Computational Science in Engineering
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Lecture – 06
Linear Algebra

So, let us continue the discussion of this dimension span and the basis and we are looking at these different subspaces of the fundamental subspaces of a linear system and they are like column space or null space column space of A transpose and null space of A transpose or left null space and this is where we stopped.

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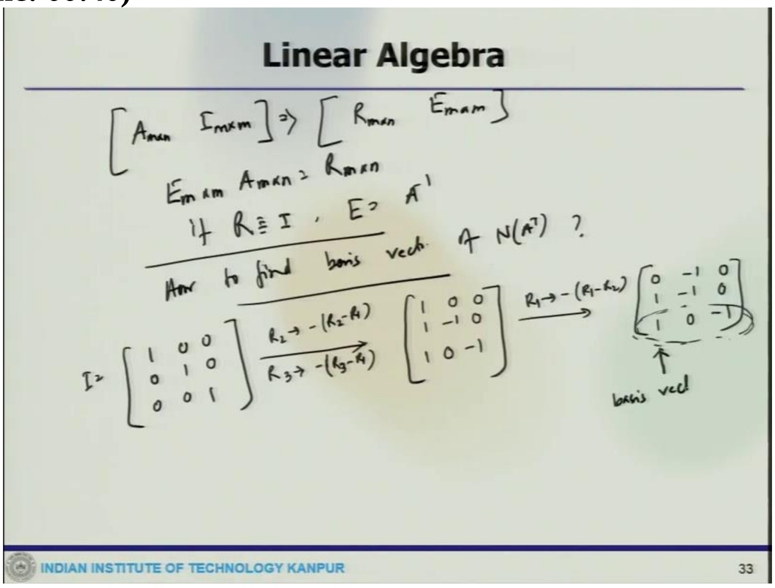

Saying that how to find out the basis vector of null space of A transpose I mean one way to look at let us take an identity matrix which is a 3×3 system $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ here if we do row operations like $R_2 \to -(R_2 - R_1)$ and $R_3 \to -(R_3 - R_1)$. So, what we get here? $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ we do $R_1 \to -(R_1 - R_2)$ and what do we get $\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. So, if you see that now, if you just look at this row reduced form this is the row or the third row which becomes 0 and here when you started with this.

So, this is where we started if R equals to I then this is the row which is reduced the reduces to 0 in this case and so, this particular vector would represent the basis vector of null space of A transpose. So, this would be the basis vector of A transpose.

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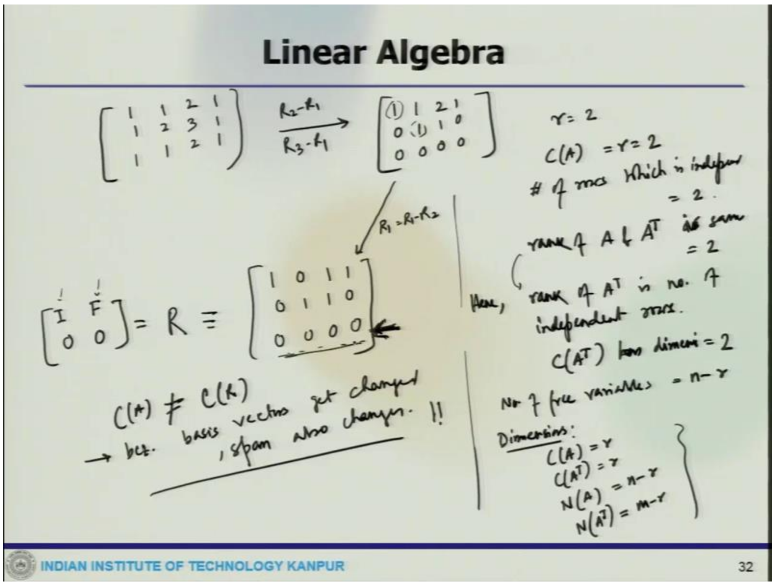

Because in the row reduced form this is the row which gets to 0 and that because of this basis vector. So, the that is the way one can find out even; so, the eventually the point here is that you can find out all this like column space, null space, column space of A transpose, null space of A transpose and their rank basis vector and everything.

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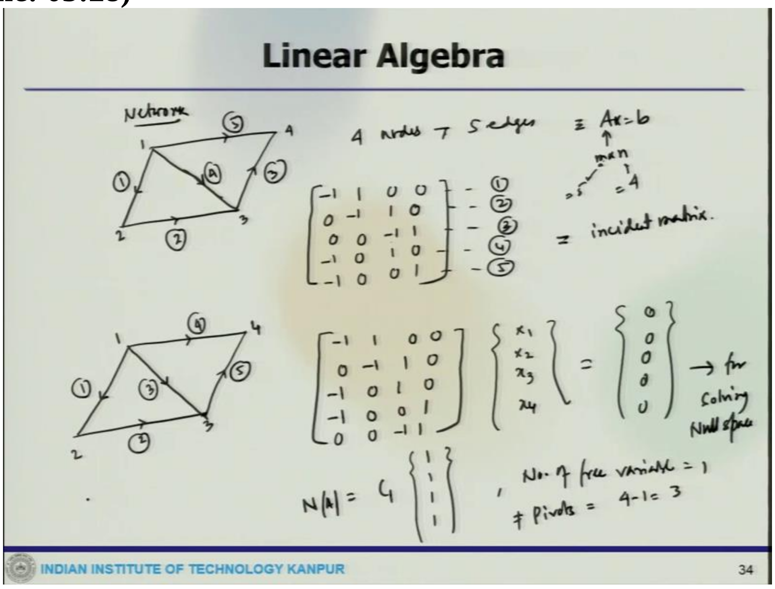

Now, important to look at some of this system where it can be applied because the type of system depends on the governing equation of the physical system. So, as I said you got this is going to represent this governing equation. So, it depends on the physical system. So, let us consider, let us say a network which can be for electrical molecular whatever be the reason. So, let us do that and we have a network like that.

So, this is vertex 1 2 3 4 and these are the edges where this could be one and the direction also given to let us say 3, this is 4, this is 5. Here we have 4 nodes and 5 edges. So, this will help into formation of a linear system like Ax = b where A would be order of m × n and here n would correspond to number of columns and correspond to node. So, this would be 4 and rows would correspond to the number of edges which is 5.

So, since we know that direction so, we know that components of the coefficient matrix A like we can set it up here. So, in between 1 and 2 since it is going from there, so 1 0 0 now, 2 to 3 similarly 3 to 4 then 1 and 4 and then this is 0 0 1. So, this corresponds to edge 1 this corresponds to edge 2 this is 3 edge 1 is going 1 to 2, edge 2 to 3 edge 3 3 to 4 edge 4 1 to 3 edge 5 1 to 4. So, this is called the incident matrix now, the same network we can change these things to a different configuration.

For example, let us say we do like this or this is the edge 1 2 3 4 this is 1 this is 2 then we say this is 3 this is 4 and this is 5. So, we can change that for that the incident matrix would be like $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. So, let us solve the system

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, this is for solving null space. So, one can see that the null space vector could be of this kind which is like $N(A) = C_1 \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$.

So, this is what it is going to be? So, here the number of free variables would be 1. So, you get number of pivots, which would be 4 - 1 is 3. So, that means, when you do the operations here, we will get 3 independent columns.

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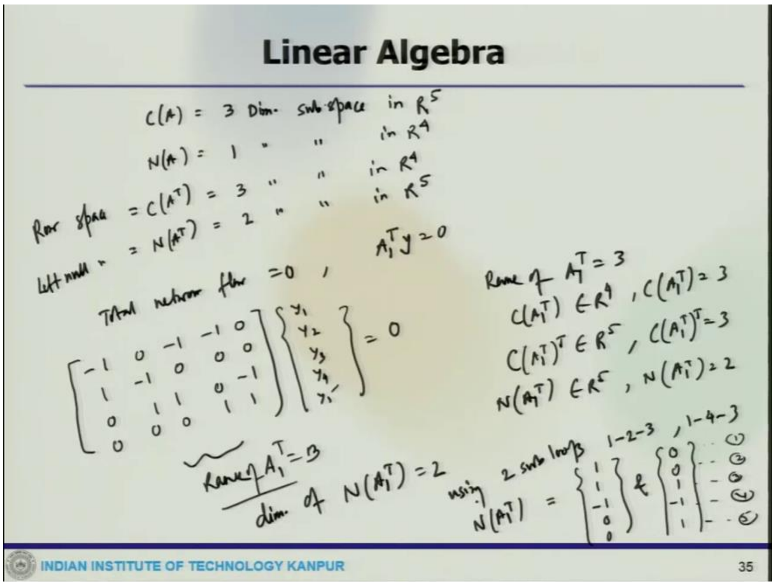

So, the column space here is a show for this particular case the column space is a 3 dimensional subspace in R^5 and null space is 1 dimensional subspace in R^4 and row spaces or which is the column space of A transpose or one can say row space which is again 3 dimensional subspace in R^4 and left null space or null space of A transpose is which is here $m \times n$ which is 5×4. So, $n - r$, so, which would be m minus here you can see this is $m - r$. So, this would be 5 - 3 it is a 2 dimensional subspace in R^5.

So, from this particular example, you can see that all these things now, total network flow is 0 thus we can solve for $A_1^T y = 0$. So, what we can write here?

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = 0$$

So, here the rank of A_1^T is 3. So, column space of A_1^T belongs to R^4. So, when I am supposed to have A_1^T here is 3 now, column space of A_1^T of transpose which is belongs to R^5 which is A_1^T to transpose is 3.

Null space of A_1^T belongs to R^5 which is 2. So, the rank here is rank of A_1^T is 3. Now, the dimension of null space of A_1^T is 2. Now, to find out the basis vector by just visualizing would take into account I mean one can do the elimination and find out but by visualize and one can find out if you take two sub loops one is that 1-2-3 and 1-4-3.

So, these are two sub loops if you take you can get the basis vectors by using two sub loops 1-2-3 and 1-4-3 the basis vector of $N(A_1^T)$ can be found which would be like $\left\{\begin{matrix}1\\1\\-1\\0\\0\end{matrix}\right\}$ and $\left\{\begin{matrix}0\\0\\1\\-1\\1\end{matrix}\right\}$.

So, this corresponds to edge 1 2 3. So, these are the basis vectors from if you so, this is by using two sub loop.

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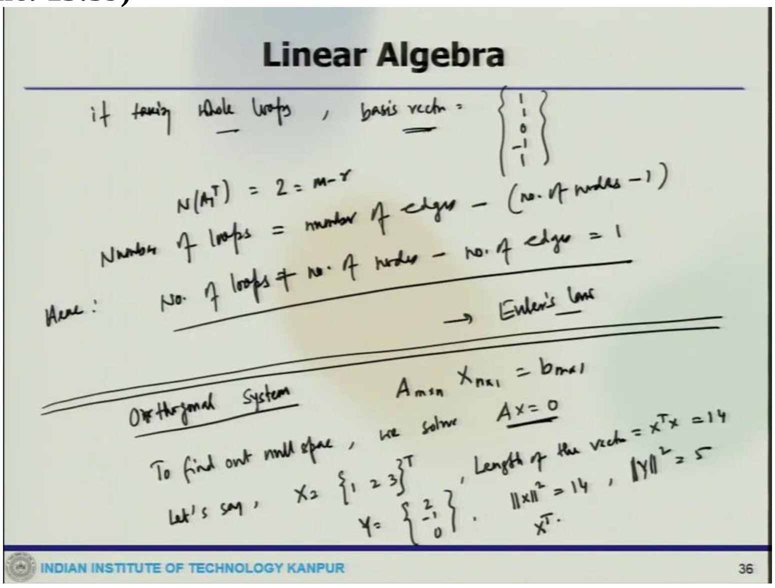

If taking whole loops then the basis vector would be $\left\{\begin{matrix}1\\1\\0\\-1\\1\end{matrix}\right\}$ which is a combination of the basis vectors of in a transpose but not a basis vector itself. So, we can see the addition of these two or the combination of these two is going to give us a basis vector. But this guy is not a basis vector of A transpose, but their combination is going to give us a basis vector of these when you consider the whole loop here.

So, $N(A_1^T)$ has 2 which is m – r which means the number of loops equals to number of edges minus number of nodes -1. Hence, we can say number of loops plus number of nodes minus the number of edges equals to 1. So, this is well known Euler's law. So, that is how you get to the kind of trying to find out all these fundamental subspaces and so on now we can move to the other part of the things which are called like orthogonal system.

Now let us say we have $A_{m\times n}\, X_{n\times 1} = b_{m\times 1}$. So here to find out null space we solve Ax = 0. So, from there we get this now for orthogonal vectors dot product must be 0 let us say we taken vector X which is $\{1 \quad 2 \quad 3\}^T$ which is the then the length of the vector would be X transpose this which is putting. similarly, if you have an Y which is $\begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix}$ then this is 14. This is 5 and we can see X transpose Y is 0.

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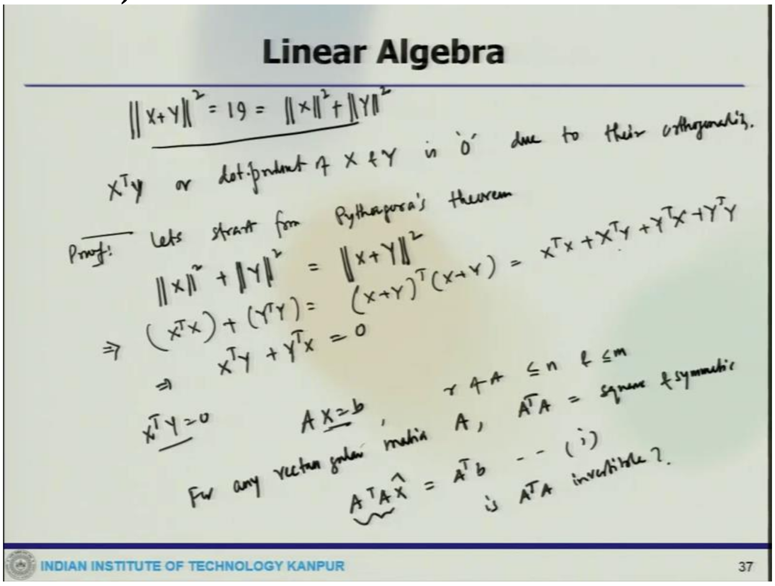

And we can have X + Y which is also 19 which is x square. So, you can see there X and Y follow Pythagoras theorem and hence we can see X transpose Y or the dot product of X and Y is 0 due to their orthogonality. Now, we can actually prove that so we can start so to see that let us start from Pythagoras theorem where we will say square equals to X + Y square. So, this is Pythagoras theorem.

So, what we can write is that

$$(X^TX) + (Y^TY) = (X+Y)^T(X+Y) = X^TX + X^Ty + Y^TX + Y^TY$$

So, which gets to $X^Ty + Y^TX = 0$. So, for any vectors X and Y $X^Ty = -Y^TX$. So, hence we can say $X^Ty = 0$ or the dot product of 2 vectors satisfying Pythagoras theorem alternately condition orthogonality condition is satisfied now.

We have a solution now, where like we get for Ax = b and we have a solution for this so b lies in the column space of A and the rank of A is less than n and less than equals to m. So, for any rectangular matrix A, A transpose A is always a square and symmetric. So, we can have solution of $A^TA\hat{x} = A^Tb$. So, the question is that is this guy is A^TA invertible. So, that is the question if that is not the case, then this should be difficult.

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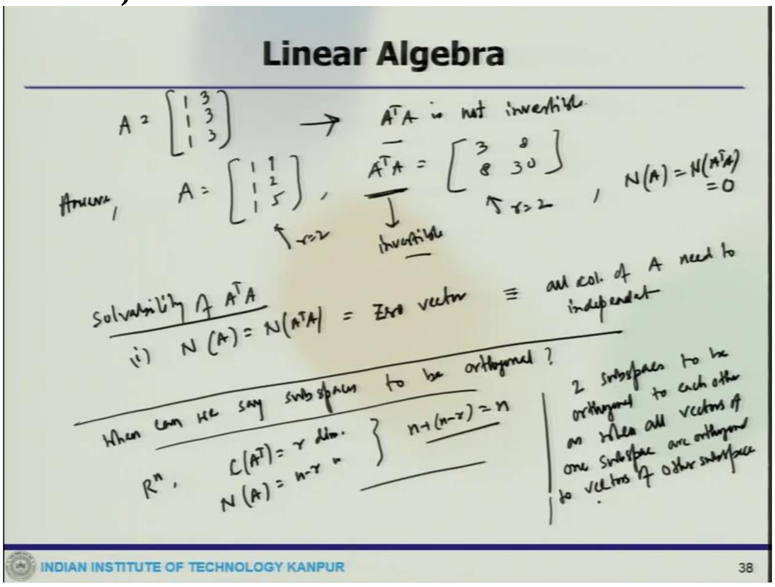

So, let us suppose, we have $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$ so, then obviously here $A^T A$ is not invertible that is clear. So, but thus in this condition equation 1 is not. So, with this condition this equation 1 is not solvable. However, if you have a situation where A is like $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$ then $A^T A$ is $\begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$. So, here both A and $A^T A$ has ranked 2 this guy has a rank 2 these guys has also rank 2 and hence null space of A is null space of $A^T A$ is 0.

So, which means this $A^T A$ is invertible or this will give this solvability of $A^T A$. So, when this is invertible, we can have the solution. So, one can say that solvability of $A^T A$ is possible one is that null space of A should be null space of A transpose the N is in zero vector that is all columns of A need to be independent which essentially means all columns have a need to be independent.

So, $A^T A$ will give the magnitude of the coefficient matrix and we can have the solution. So, see these things like orthogonality and all these things they are all kind of sort of connected with finally the solution of Ax = b because that is what we are interested in now, then the point comes when can we say subspaces to be orthogonal? So, that is a question that now we need to answer.

So, row space and null space comes in R^n and column space of A^T is r dimensional and null space of A is n - r dimensional. Now, if they are orthogonal as they are the sum of their dimensions give the overall dimension which is like n + n - r which is n. So, if that is the situation, we say two subspaces to be orthogonal to each other, so, that when we can say that means two subspaces to be or orthogonal to each other as when all vectors of one subspace are orthogonal to vectors of other subspace.

However, if two planes or subspaces are orthogonal, it does not mean all the vectors in one plane is perpendicular to those in another. Now, here that three trivial conditions of being a subspace is kept in mind two subspaces to be orthogonal to each other cannot meet at a non-zero vector which means the so, zero vector is a trivial constituent of any subspace and hence, only possible intersection point for two orthogonal subspace is the zero vector or origin. So, this is a really important statement.

Again, let me rephrase that. We say that when two subspaces are orthogonal to each other, as in all vectors of one subspace to be orthogonal to the vectors of the other subspace. But however, if two planes or subspaces are orthogonal, it does not mean all the vectors in one plane is perpendicular to those in another. Now, here that three trivial conditions of being a subspace is kept in mind, which we have already discussed that two subspaces to be orthogonal to each other cannot meet at a non-zero vector.

So, that means zero vector is a trivial constituent of any subspace and hence only possible interaction point for these two orthogonal subspaces is zero vector origin. So, these are the said situations where the orthogonality conditions are there for subspaces and the need to satisfy. So, I will stop the discussion here and continue the discussion in the next session.