Computational Science in Engineering
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Lecture - 05
Linear Algebra

The discussion on linear algebra now, what we have looked at is that the what is the condition for which we can find out a solution for Ax = b and we have already talked about that that b has to lie in that column space of A that means, the particular matrix A if it is the order of m × n then there is a column vectors they will form a space and assuming that there would be certain vectors which are may not be independent they are dependent.

But even then, I mean using the independent vectors through there is a column space which exists there and to have a solution for a particular system, the B has to lie in the column space and then when you have that, then there are two components of the solution one could be the solutions with the particular variables of the free variables or one could be the preferred variables and what we have looked at when you do the elimination of the particular matrix or the linear system.

So, we get the row reduced form or Echelon form and from there, we can identify the pivot variables and then the solution will have two components one from come from the pivot components and one from the free variables. So, this is what we have looked at.

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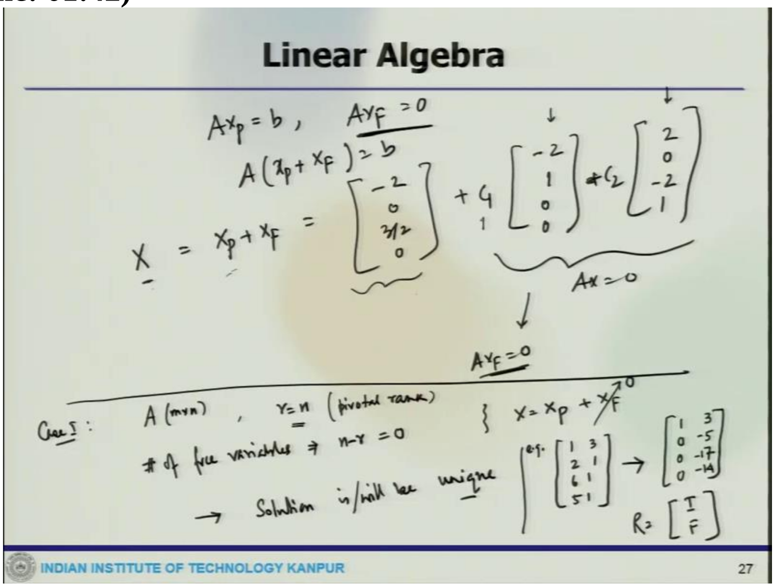

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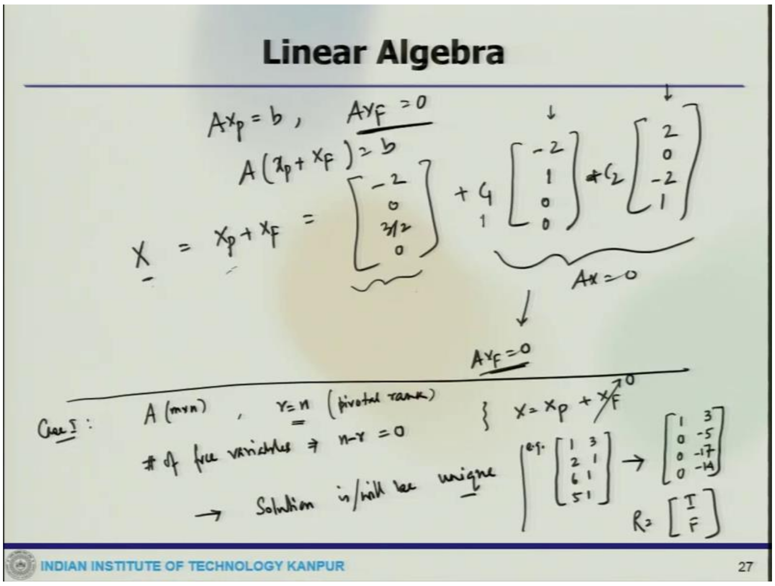

And we even stopped here and this is what if you just recall that from the augmented system, the total solution vector, you will have a pivot variable and you will have the free variable and the column which corresponding to free variable they will contribute to this solution and finally, the total solution would we have this kind of nature. So, this is coming from the pivot variable and these are the free variable and for the free variable, this is going to giving a solution for the $Ax_F = 0$.

So, these guys is going to contribute for this one, which is the null space vector and these are the basis vector and any combination these are C_1 and C_2 are the arbitrary scalar. So, any combination of that is going to form the null space vector. So, this is what we have looked at it now, even there are different conditions which may appear and one has to deal with those conditions in a different way.

For example, let us take certain cases let us say case 1 where obviously the matrix is order of m × n as we have said and its rank is n so, this is the rank or whatever you called is pivotal rank. So, the number of free vectors or number of free variables, so, number of free variables that would be n - r, which is essentially 0, because this is so, the solution would be only contributing from the pivot part there is no free variable.

So, this is not contributing anymore that would be 0. So, what it means, the solution is going to be is or will be unique. So, this is quite important that the solution that we get that would be unique. So, we can see an example here a quick example which can give you an idea like you take a matrix like this, where it is a $\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ and if you do the operations, so, this guy is going to go down to $\begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 0 & -17 \\ 0 & -14 \end{bmatrix}$. So, what you have the row reduced form essentially, I and F and the solution is going to be unique. Now, that is particular situation where you have this case.

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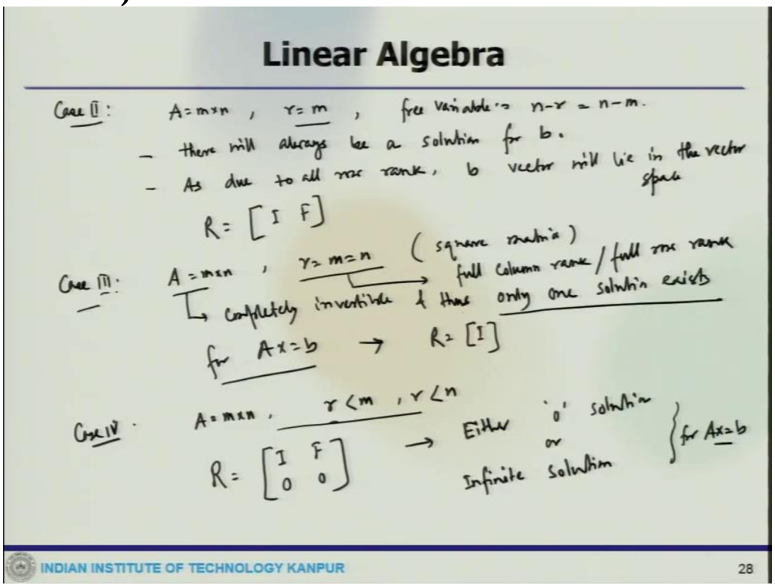

Now, we can move to a situation where you have case 2, where again the matrix is order of m by n and your rank is now m. So, free variables would be would be n - r, which is n - m. Now, in this case, there will always be a solution for a b. So, that means, there will always be a solution for b now, what you can see here, since the rank is equals to m or the exactly the number of rows or you can call it a row rank, so, as due to all row rank, so, b vector will lie in the vector space.

So, the way if you take an example and reduce it the row reduced form will look like that, because here you mind it, I mean you can take some examples and look at it, but this is because of you have a complete low row rank. So, the previous one case 1 we have taken complete column rank or the pivotal rank and next case is that row rank. Now, you can have a condition where things are like your m into n and your rank is m = n which is that means, this is a square matrix.

So, this is when r = m = n one can say this is either full column rank or full row rank. So, either of them are fine, you can say it is so, the both of them are here. Now, so, which means that A is completely inveritable and thus only one solution exists for Ax = b which means, my row reduced from will have this. So, only one solution exists for this. So, when you have full rank either full column rank or row rank, this is what it is going to happen and there could be another situation where you can have for A = m × n, either r could be less than m and r < n.

So, this will be the rank which is in neither full row rank not full column rank. So, if you do have the row reduced form, so, they will look like this and this is the kind of cases what we have been already taking some examples and looking at it. So, here what will happen in this situation either zero solution or there would be infinite solution for Ax = b, so, these are the two possibilities which may occur or appear here. So, when you see this kind of situation.

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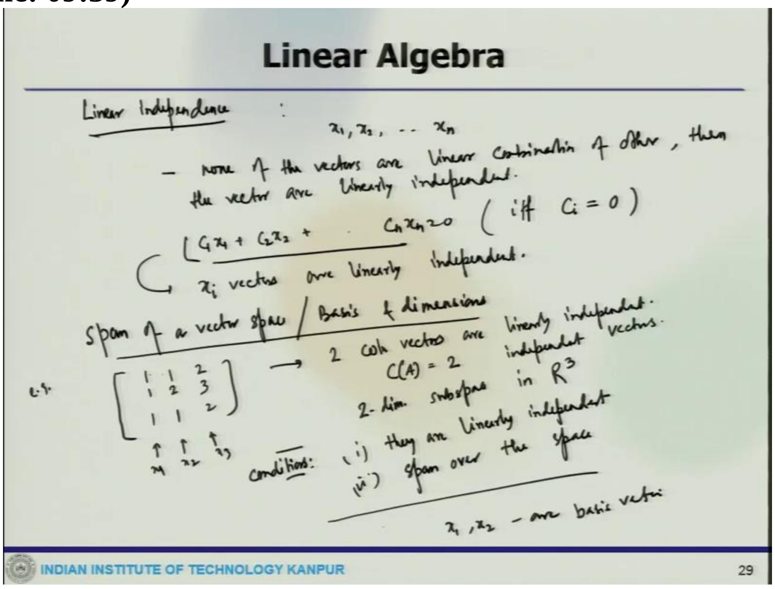

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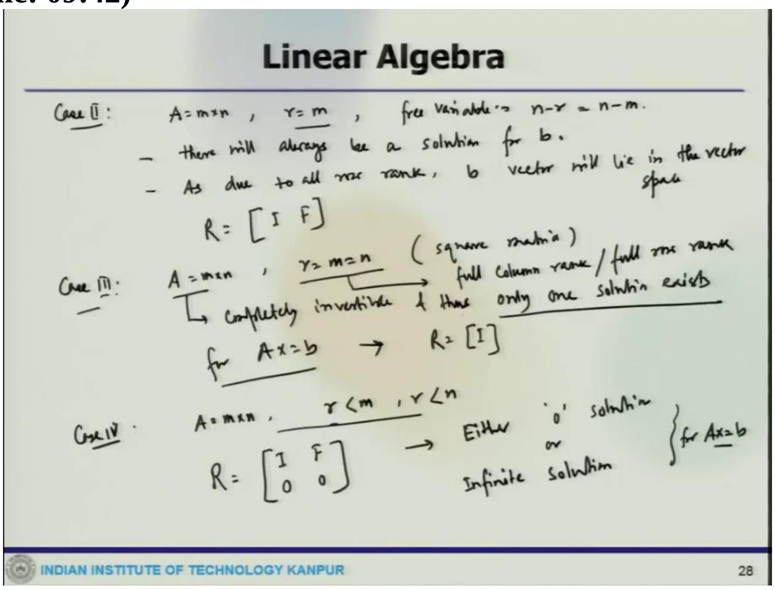

So that brings to an important issue of that, and all what we are saying here, the solution of x = b and then we are talking about b has to lie in column space and the consists of some column vectors. The vectors have to be independent. So, that is where we bring the formal definition of linear independence. So, essentially, when do we call a set of vectors to be linearly independent?

So, let us say there are a set of vectors which are x_1, x_2 to x_n and basically if none of the vector is a linear combination of the other, so, what we are saying that none of the vectors are linear combination of other. So, which is not necessarily all of the other then the vectors are linearly independent. So, which would you have been also looking at the column space vector if all the columns are not kind of, I mean any of the column is not a linear combination of the other columns, then the all the columns are independent?

And we can get full pivotal rank which would be the case for like situation where r = n like that. So, in mathematically what we can write we can write for any scalar $C_1\ x_1 + C_n\ x_n$ where this is possible for linear independence if and only if all C_i are 0. So, then what do you can say that all the x_i vectors are linearly independent. So, there is no linear combination in order to find another in case of a linearly independent set of vectors. Now, the thing when there is a span of a vector space or basis and dimension.

So, here again when you try to explain that we will start with an example. For example, let us say $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$. So, this corresponds to x_1 this is x_2 x_3 now, here there are 3 vectors and they are spanning over 3 Dimensional space. Now, any addition scalar multiplication does not cause change from 3 dimensional spaces. Now, that is one situation the other case in this column vectors what do you see here?

If you see carefully these 2 if we add it then the third one we can get or from third one to subtract the second one we can get the first one or third one to first one and we can get this which clearly means that 2 column vectors any 2 column vectors they are linearly independent not the third one. So, the column space will have 2 independent vectors and this is a 2 dimensional subspace in R^3.

Now, the conditions of being basis vectors so, they are conditions of being basis vectors 1 they are linearly independent 2 they span over the space. So, these are the conditions for being basis vectors. So, here what do we have the x_1 and x_2. So, this would be basically let us say x_1 and x_2 are the basis vectors here x_1 and x_2 are basis vector, so, x_3 would be dependent.

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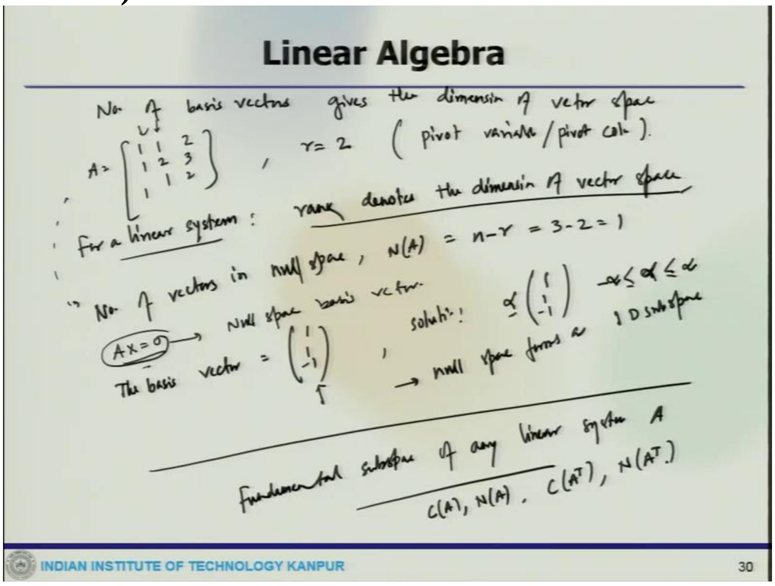

So, the number of basis vectors when you get the, so, the number of these basis vectors in gives the dimension of the vector space, same thing here, because we said that the column space has 2 dimensional subspace and because the 2 vectors which are going to be linearly independent, now, here for this particular example, this A which is $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, the rank is 2. So, that means that many pivotal variable or pivot columns are there. So, for a linear system the rank denotes the dimension of the vector space.

So, that means, for this is very important to note for linear system rank denotes the dimension of vector space. So, once we find out that rank, which means there are a number of pivots or the pivot columns or whatever. So, when you find out that, define the vector space or other column space. Now, similarly, we can find out the number of vectors in null space, which is going to basically give us the 0 solution for a linear system N(A) would be n - r, which is 3 - 2 one for this particular example.

Now, here the basis vector for the null space would be, one can clearly see that, because there is just to we have already discussed these things, how to find out the null space basis vector, we can do that Ax = 0 solution from there, we can find out the null space basis vector. But since the example that we have taken here looks quite simple to visualize and one can say, the basis vector here, is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ because if these 2 are considered to be independent that the third one is going to give us the zero solution.

So, and the solution would be $\alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ where $-\infty < \alpha < \infty$. Here there is only one, this is vector and any other combinations with alpha or C whatever the scalar, this could form the null space here null space forms a 1D subspace because the number of basis vectors is going to form the vector space. So, here the number of basis vector is 1 and we can have multiple solutions with the new scalar multiplying that.

So, this is the basis vector and there is any scalar which can give infinite number of solutions. But this is there are only one basis vector which is the present the zero solution to that this form so, one dimensional subspace and this is very, very important.

How these column vectors in a linear system contribute to all these different spaces for the column space or null space or column space? Now we can with that we can actually go to a like we look at the fundamental subspace of. So, we go to the fundamental subspace of any linear system. So, we will look at what are those. So, we have already looked at that, like there are 4 fundamental subspaces one would be column space, one would be null space, then you can have column space of A transpose the null space of A transpose.

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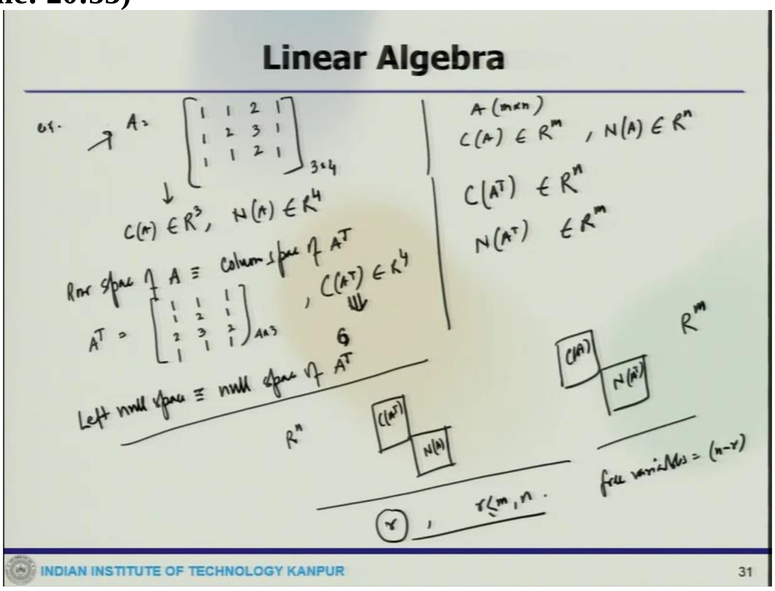

So, we can see just taken 3×4 system let us take a system. So, the idea here is that when you take an example, you can easily visualize what is happening here and then through that, you can come to a solution quite quickly. Now, here as we say, so, the any system for linear system of $m \times n$ that column space belongs to R^m the null space would belong to R^m.

So, now, for this particular example, here the column space would belong to R^3 and null space would belong to R^4. Now, what happens to row space of A which is equivalent to column space of A transpose that means, if I have a system if I just do the transport that become the row space of A in this particular case, the A transpose would be $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ which is 4×3 which means the column space of A transpose belongs to R^4.

And that saying that that column space of any $n \times m$ system that column space of A transpose belongs to R^n. Similarly, here if you look at the left null space which is the null space of a A transpose. So, this belongs to R^m. So, if we put these guys in sort of an kind of picture here, this would be my column space of A transpose null space of A. So, this is in our n or if I see that so, this is column space of A null space of A transpose that belongs to R^m.

So, this is the picture what you can see when you talk about these fundamental subspaces. Now, another thing important thing is that the dimension of any matrix is given by its rank r and let us say $r < m$ or less than equals to m or n. So, that is impossible. So, for any this $m \times n$ system, if the rank is r, the number of free variables would be always $n - r$. So, let us take this guy and look at these different fundamental subspaces.

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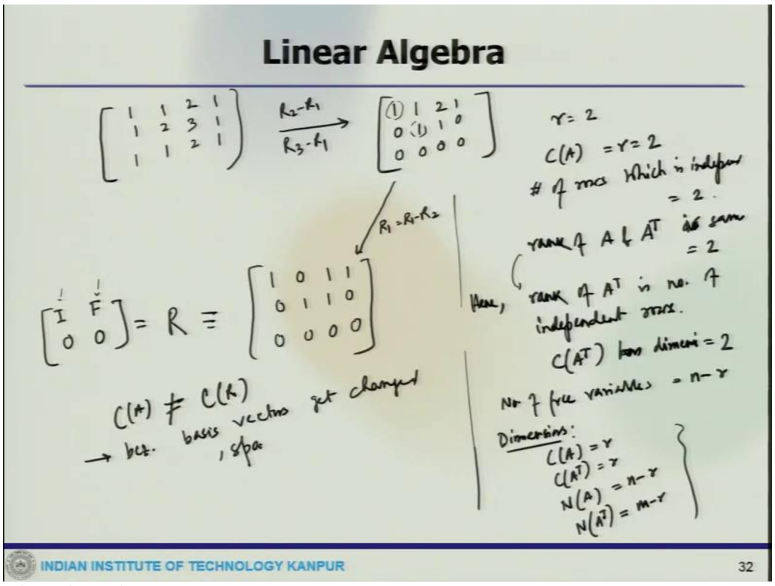

So, we have $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$. So, what we need to do again do the elementary operations let us say R_2 - R_1 and R_3 - R_1. So, what we get is that $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, there are pivots here. So, r would be 2 because there are 2 pivot variables. So, the dimension of the column space would be rank and that would be 2 and the number of rows which is also independent. So, the number of rows which is independent equals to 2 again a rank of A and A transpose is same.

So, that would be 2 and hence the rank of A transpose. So, the rank hence the rank of A transpose is number of independent rows. So, the column space of A transpose also has a dimension equals to 2. So, here the number of free variables or would be n - r which is the dimension of the null space. So, now, one can find out the dimensions of these fundamental spaces for column space, this would be the rank and column space of A transpose that is also R.

Now, that dimension of N(A) or the null space, which is the number of free variables which is n - r and dimension of the null space or a transpose which would be m - r. So, these are the 4 fundamental subspace and their dimension what one can find it out now, here if we do little bit of more operation like $R_1 = R_1 - R_2$ and what we can get here is that $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, which is saying equivalent to the row reduced form and that is nothing but $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ this is corresponding to free variables.

This is independent variables or pivot variables. Now, column space of a matrix and the row reduced form are not same as even though row operations are done, columns are being changed. So, that is very very important to note. Hence, basis vectors are being changed thus. So, what we are trying to see here is the column space of A matrix and the row reduced form of not the same as even though we have done the row operations.

But columns are being changed. So, the basis vectors are being changed. So, the column space of A is not the same as row reduced form column space of row reduced form. So, here because basis vectors get changed. Hence, the span also changes. So, this is quite important to note that this would happen when we do that.

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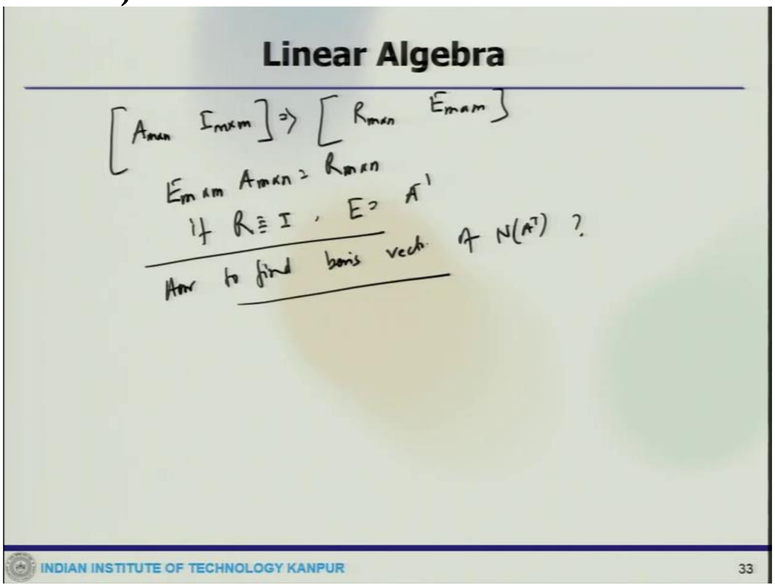

So, now in this particular case, so, what we can say that $[A_{m\times n} \quad I_{m\times m}]$ which is $[R_{m\times n} \quad E_{m\times m}]$. So, now, here if you look at the $E_{m\times m}\, A_{m\times n} = R_{m\times n}$. If R= I, then E is A^{-1}. Now, the point here is that how to find basis vector of null space of A transpose. So, that is the question which raises here. So, we will stop the discussion here and continue from here in the next session.