Computational Science in Engineering
Prof. Ashoke De
Department of Aerospace Engineering
Indian Institute of Technology - Kanpur

Lecture - 04
Linear Algebra

So let us continue the discussion on the solution of Ax = b and what we are trying to I mean answer is that whether what is the possible vector b, where we can find solution or to have a solution, I mean essentially the condition for b that this has to belong to the column space. So, then only we can have a solution.

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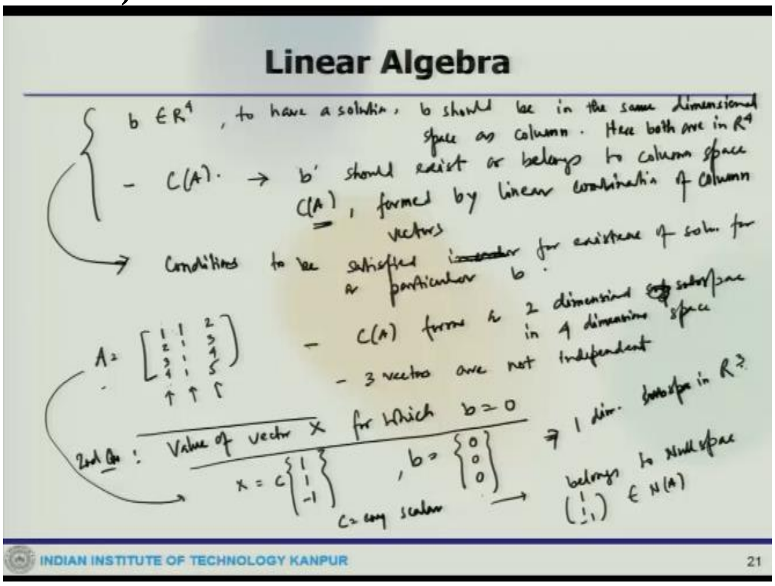

Then the next question that we have raised here is that what is that vector which can give us the zero solution? So, the vectors which will actually provide the zero solution that would belong to null space.

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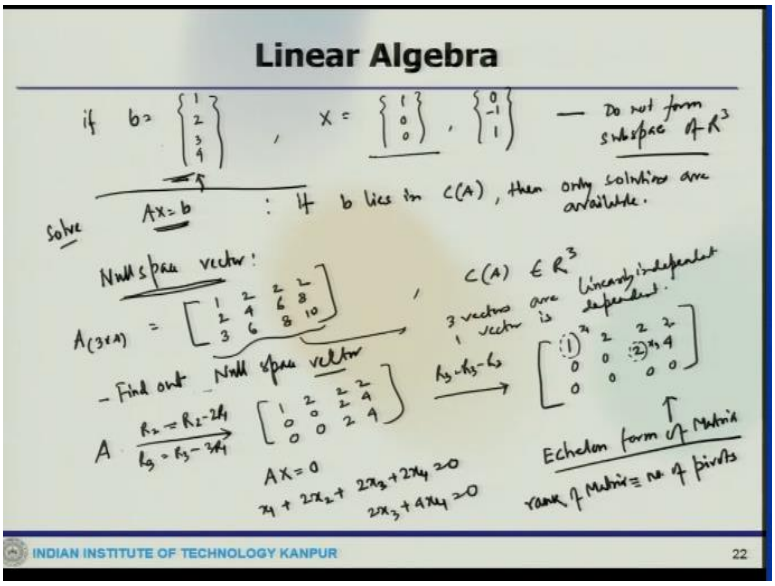

Now continuing with the same example or keeping the same example in mind, let us say if our $b = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{Bmatrix}$ then the possible solutions could be $X = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ that is one possibility or it could be $\begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$. So, that means as I said, for a particular b the vectors would be so the question is formation of the subspace by two solutions for a specific b. So, the solution vectors here will not form a subspace of R^3 because the plane which are found by the vectors, that does not pass through origin and also zero vector does not exist.

So that means we have a particular b, so previously we have seen for zero solutions, so this is a particular solution vector, but here we have a particular b and we have a solution vector. Now, these solution vectors they do not they basically do not form subspace of R^3 that means to form a subspace of R^3 we have already mentioned these conditions what is possibly here if you just refresh that, to have a subspace of that, the plane found by this vector that does not pass through the origin or there is no zero vector which actually exists there.

Now, our goal is to get a solution for Ax = b or solve Ax = b and what is important here? If b lies in column space of A then only solutions are available. So, that means to have a solution, we have to lie in the column space and then only solution is possible. Now, there is another one which is that null space vector, so the column spaces are found by column vectors of the null space vector is the column spaces are found by column vectors of coefficient matrix.

So we can again look at an example let us say we take a different example of 3×4 system which is like $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$. So, what we can actually look at here is that this particular example column space of A belongs to R^3 and also if you look at carefully that there are 4 columns, but or column vectors, so out of these 3 vectors are linearly independent and 1 vector is dependent that means that any combination of that other 3 would form the fourth one.

So here what we are trying to find out is that we want to find out the null space vector so, typically or in general this is often through the elimination process again an idea is to form the upper triangular matrix. So, here if we do an operation like R_2 so $R_2 = R_2 - 2R_1$ and also $R_3 = R_3 - 3R_1$ then we get to $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$. Now the further operation on R_3 which is $R_3 - R_2$ which will give us $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So, this is not an obviously this is not a proper upper triangular matrix. So, this is rather called as Echelon form of matrix. So, since it is not a symmetric one so, we get an Echelon form of the matrix or matrix A. Here if you look at that there are 2 pivots which are there this corresponds to x_1 and this corresponds to x_3. Now, what we are trying to find out we are trying to find out the null space vector. So, that means the solution that we are shooting for is $Ax = 0$.

If we do this here, what we end of getting is that

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

and

$$2x_3 + 4x_4 = 0$$

and we get x_1, x_2 and x_3 these are the 2 pivots which are sitting in this Echelon matrix actually, or in the Echelon form of this matrix. Now, we get once we do this elimination process, we end of getting these pivots and the pivots actually whatever the pivots which are there, these pivots are going to determine the rank of the matrix. So, the rank of matrix would be the number of pivots here 2.

So, the immediately one can say that the rank of these particular matrix which is taken as an example, is 2 straightaways since the number of pivots at 2 and the columns these are the columns first column and the third column, the corresponding columns with the pivots are called the pivot columns and other columns like these and these are called the free columns.

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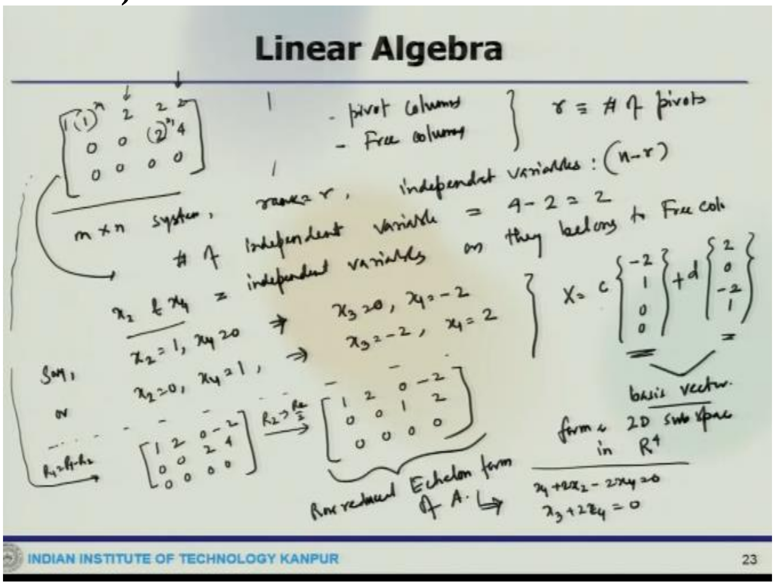

So as soon as we determine, so we have like let me write down this Echelon form which is there and then it would be easier $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, this is one that this is the one correspond to x_1 and x_3 so, with the pivots these are called the pivot columns where the pivots sitting and others are the free columns and r is the rank of the matrix which will be the number of pivot sitting there in the Echelon from.

So, if we have now, if we try to generalize that if we have a m by n system and rank is r, then we have independent variables which are n - r. And so, for this particular example, the number of independent variables equals to 4 - 2 which is 2. Now, here if you see, the free columns are the second column and the fourth column so the variable x_2 and x_4 are taken as independent variables as they correspond or belong to free column.

So, we can take a new value for x_2 and x_4 and we can find out the solution for let us say, we take $x_2 = 1$ and $x_4 = 0$ which will give us $x_3 = 0$ and $x_1 = -2$. Similarly, we can say or we can have $x_2 = 0$ or $x_4 = 1$ then that will give us $x_3 = -2$ and $x_1 = 2$. So, we can get any solution in this particular form where X would be

$$X = c\begin{Bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{Bmatrix} + d\begin{Bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{Bmatrix}$$

So, the solution would be in this particular form here c and d, are any scalar which will give us any combination of the solution.

Now, these 2 vectors which we get here this guy and this guy these are called basis vector and since there are they form a 2 dimensional subspace because there are only 2 basis vector. So, it depends on how many basis vectors you have. So, since there are only 2 basis vectors so they form 2D subspace in R^4 or 4 dimensional vector spaces so, there are 2 basis vectors. So now, we can a little bit do further more calculations or like from here if we do further operations like $R_1 = R_1 - R_2$.

So, some more elimination; just above and below pivot which will get us $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Now, what we will do is $R_2 = R_2/2$, $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, this guy here is called row reduced Echelon form of A. And once we do that, from this guy, the equation becomes from here, if we write the equation becomes $x_1 + 2x_2 - 2x_4 = 0$ and $x_3 + 2x_4 = 0$.

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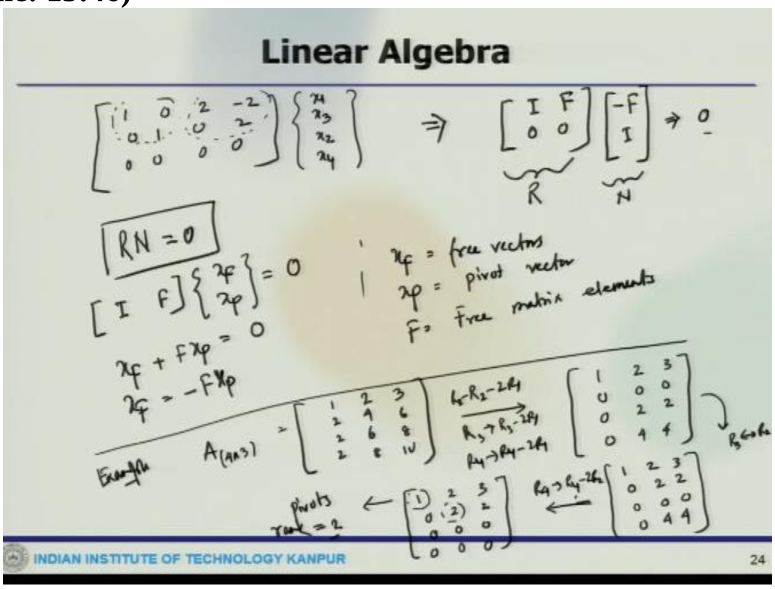

So, once we rearrange the things, what we can write is that let us say $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, what we have done is that we just interchange the columns. So, which means this would be x,

$X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$. So, here, we can see this is a block of identity matrix this is another block of free variable. So, one can write this one as an identity matrix free variable 0 and 0.

So here, if we do that guy with minus F and I then this would give us always 0 and this particular form is called R, which is the row reduced Echelon form and this is my null space vector, which will give zero solution that means the row reduced Echelon form multiplied with the null space vectors should be giving zero, N here is the null space or called the null space vector or other way I can write

$$[I \quad F]\begin{Bmatrix} x_F \\ x_P \end{Bmatrix} = 0$$

So $x_F + Fx_P = 0$ and we get $x_F = -Fx_P$ here x_F are the free vectors x_P are the pivot vectors and F means free matrix element, so this is how you can get the row reduce form and from there you can find out the null space vector. So, let us take another example and find that, let us say A is just 4×3, so just reverse this one, this would be $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$. so, we have 4 rows 3 columns. And again, if we do the operations here, let us say R_2 is R_2 - $2R_1$, R_3 is R_3 - $2R_1$ and R_4 is R_4 - $2R_1$, what we get here is that $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$.

Now we do the exchange of row like R_3 is R_2 then what we can write $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ and then further operations like what we do R_4 which is R_4 - $2R_2$ which is like $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Here again, you can see this guy and this guy, these are sort of pivots and the rank is 2. Now interestingly, this particular example is the transpose of the previous one. So previously here, we started with 3×4 and the same one, we just reverse the 4×3 and you can see the rank of that matrix. So, whether it is A transpose or A, the rank remains same.

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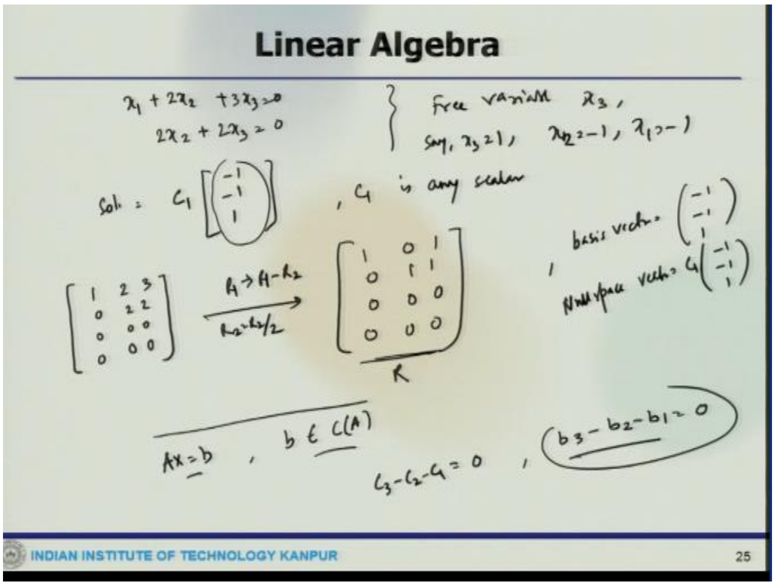

From here, we can write $x_1 + 2x_2 + 3x_3 = 0$ and $2x_2 + 2x_3 = 0$ here, we have free variables, which is x_3 and let us say $x_3 = 1$ then we get $x_2 = -1$, $x_1 = -1$. So, the solution would be $C_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ where C_1 is any scalar. Now, from this particular matrix we will do a little bit more operation where we have $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, do a little bit of more operations like R_1 is $R_1 - R_2$ and $R_2 = R_2/2$ what do we get $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So, this is what it is R? Row reduced Echelon form and here the vector which is this. This is the basis vector which is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ which is giving us the zero solution or the vector which belongs to the null space vector and the null space vector is any linear combination of that, which is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ and this is the basis vector and you can see the one vector which is forming this null space. So, this is forming essentially 1 dimensional subspace. So, this example is this, so column space lies in R^4 and null space in R^3.

So, the null space is forming the basis vector is forming 1 dimensional subspace in R^3. Now, we continue with the previous example to find out the solution for $Ax = b$. Now, obviously, for solution of any $Ax = b$, b has to belongs to column space of A. And the previous example what we can see the previous example means we are talking about this one only 4 × 3 system where we can continue, we can say that here the this is a column 1, this is column 2, this is column 3, then we can say C_3 - C_2 - C_1 would be 0.

So, which means there is a restraint of that b_3 - b_2 - b_1 would be 0 so, this is one can see by looking at this guy, that C_3 - C_2 - C_1 so, this would be the restraint there, but this particular thing, this conclusion should arise from the elimination process.

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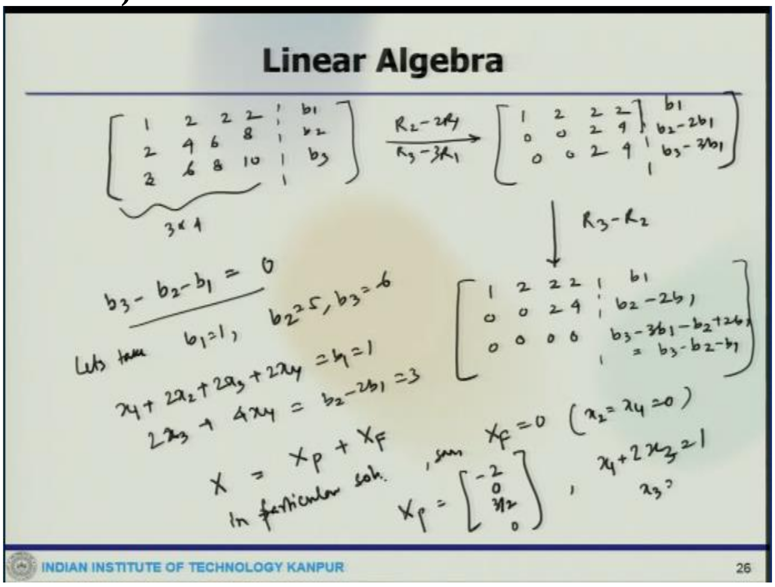

So let us augment the matrix and do the thing so, if we will augment the matrix what we can write is that $\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 \vdots & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix}$. so, we are going back to this 3 × 4 system. Now, here the operation is R_2 - $2R_1$, R_3 - $3R_1$ which will give us $\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 \vdots & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$. Now, the further operations on this which is R_3 - R_2 which will get us $\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 \vdots & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - b_2 + 2b_1 \end{bmatrix}$ which would be $b_3 - b_2 - b_1$.

So, from the last row it is evident that $b_3 - b_2 - b_1$ this has to be 0 so, this is a necessary condition to obtain a solution. So, if say let us take $b_1 = 1$, $b_2 = 5$ and $b_3 = 6$. so, the equations become $x_1 + 2x_2 + 2x_3 + 2x_4 = b_1 = 1$ and $2x_3 + 4x_4$ would be $b_2 - 2b_1$ which is 3. So, the final solution vector should be the sum of pivot variable and the free variable. In particular solution the free variables let us say for in particular solution say $X_F = 0$ which is $x_2 = x_4 = 0$. Then what happens is that X_p will become $X_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$. so, which is $x_1 + 2x_3 = 1$ where $x_3 = 3/2$.

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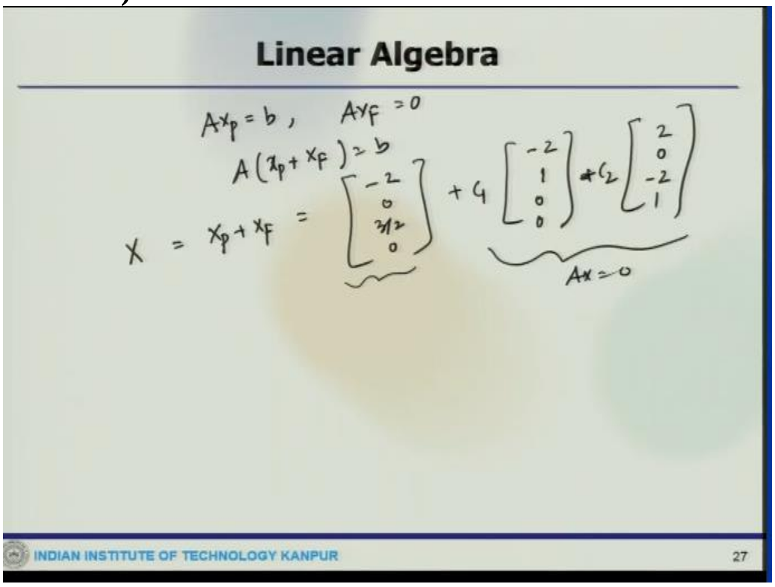

Again you have $Ax_p = b$ and $Ax_F = 0$ so, $A(x_p + x_F) = b$, so the complete solution would be $x_p + x_F$ which is $\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + C_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So, this one we get by setting Ax = 0 and this one by making free variable to be so, the free variables are basically the null space vector and the solutions will be available provided rest of the conditions are satisfied.

So, you can see that the solution could be getting into the both the particular and the pivot variable. So, this is how you can actually get a solution of the combination of the free variable and the pivot variable. We will stop it here and continue the discussion in the next class.