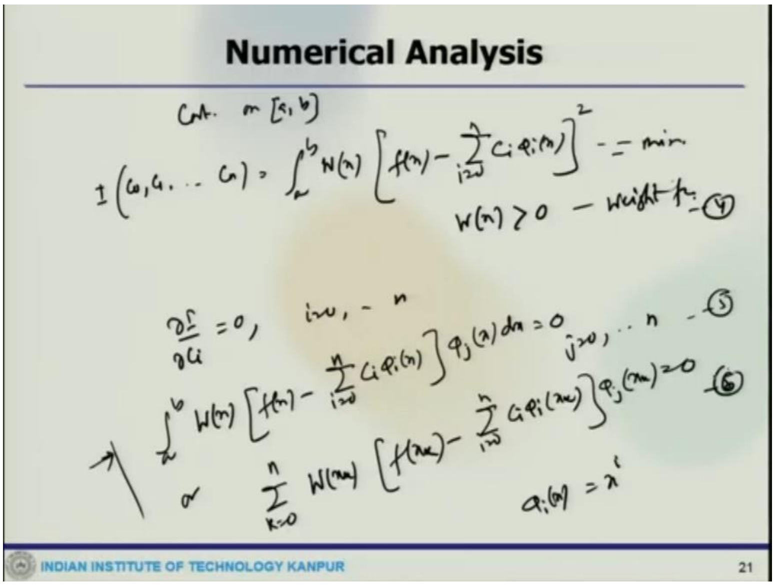

Now, for function which are continuous on a and b, we determined

$$I(\ c_0, c_1, \ldots, c_n) = \int_a^b W(x)\left[f(x) - \sum_{i=0}^{n} c_i \phi_i(x)\right]^2$$

This is also minimum where $W(x) > 0$ is the weight function, so this is the weight function. The necessary condition for, let us say, we say this is equation 3 and this is 4. So, the necessary condition for 3 and 4 to have a minimum value is that,

$$\frac{\partial I}{\partial c_i} = 0$$

for i goes to 0, 1 to n.

So, which gives a system of (n + 1) linear equation and where we have (n + 1) unknown and which takes the form like

$$\int_a^b W(x)\left[f(x) - \sum_{i=0}^{n} c_i \phi_i(x)\right] \phi_j(x) dx = 0$$

where j goes from 0 to n. So, the equations are called the normal equations these equations, I mean, either this or one can write in this equation like that

$$\sum_{k=0}^{N} W(x_k)\left[f(x_k) - \sum_{i=0}^{n} c_i \phi_i(x_k)\right] \phi_j(x_k) = 0$$

So, these are called the normal equations.

Now, for larger normal equation become ill condition where $\phi_i(x) = x^i$ and this difficulty can be avoided if the function $\phi_i(x)$ are so chosen that, they are orthogonal with respect to the weight function over on a and b.

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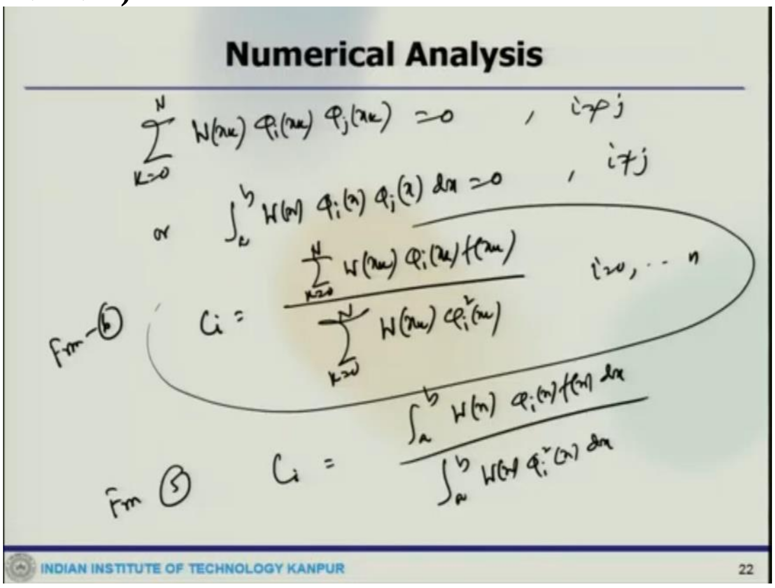

Such that what we write that

$$\sum_{k=0}^{N} W(x_k)\phi_i(x_k)\phi_j(x_k) = 0$$

i not equal to j or one can say

$$\int_a^b W(x)\phi_i(x)\phi_j(x)dx = 0$$

where, i not equal to j. So, if the function $\phi_i(x)$ is orthogonal then, what we can get? That

$$C_i = \frac{\sum_{k=0}^{N} W(x_k)\phi_i(x_k)\phi_j(x_k)}{\sum_{k=0}^{N} W(x_k){\phi_i}^2(x_k)}$$

where i goes from 0 to n and from other expression like from this one we can get from this guy. So, let us say this is 5, this is 6, so, this is from 6 now, from 5 what do we get

$$C_i = \frac{\int_a^b W(x)\phi_i(x)f(x)dx}{\int_a^b W(x){\phi_i}^2(x)dx}$$

so now, this is what I mean, you get on this.

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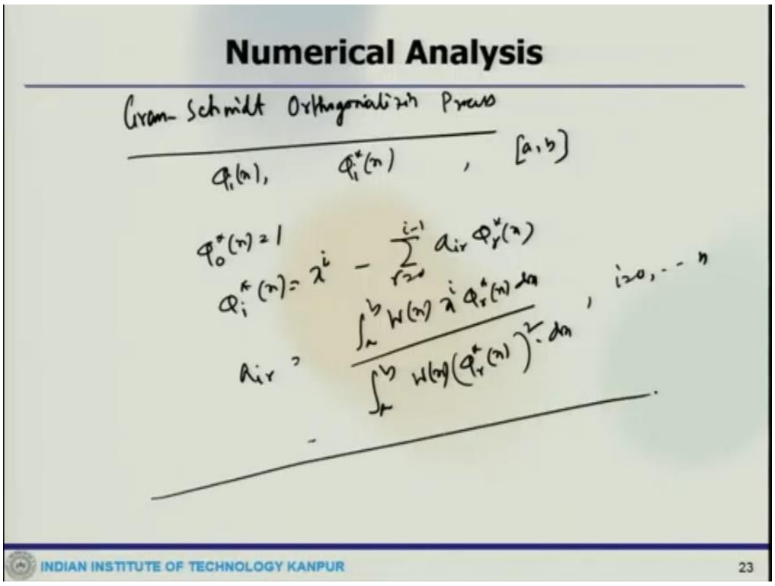

Now, we can look at in Gram Schmidt orthogonalization process which is also you have talked about this. So, given a function $\phi_i(x)$ the polynomial ${\phi_i}^*(x)$ orthogonal on a and b, then we can find out we have like, ${\phi_0}^*(x) = 1$,

$${\phi_i}^*(x) = x^i - \sum_{r=0}^{i-1} a_{ir}\, {\phi_r}^*(x)$$

where,

$$a_{ir} = \frac{\int_a^b W(x) x^i {\phi_r}^*(x) dx}{\int_a^b W(x) \left({\phi_r}^*(x)\right)^2 dx}$$

where, i goes from 0 to n. So, what a discrete set of points we replace the integrals by this essentially the summation.

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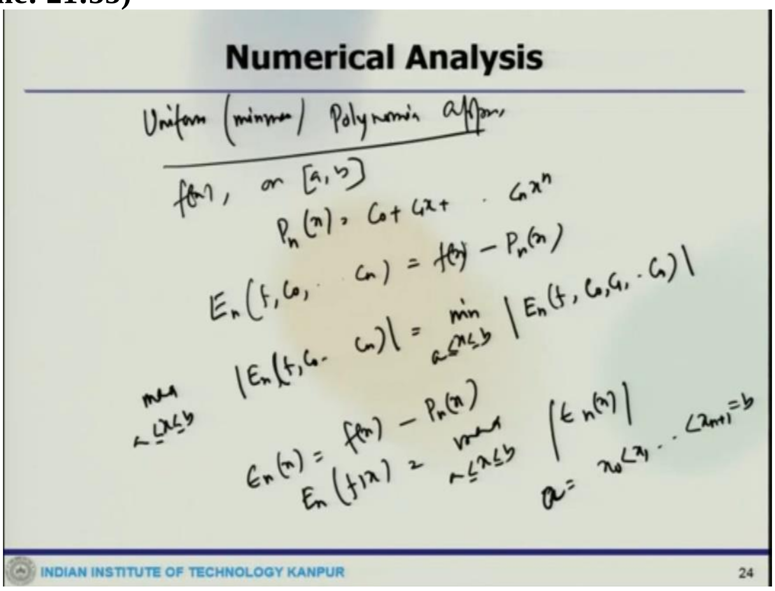

Now, another thing which is possibly can happen is that uniform or minmax polynomial approximation, so that is another thing. So, taking that approximating polynomial for a continuous function of $f(x)$ on a and b in the form like

$$P_n(x) = C_0 + C_1 x + \cdots + C_n x^n$$

So, we determined $C_0, C_1, \ldots C_n$ such that the derivation which is

$$E_n(f, C_0, C_1, \ldots C_n) = f(x) - P_n(x)$$

which satisfies this condition that

$$\max_{a<x<b} |E_n(f, C_0, C_1, \ldots C_n)| = \min_{a<x<b} |E_n(f, C_0, C_1, \ldots C_n)|$$

So, what if we denote

$$\epsilon_n(x) = f(x) - P_n(x)$$

and

$$E_n(f, x) = \max_{a<x<b} |\epsilon_n(x)|$$

then, there are at least (n + 2) points like in $a = x_0 < x_1 < \cdots < x_{n+1} = b$which would be where Chebyshev equi-oscillation theorem says.

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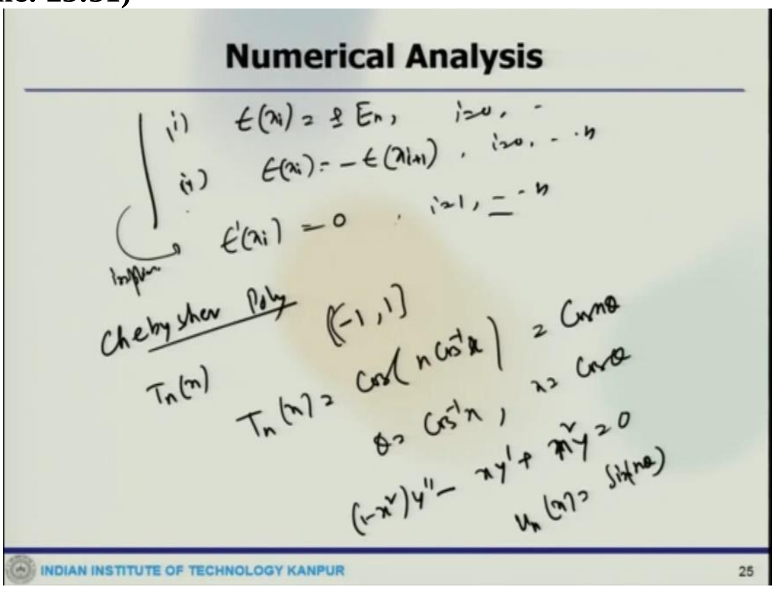

That if

i) $\epsilon(x_i) = \pm E_n$ where, i goes to 0 to n

ii) $\epsilon(x_i) = -\epsilon(x_{i+1})$ where, i = 0 to n.

So, the best uniform or minimax polynomial approximation is uniquely determined under the conditions of these 2. It may be said that, I mean, it may be the second one actually implies that, $\epsilon'^{(x_i)} = 0$ for i = 1 to n. So, this is what it implies. So, this is what another thing that one can get is that Chebyshev polynomial.

So, the Chebyshev polynomial of first kind $T_n(x)$ which is defined on [-1, 1] which can be written as that

$$T_n(x) = \cos(ncos^{-1}x) = \cos n\theta$$

Or

$$\theta = cos^{-1}x$$

Or

$$x = \cos\theta$$

So, this polynomial satisfies the differential equation

$$(1 - x^2)y'' - xy' + n^2y = 0$$

So, one independent solution gives $T_n(x)$ and the second independent solution is given by

$$u_n(x) = \sin n\theta$$

we note that $u_n(x)$ is not a polynomial.

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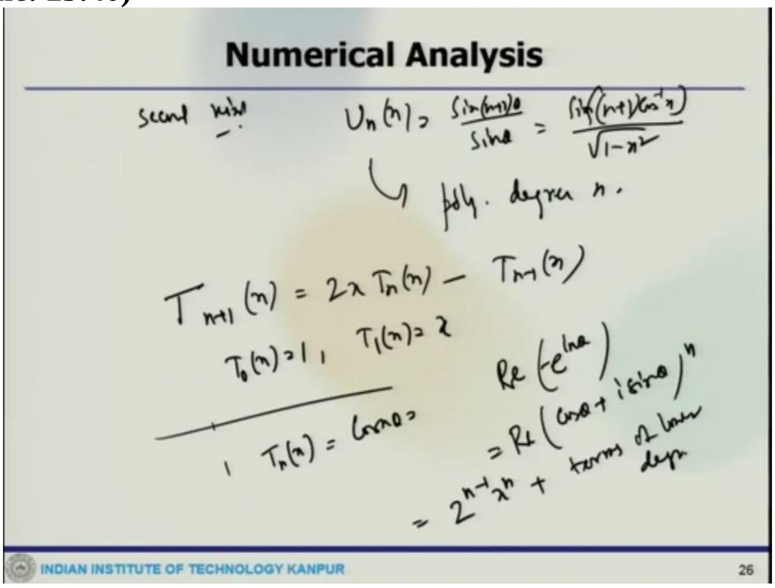

The Chebyshev polynomial of second kind denoted by the second kind is denoted by, $u_n(x)$ which is

$$u_n(x) = \frac{\sin(n+1)\theta}{\sin n\theta} = \frac{\sin((n+1)cos^{-1}x)}{\sqrt{1-x^2}}$$

So, this is a polynomial of degree n. So, Chebyshev polynomial of $T_n(x)$ satisfies the recurrence relation like,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

where,

$$T_0(x) = 1$$

and $T_1(x) = x$. So, you can similarly extract the other, if $T_0(x) = 1$. So, also, what you can have been that

$T_n(x) = \cos n\theta = Re\left(e^{in\theta}\right) = Re\,(\cos\theta + i\sin\theta)^n = 2^{n-1}x^n + terms\ of\ lower\ degree$

and the Chebyshev polynomial $T_n(x)$ has certain properties which like, one can see.

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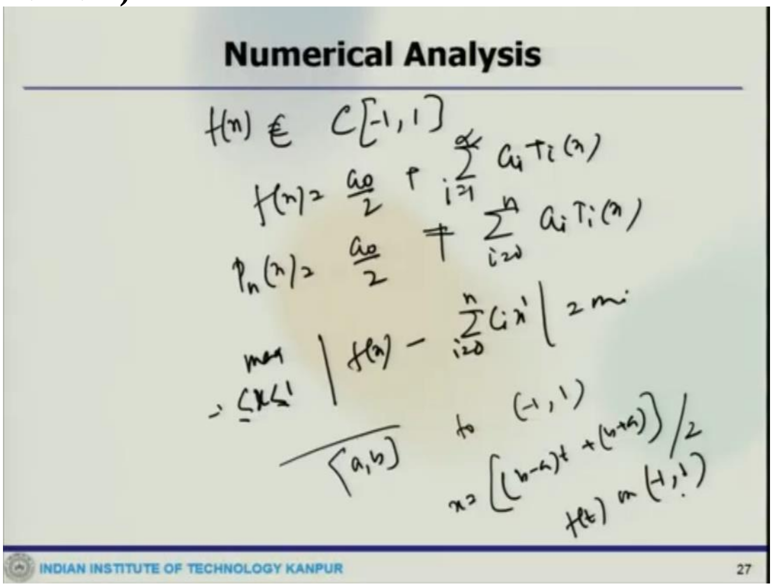

In any of these books that this will follow some properties one I quickly like to touch upon is that the, so this Chebyshev polynomial approximation. So, let us Chebyshev series expansion $f(x)$ belongs to C[-1,1] which is written as

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i T_i(x)$$

then the partial sum would be

$$P_n(x) = \frac{a_0}{2} + \sum_{i=0}^{n} a_i T_i(x)$$

It is very nearly the solution of the min max problem where

$$\max_{-1<x<1} \left| f(x) - \sum_{i=0}^{n} C_i x_i \right| = minimum$$

So, to obtain the approximate polynomial of $P_n(x)$, we follow certain steps like, we transform the interval $[a, b]$ to [-1,1] and then using linear transformation like

$$x = \frac{[(b-a)t + (b+a)]}{2}$$

and then obtain a new function $f(t)$ on (-1,1). Then obtain the power series solution for $f(t)$ and then we can write each term.

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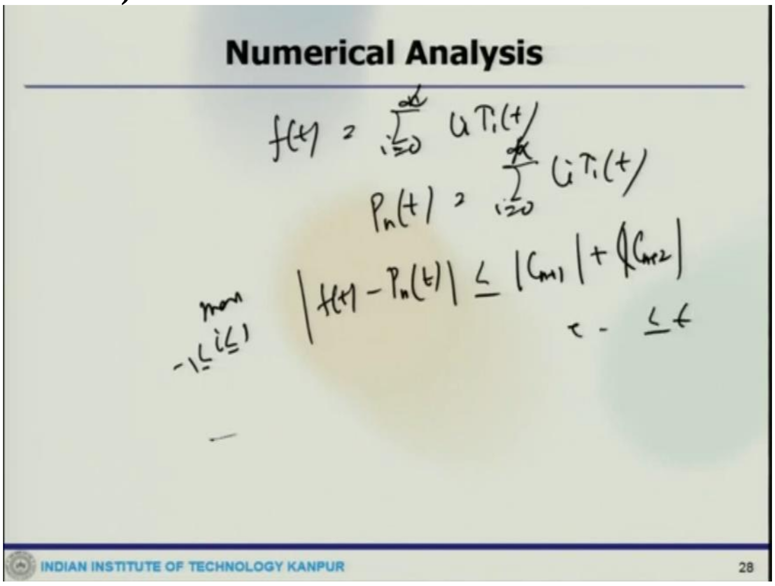

Like,

$$f(t) = \sum_{i=1}^{\infty} C_i T_i(t)$$

and the partial sum like

$$P_n(t) = \sum_{i=0}^{n} C_i T_i(t)$$

and then the good uniform approximation of $f(t)$ is, in the sense like

$$\max_{-1 \le i \le 1} |f(t) - P_n(t)| \le |C_{n+1}| + |C_{n+2}| + \cdots \le \epsilon$$

So, this is how the Chebyshev polynomial approximation can be obtained. So, you can see there are different interpolation function and how the approximation is done.

And these are useful for different applications. And user has to decide which is best suited for this particular approximation. So, we will stop the discussion here and continue with the other discussion in the next session.