Computational Science in Engineering

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Lecture – 07
Linear Algebra

So, let us continue the discussion on linear algebra. So, we have now looked at different subspaces and their dimension, just like four fundamental subspaces, which are there. For example, for a linear system of A, you get to have four fundamental subspaces one is column space, one is null space one is column space of A^T, null space of A^T and we can find out their dimensions then the basis vectors and all sort of things. So, we have looked at also the orthogonality condition. Now, we are going to continue the discussion on projection.

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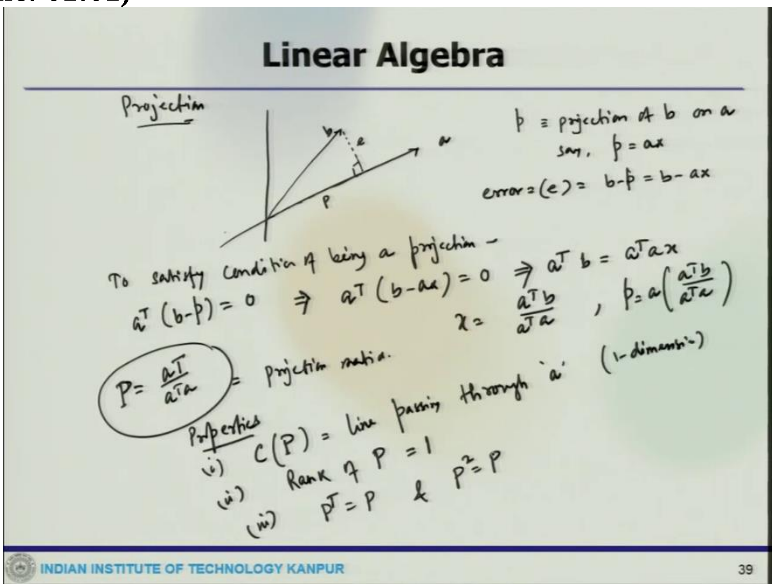

So, what we look at here is the now the projection and how we look at let us say we start with some vector, let us say you have like this, you have in this is a and this is a vector b then the projection to be here and so, this is again perpendicular. So, this is the projection p and this is the e. So, you can see the vector here you can say that the p is projection or said to be projection of b on a let us say $p = ax$ then what is the error so error is e which is $b - p = b - ax$.

Now, to satisfy a condition of being projection to satisfy a condition of being a projection what we write is that $a^T(b - p) = 0$ which means $a^T(b - ax) = 0$ which is $a^T b = a^T ax$. So, what we get that $x = \frac{a^T b}{a^T a}$. hence, projection would be $p = a\left(\frac{a^T b}{a^T a}\right)$. So, let us write it in that way.

So, that we can avoid some confusion. So, this capital P which is $p = \frac{a^T}{a^T a}$ this is called the projection matrix. Now, again to satisfy condition of projection this needs to be one dimensional the properties of projection matrix are so, there are some properties numbers

i) the column space of these capital P is line passing through passing through a which is one dimensional

ii) the rank of P would be 1

iii) P transpose equals to P and P square should be equals to P.

So, now, let us translate the idea of Ax = b now, we may not have the solution, but we can try to find out the nearest solution. So, if there is no solution, that is b does not lie in column space of A. So, here we are trying now find Ax = b. So, there is no solution means b does not lie in the column space of A.

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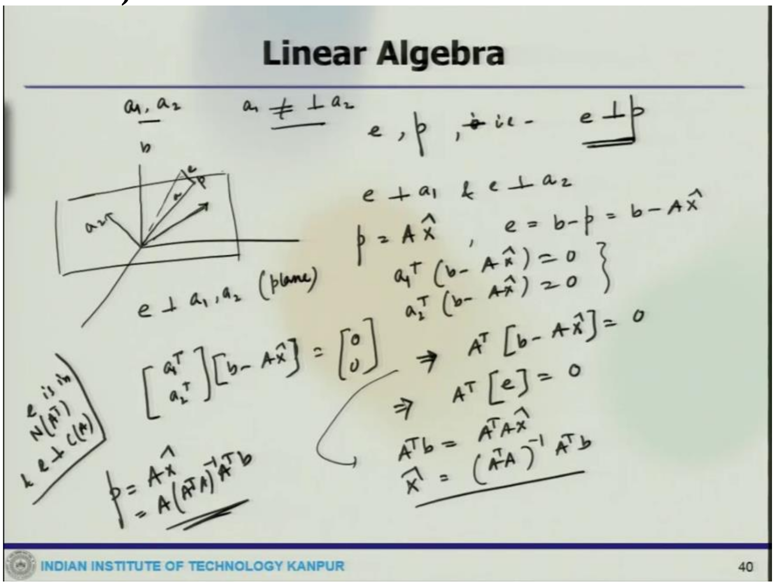

Again, it may so happen that let us say small a_1 or a_2 or column vectors of a and where a_1 and a_2 are not perpendicular to each other, but they are independent. So, here a_1 not perpendicular to a_2, but they are independent. So, we may not lie in the plane produced by $a_1 a_2$ but may have a projection p on it. So, the way we can see this like let us say this is a_2 and this probably this guy this is b then this is a this could be error, this is p and we can have a sort of plane like this.

So, it is now here the error vector and the p projection vectors are perpendicular to or perpendicular. So, that means that e is perpendicular to p. So, we have the error vector and projection vector they are perpendicular and we can now p lies in a plane which is found by a_1

and a_2 and since a_1 and a_2 are independent. So, what will happen is you would be perpendicular to a_1 and e would also perpendicular to a_2 independently. So, what we can write that $p = a\hat{x}$ again the error would be $b - p$ which is $b - a\hat{x}$.

So, since we have e is perpendicular to a_1 and a_2 that is basically the plane what we can write ${a_1}^T(b - a\hat{x}) = 0$, ${a_2}^T(b - a\hat{x}) = 0$. So, they are using two conditions of the orthogonality condition. Now, if we combine them then what we can write is that you want transpose ${a_2}^T(b - a\hat{x}) = 0$. So, in turn, we can write $A^T(b - A\hat{x}) = 0$ for the sum now, this guy we can write e is 0.

So, now, from here, we can again write $A^T b = A^T A\hat{x}$. So, $\hat{x} = (A^T A)^{-1} A^T b$. So, my projection matrix or projection vector is $p = a\hat{x}$. So, this would be $p = A(A^T A)^{-1} A^T b$. Now, what is important here is that e lies in the e is in a null space of A transpose and e is perpendicular to the column space of A.

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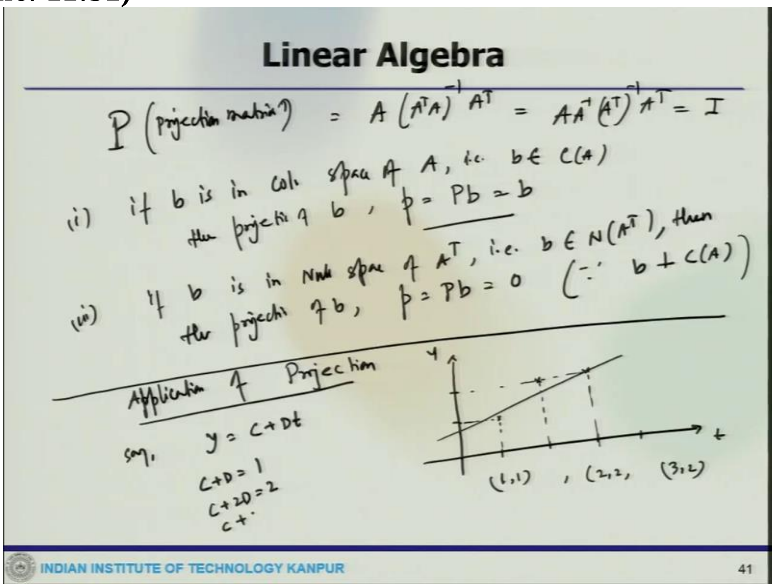

So, making the projection matrix what we can get the projection matrix so, this is the projection matrix capital P which would become $P = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1} = I$. Now, what is there if b is in column space of A so, that is b belongs to column space of A the projection of b that equals to Pb which is b.

Ssecond, if b is in null space of A transpose that is b belongs to null space of A transpose then the projection of b which is p equals to Pb is 0 since b is perpendicular to column space of A. So, this condition they need to be satisfied. Now, we can see some application of projection that we can see. So, let us take some function say we have a function $Y = C + Dt$ and we can plot that this is t and this is Y let us see one. So, we can say like this. So, that could be there. So, this is could be there. So, let us take these points, which is for example, let us say (1, 1) (2, 2) (3, 2). So, they formed a system like C + D = 1, C +2D = 2, C + 3D = 2.

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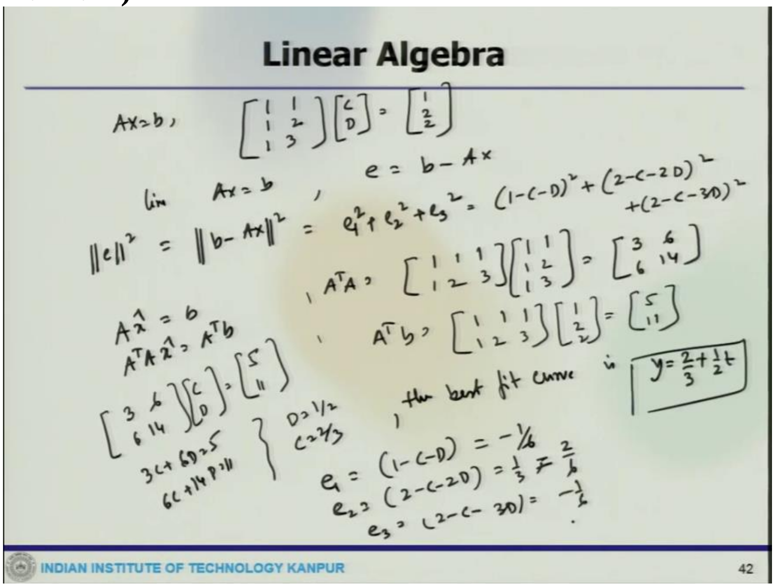

Now, in terms of linear system, where we can write Ax = b, we have $\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Now, in order to solve this, one approach is to draw a line which is best fit, that is distance of each point from the line is minimum, this is the least square method where error is minimized. So, let us fit the curve in line here, the line Ax = b. So, error would be $e = b - Ax$. So, what we can say magnitude of error square is $\|e\|^2 = \|b - Ax\|^2 = e_1^2 + e_2^2 + e_3^2$.

So, which one can write $(1 - C - D)^2 + (2 - C - 2D)^2 + (2 - C - 3D)^2$. So, we can or visualizing the closest or approximate solution we can say that $A\hat{x} = b$. So, $A^T A\hat{x} = A^T b$. Now, $A^T = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}3 & 6\\6 & 14\end{bmatrix}$. So, what we can write here $\begin{bmatrix}3 & 6\\6 & 14\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}5\\11\end{bmatrix}$ because $A^T b = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$. So, here we get 3 C + 6 D = 5, 6C + 14 D = 11.

So, after solving this, what do we get $D = \frac{1}{2}$, $C = \frac{2}{3}$. So, the best fit curve is $Y = \frac{2}{3} + \frac{1}{2}t$. So, that is the best fit curve and the errors like

$$e_1 = (1 - C - D) = -\frac{1}{6}$$

$$e_2 = (2 - C - 2D) = \frac{1}{3} = \frac{2}{6}$$

$$e_3 = (2 - C - 3D) = -\frac{1}{6}$$

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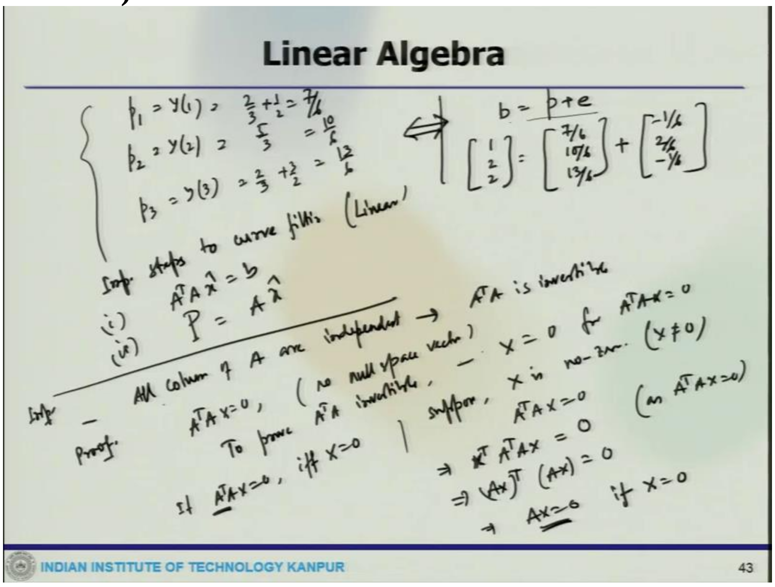

So, we can write

$$p_1 = y(1) = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

$$p_2 = y(2) = \frac{5}{3} = \frac{10}{6}$$

and so,

$$p_3 = y(3) = \frac{2}{3} + \frac{3}{2} = \frac{13}{6}$$

Here p_i are the predictions of the points on the best fit lines. So, p_1, p_2, and p_3. Now, it verifies the result of $b = p + e$. So, which is $\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}7/6\\10/6\\13/6\end{bmatrix} + \begin{bmatrix}-1/6\\2/6\\-1/6\end{bmatrix}$. So, you can see the projections to the best fit line and some also satisfied the condition or it verifies.

So, they are kind of equivalencies there. Now, there are two important steps to curve fitting that is linear one is that $A^T A\hat{x} = b$ and second is projection matrix would be A transpose. So, obviously, when you say that the important thing is to note that all columns of A are independent and then only $A^T A$ is invertible. So, this is also the condition for projection system.

So, without this we cannot find that. So, you can have a small quick look how to just to prove $A^T Ax = 0$ that means, we do not have any null space vector. So, no null space vectors we have to prove that $A^T A$ is invertible. So, need to show that, so, to prove $A^T A$ invertible it is so that x is 0 for $A^T Ax = 0$ and if $A^T Ax = 0$, this is only possible if and only if x is 0 and then A^T is invertible.

So, let us say suppose x is non-zero that means $x \neq 0$ and we say $A^T Ax = 0$. So, what we can write $x^T A^T Ax = 0$. So, which means $(Ax)^T (Ax) = 0$. Hence, Ax = 0 which is only by true if x = 0 as all columns of here independent, if x is 0 and $A^T A$ is invertible and there is only zero vector in the null space. So, the most fundamental orthonormal vectors are unit vectors of the right handed axis system.

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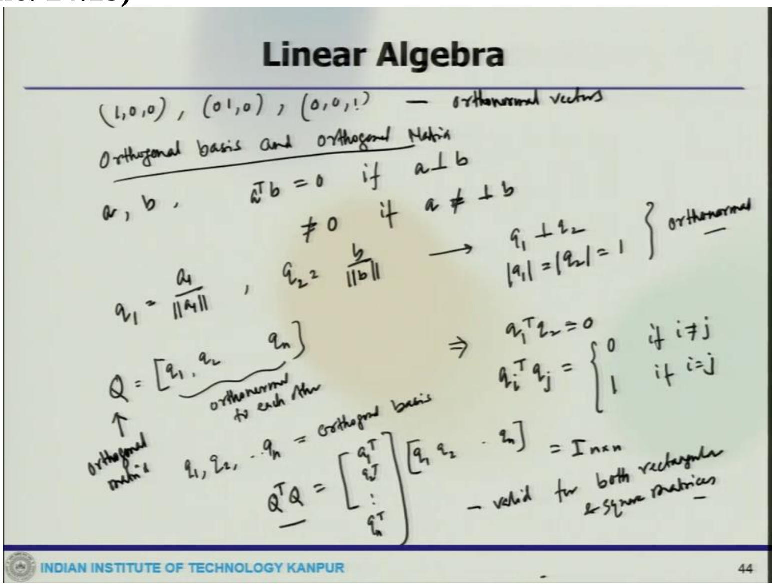

Like (1, 0, 0) (0, 1, 0) or (0, 0, 1). So, these are orthonormal vectors. They are not only perpendicular to each other, but they are individual magnitude is also 1. So, now we can look at orthogonal basis and orthogonal matrix. So, let us say we have two vectors a and b. So, $a^T b = 0$ if a is perpendicular to b and this would be not 0 if a not perpendicular to b. So, now given any vectors which are orthogonal.

So, when you take any vector which are orthogonal then we can say or find out let us say, $q_1 = \frac{a_1}{\|a_1\|}$, $q_2 = \frac{b}{\|b\|}$. So, here a and b are orthogonal and once you do that here q_1 and q_2 are again going to be orthogonal but at the same time their magnitude is are also. So, q_1 is perpendicular to q_2 magnitude of q_1 is magnitude of q_2 which is also 1.

So, q_1 and q_2 are orthonormal. So, they are called orthonormal. So, any vectors they can be orthogonal to each other, but unless their magnitude is also is one they cannot be said orthonormal to have the orthonormal vectors we not only have these vectors to be orthogonal to each other also their magnitude to be also 1. So, let us say a matrix we have a Q matrix where the vectors are q_1, q_2, q_n.

So, these are all orthonormal to each other. So, here Q is called the orthogonal matrix this is called the orthogonal matrix and these guys this $q_1, q_2, \ldots q_n$. These are called orthogonal basis. So, what we have that ${q_1}^T q_2 = 0$ and if we extend that for this n dimensional vectors or in vectors we can write ${q_i}^T q_j$ is 0 if i not equals to j, 1 if i = j.

So, this orthogonal matrix what we can write

$$Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} = I_{n \times n}$$

and this criteria of $Q^T Q$ becoming an identity matrix is also. So, this is valid for both rectangular and square matrices. So, this is valid for that.

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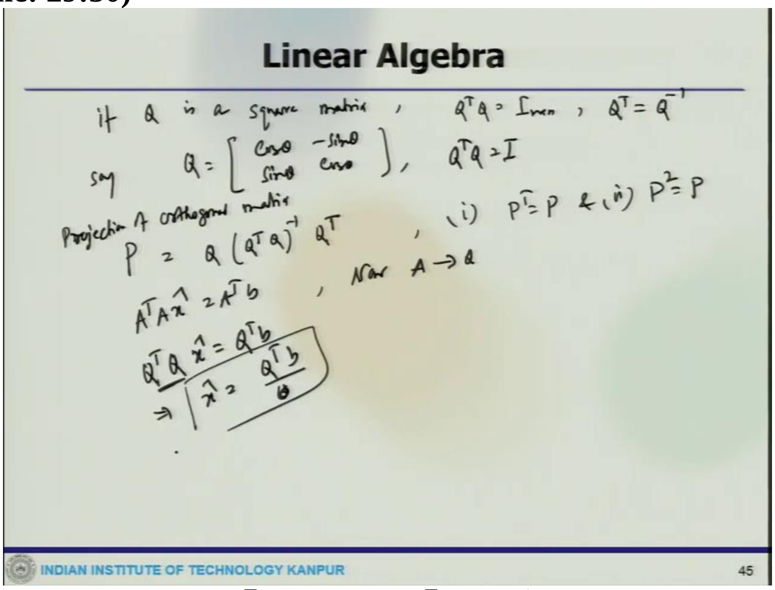

Now, if Q is a square matrix then $Q^T Q$ is $I_{n \times n}$, $Q^T = Q^{-1}$. Let us say Q is a matrix

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So, $Q^T Q$ is I. Now, also let us consider the projection of the orthogonal matrix. So, that would be P which is

$$P = Q(Q^TQ)^{-1}Q^T$$

So, now, this satisfies the properties

i) $P^T = P$

ii) $P^2 = P$

to take in consideration of the normal equation, we write $A^TAx = A^Tb$.

Now, is your Q then we can write $Q^TQ\hat{x} = Q^Tb$. So, $\hat{x} = \frac{Q^Tb}{Q^TQ}$ so, Q^TQ is identity, so, we get this. So, we can see how orthogonality and then from orthogonal to this orthonormal condition, they actually satisfy these conditions of the properties of orthonormality and all this will stop here and continue the discussion in the next session.