Computational Science in Engineering
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Lecture – 38
Numerical Analysis

So, let us continue to look at this solution of the ODE equation that we have started and we have started looking at different explicit method. And then while talking we just came to the RK based method, which is general Runge Kutta based method and where we can derive higher order also.

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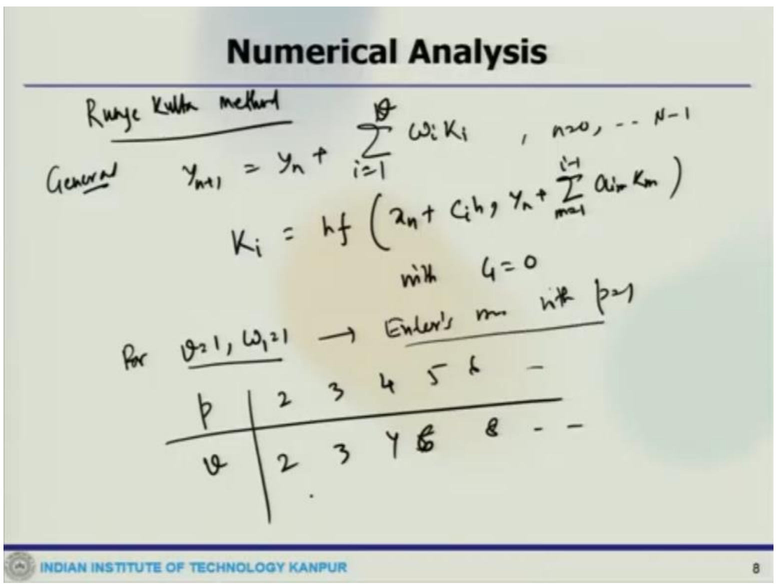

So, this is what we have looked at the general expression for RK based method. And then we have said that, for a particular v and w_1 this becomes the Eulerian method, which is the lowest order RK method.

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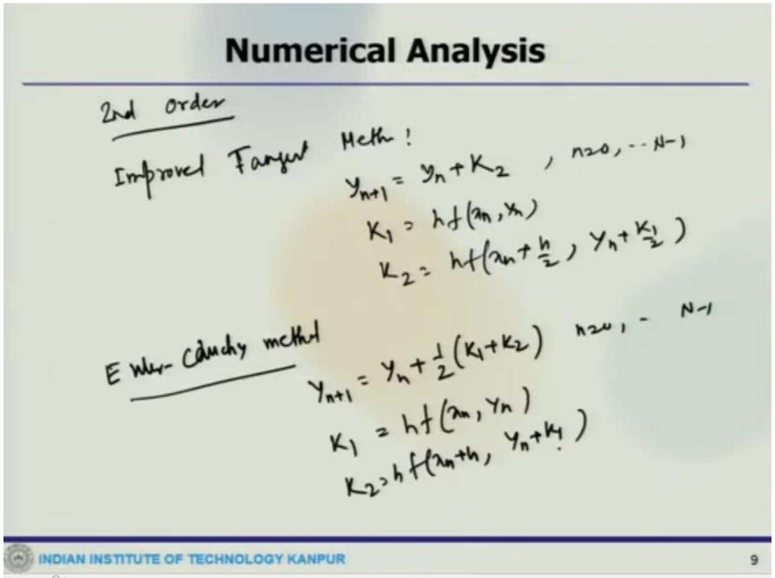

So, we will look at some of these likes in the second order methods like RK-2, so which could be like one can say there could be second or third method like some of this improved tangent method, where we express

$$y_{n+1} = y_n + K_2$$

where n goes to 0 to (N – 1) and

$$K_1 = hf(x_n, y_n)$$

and

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

or if it is Euler Cauchy method. So, in that case you express like you will

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

where n goes from 0 to (N – 1). Then

$$K_1 = \ hf(x_n, y_n)$$

$$K_2 = hf(x_n + h, y_n + k_1)$$

So, this is how you get second order method.

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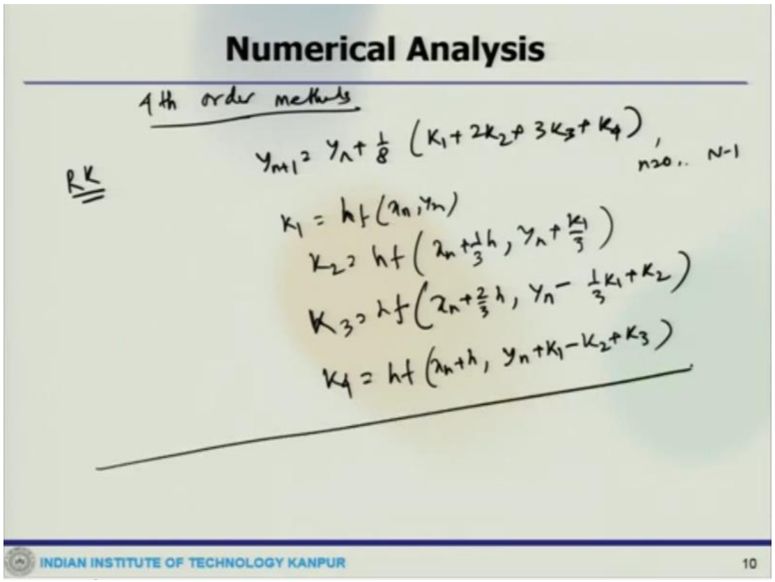

Now, you can look at also some fourth order kind of methods like there are also some like third order method which one can find out, but we will look at some fourth order methods, where you can have RK based method like then we will write RK is

$$y_{n+1} = y_n + \frac{1}{8}(K_1 + 2K_2 + 3K_3 + K_4)$$

where n = 0 to (N – 1).

$$K_1 = \ hf(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{k_1}{3}\right)$$

$$K_3 = hf\left(x_n + \frac{2}{3}h, y_n - \frac{k_1}{3} + k_2\right)$$

$$K_4 = hf(x_n + h, y_n + k_1 - k_2 + k_3)$$

this is RK based method.

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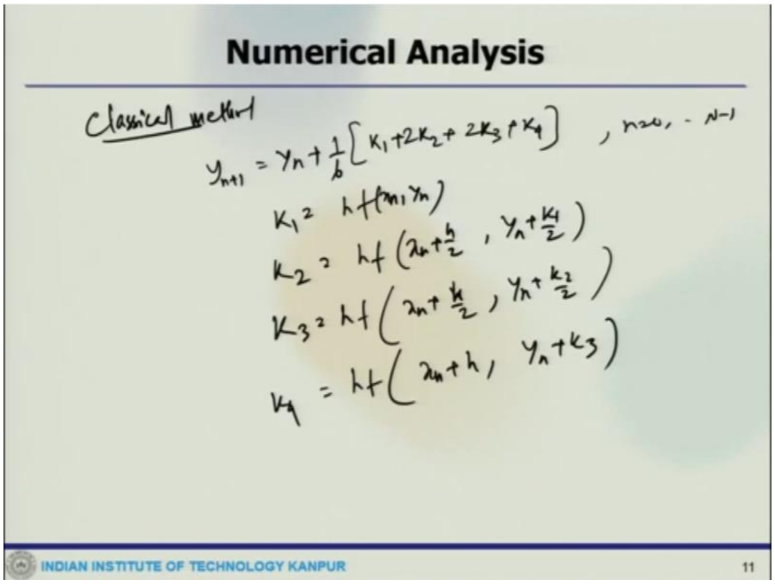

If you go to some classical fourth order where you can write

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 3K_3 + K_4)$$

where n goes from 0 to (N – 1),

$$K_1 = \ hf(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_3}{2}\right)$$

$$K_4 = hf(x_n + h, y_n + k_3)$$

So, this is typical or classical fourth order based methods.

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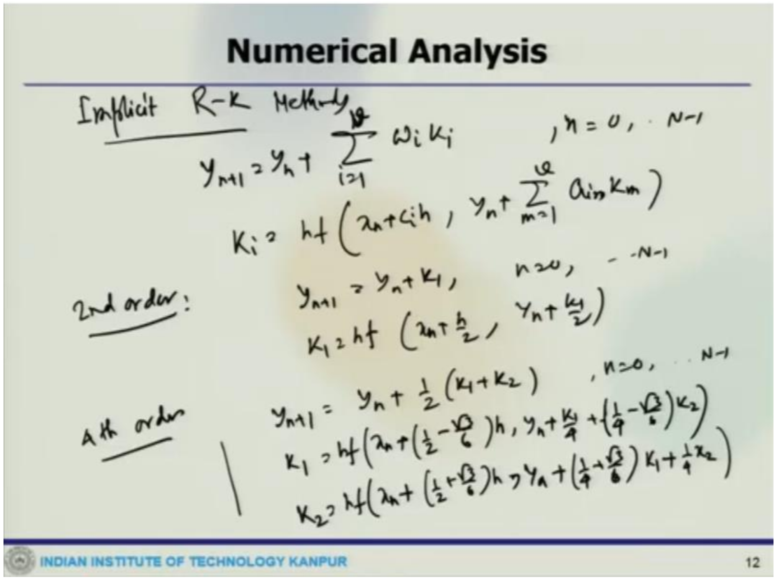

So, now what we can look at is that we can also look at implicit RK methods Runge Kutta methods. So, here the equation which is written in the general RK based system, this equation what is written here. So, this is now modified and modified like this kind of. So, this is the way that we modify

$$y_{n+1} = y_n + \sum_{i=1}^{v} w_i k_i$$

where n goes 0 to (N – 1) and

$$k_i = hf\left(x_n + c_i h, y_n + \sum_{m=1}^{v} a_{im} k_m\right)$$

So, now v is the function evaluation implicit RK methods of order of 2 can be obtained.

So, like you can see if it is second order how you get it? So, this will look like

$$y_{n+1} = y_n + k_1$$

n goes from 0 to (N – 1) and

$$K_1 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

where if it is fourth order. So, these are the different ways that one can evaluate all these differential equations and you can see how you can find out the system solution of a differential equation. So, this is implicit fourth order RK based method that this is how you evaluate the term. Now, these are some of the implicit and explicit method.

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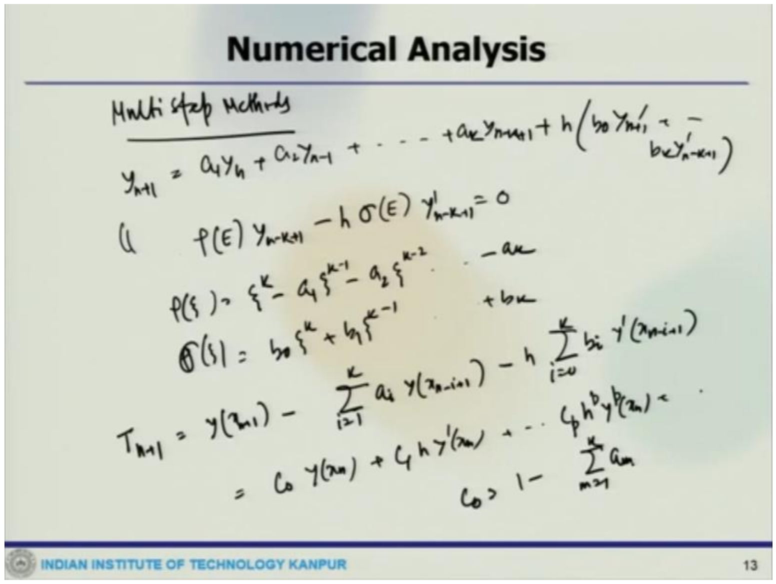

Now, we can look at some of the like multi step method. Now, because the one which we have already discussed so, far these are the simple methods. In general case step or multi step method for solution or initial value problem which is given as a Kth order difference equation like with

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k+1} + h(b_0 y'_{n+1} + \cdots + b_k y'_{n-k+1})$$

So, this is what we can in symbolically write, the method is called an explicit or predictor method where $b_0 \neq 0$ is called the implicit or character method. So, the local truncation error or could be in form like

$$T_{n+1} = y(x_{n+1}) - \sum_{i=1}^{k} a_i y(x_{n-i+1}) - h \sum_{i=1}^{k} b_i y'(x_{n-i+1})$$

So, this is what you can write. And now, you can expand the right hand side term and Taylor series expansion and rearranging what you can write

$$= C_0 y(x_n) + C_1 h y'^{(x_n)} + \cdots + C_p h^p y(x_n)$$

like this, where

$$C_0 = 1 - \sum_{m=1}^{k} a_m$$

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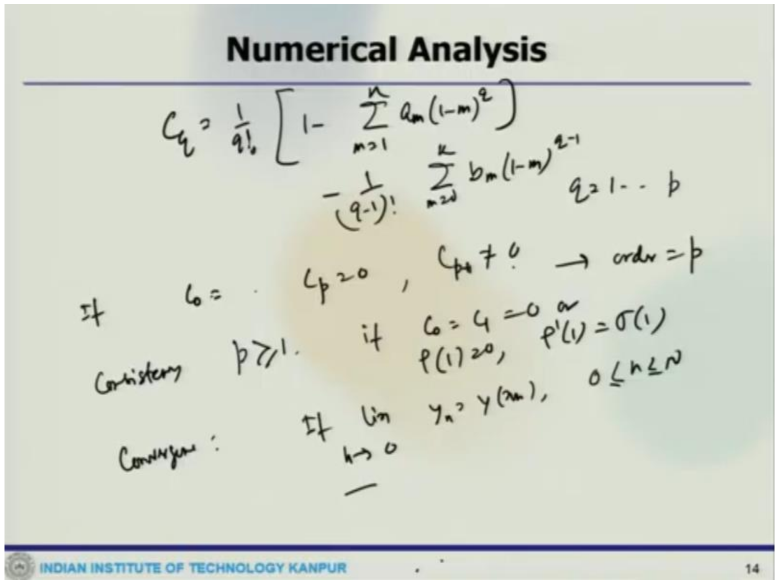

And

$$C_q = \frac{1}{q!}\left[1 - \sum_{m=1}^{k} a_m\,(1-m)^q\right] - \frac{1}{(q-1)!}\sum_{m=1}^{k} b_m\,(1-m)^{q-1}$$

q = 1 to p. And now if $C_0 = C_p = 0$, then order of this method would be p th order. Now for consistency p would be greater than 1 then the multi-step method he says to be consistent if $C_0 = C_1 = 0$ or $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$.

Convergence so, if in the limit

$$\lim_{h\to 0} y_n = y(x_n)$$

then this said to be and also the rounding errors are arising from all the initial conditions tend to 0 then the method is said to be convergent. So, now we can look at some methods which are based on this kind of multi-step kind of method.

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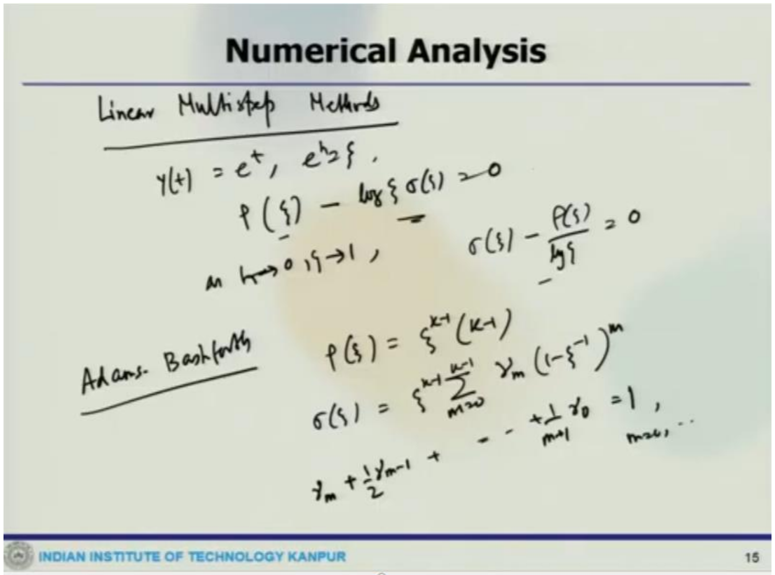

So, for example, we can look at some linear multi step methods like we can find out this like putting let us say we put $y(t) = e^t$ and $e^h = \xi$, in our equations, which is given in the general system here. Then what we get is that

$$\rho(\xi) - \log \xi \sigma(\xi) = 0$$

as h tends to 0, ξ tends to 1. So, we can we can find the maximum order for this. If $\sigma(\xi)$ is specified then this can be used to determine the $\log \xi$ of degree K.

And similarly, if $\rho(\xi)$ is given then we can use the equation that

$$\sigma(\xi) - \frac{\rho(\xi)}{\log \xi} = 0$$

So, that is the way also we can find out that thing. Now, the one of the methods of such kind is very known as Adam Bashforth what do you say that

$$\rho(\xi) = \xi^{k-1}(k-1)$$

Where

$$\sigma(\xi) = \xi^{k-1} \sum_{m=0}^{k-1} \gamma_m (1 - \xi^{-1})^m$$

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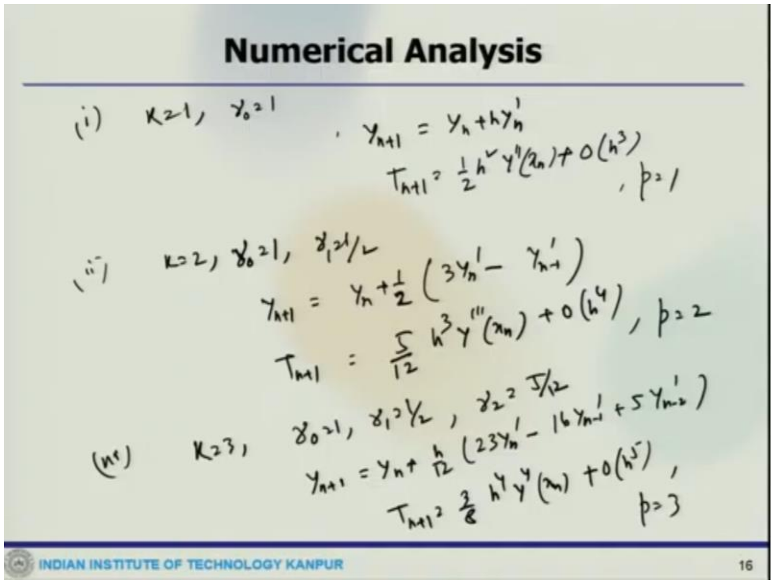

So, what we can have like for K = 1 and $\gamma_0 = 1$, we get

$$y_{n+1} = y_n + hy'_n$$

and that error would be

$$T_{n+1} = \frac{1}{2}h^2y''(x_n) + O(h^3)$$

here the p is 1. Similarly, K = 2, $\gamma_0 = 1$ and $\gamma_1 = \frac{1}{2}$. We get

$$y_{n+1} = y_n + \frac{1}{2}(3y'_n - y'_{n-1})$$

And the error would be

$$T_{n+1} = \frac{5}{12}h^3y'''(x_n) + O(h^4)$$

are the order of the p = 2

Now, similarly we can have like K = 3 then $\gamma_0 = 1, \gamma_1 = \frac{1}{2}, \gamma_2 = \frac{5}{12}$ then you get

$$y_{n+1} = y_n + \frac{h}{12}(23y'_n - 16y'_{n-1} + 5y'_{n-2})$$

and that truncation error would be

$$T_{n+1} = \frac{3}{8}h^4y^4(x_n) + O(h^5)$$

So, this has the third order estimation. So, like this we can devise a different kind of method. Now, which would be quite handy and that one can find the solution for this kind of things. Now, what we can go now?

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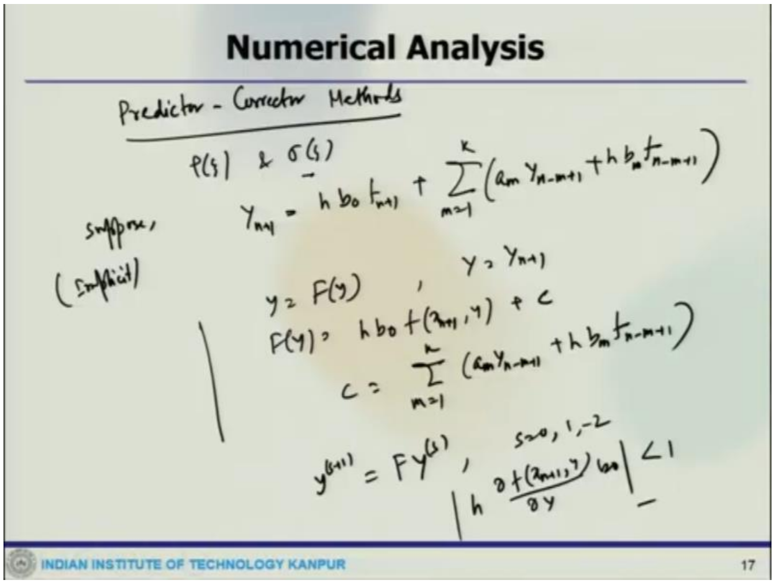

Basically, look at a slightly different kind of method which is called the predictor corrector methods. So, what do we do that when we have the $\rho(\xi)$ and $\sigma(\xi)$ are of the same degree we produce an implicit or a corrector method, if the degree of $\sigma(\xi)$ is less than a degree of $\rho(\xi)$, then we have an explicit predictor kind of method. So, the corrector method produces a nonlinear equation for solving at x = n + 1.

However, the predictor method can be used to evaluate y_{n+1}. So, this approach is known as predictor corrector approach, let us say suppose, we have

$$y_{n+1} = hb_0 f_{n+1} + \sum_{m=1}^{k} (a_m y_{n-m+1} + hb_m f_{n-m+1})$$

So, to find out y_{n+1} this equation may be written as $y = F(y)$ and y is written as y_{n+1},

$$F(y) = \; hb_0 f(x_{n+1}, y) + c$$

where c is $\sum_{m=1}^{k} (a_m y_{n-m+1} + hb_m f_{n-m+1})$.

So, an iterative method can be used to solve this particular set of equation and which is suitable for first approximation and all such these things like

$$y^{(s+1)} = Fy^{(s)}$$

where s goes from 0, 1, 2.

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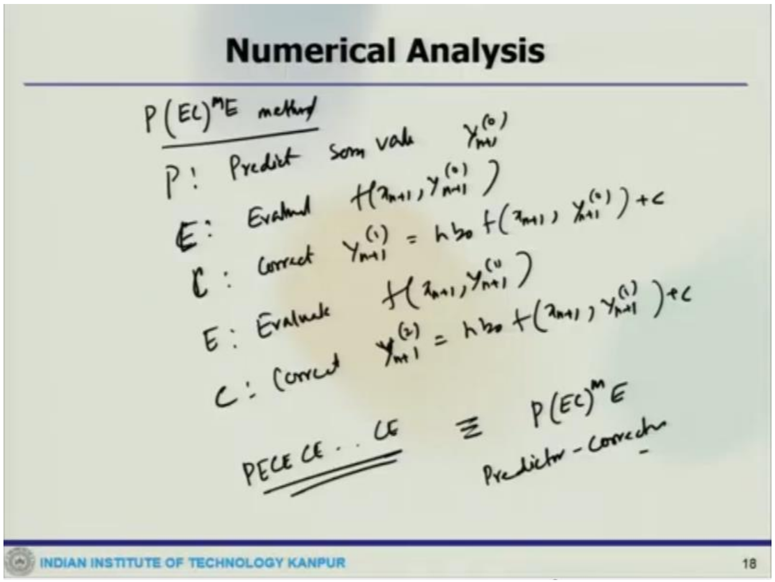

So, this is where it will go and now, like we have a $P(EC)^m E$ method. So, this is called so, explicit predictor method for predicting $y_{n+1}^{(0)}$ and use the implicit corrector method for iteratively until convergence obtain. So, what we write this particular equation here that what we have written here. So, what we write here is that, so the P is the predict some value $y_{n+1}^{(0)}$, then E is that evaluate $f(x_{n+1}, y_{n+1}^{(0)})$ then, C is correct.

$$y_{n+1}^{(1)} = hb_0 \, f\left(x_{n+1}, y_{n+1}^{(0)}\right) + c$$

E again evaluate

$$f(x_{n+1}, y_{n+1}^{(1)})$$

and again correct

$$y_{n+1}^{(2)} = hb_0 \, f\left(x_{n+1}, y_{n+1}^{(1)}\right) + c$$

so the sequence of operations like PE CE CE..CE like this. So; which is kind of equivalently denoted as PEC to the power m E, which is called predictor corrector method. So, the predictor will be of the same order of the lower order than such things.

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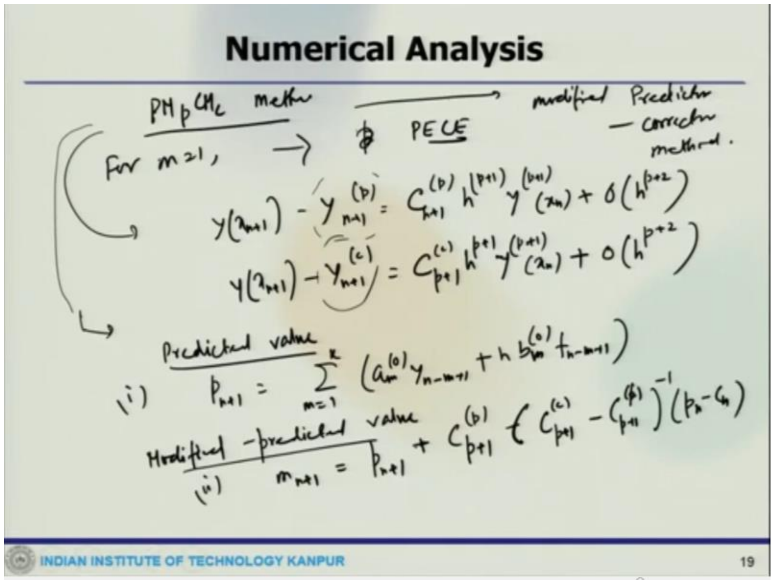

Now, similarly, you can have PM p CM c method, which is for m = 1, this becomes the predictor corrector method, if the predictor and corrector methods are of the same order p then we can use this is used the estimate of the truncation error to modify the predictor and corrector values. So, that we can write the procedure for this is called the modified predictor corrector method. So; what we do that in this so, we get

$$y(x_{n+1}) - y_{n+1}^{(p)} = C_{n+1}^{(p)} h^{(p+1)} y^{(p+1)}(x_n) + O(h^{p+2})$$

And

$$y(x_{n+1}) - y_{n+1}^{(c)} = C_{n+1}^{(c)} h^{(p+1)} y^{(p+1)}(x_n) + O(h^{p+2})$$

So, here $y_{n+1}^{(p)}, y_{n+1}^{(c)}$ represents solution values which are obtained by using predictor and corrector step so, this is predictors step and this is corrector step. So, now we can write the modified method or the steps like so, first we have to the predicted value like

$$p_{n+1} = \sum_{m=1}^{k} \left(a_m^{(0)} y_{n-m+1} + h b_m^{(0)} f_{n-m+1} \right)$$

and then we have modified predicted value.

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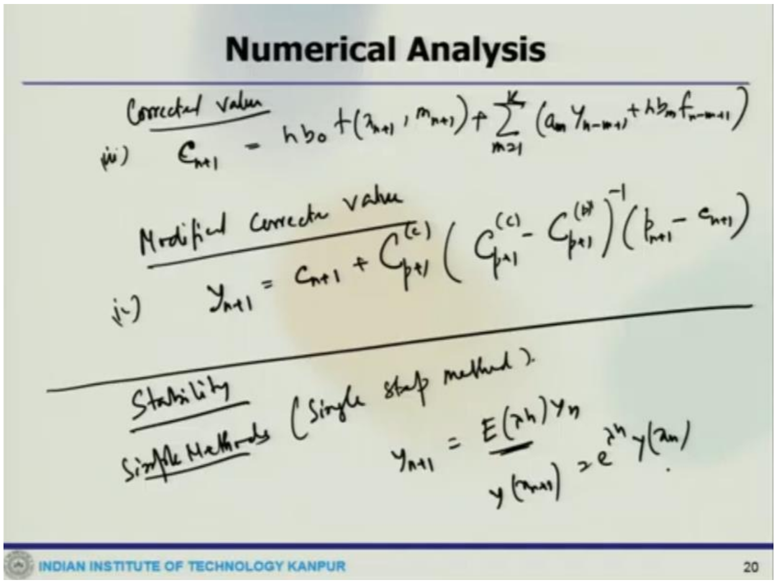

And now, we can correct the values like now we have a corrected value, which would be third level,

$$c_{n+1} = hb_0 f(x_{n+1}, m_{n+1}) + \sum_{m=1}^{k} (a_m y_{n-m+1} + hb_m f_{n-m+1})$$

and then finally we have the modified corrected value.

So, this is how you can get the different predicted corrected values. Now, we can also should have the stability of the solution because when you find out this numerical solution, the stability is important like the simple methods, what we have used the stability the single step method or the single let us say simple method or single step methods which we have initially discussed, there, we have the first order differential equation, the equation would be written like

$$y_{n+1} = E(\lambda h) y_n$$

This depends on the particular single step method and analytical solution of the test problem could be

$$y(x_{n+1}) = e^{\lambda h}\, y(x_n)$$

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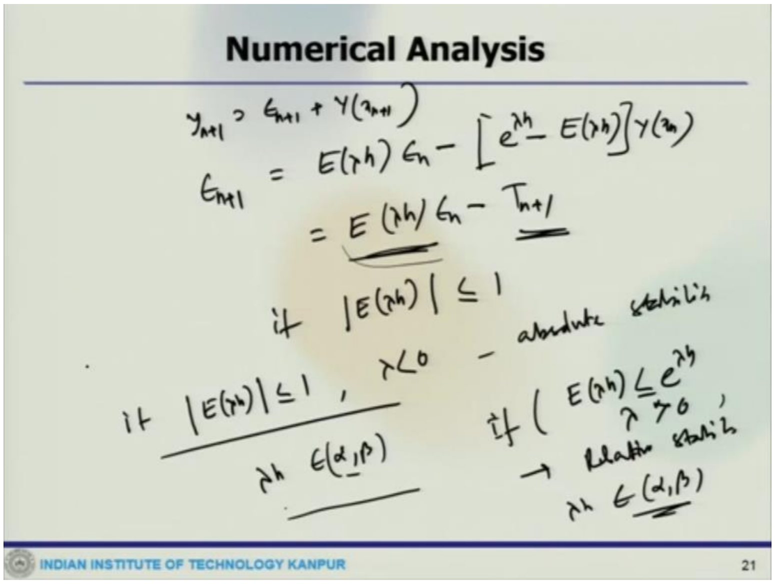

And then what and the error equation we substitute which is like

$$y_{n+1} = \epsilon_{n+1} + y(x_{n+1})$$

Now, this is the local truncation error and it is independent of ϵ_n. So, the first term on this particular error expression the propagation of the error term. So, this is the error term will not grow if $|E(\lambda h)| \leq 1$.

So, if we have $|E(\lambda h)| \leq 1$ for $\lambda < 0$, then this is called the absolute stability for the single step method absolute stability and see now, interval of absolute stability the method is absolute stable for all lambda h which belongs to alpha and beta and the interval of alpha and beta it says to be the interval of absolute stability. Now, relative stability there would be like if

$$E(\lambda h) \leq e^{\lambda h}$$

for $\lambda > 0$ for any single step method, this is called the relative stability.

And interval of the relative stability if the method is said to be relatively stable within an interval for all λh, which belong to (α, β), these interval particular interval (α, β) is said to be the interval for this relative stability. So, you can see that any method when you devise a method new in particular numerical method you can have. So, obviously, there are errors and you need to sort of kind of minimize this error.

Because there are mostly when you expand the functions and approximation you do there are truncation error there round off error. And to get a stable solution you need to have some stability condition. There are certain restrictions on the step size certain restriction sometimes will be on the time size which we will look at it and this will give you a proper accurate solution. We will stop here and continue the discussion in the next session.