Computational Science in Engineering
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Lecture - 08
Linear Algebra

So, let us continue the discussion on these orthonormal vectors and orthogonal matrix. So, we have stopped here. So, looking at this orthogonal matrix and orthonormal vectors and this is what now, what we can do now, we will look at the Gram Schmidt decomposition. So, let us say we have this kind of situation like we have a vector like this and this is let us say another vector like this is. So, this direction is the a.

So, we say A this is b, then this would be the projection this is C. So, these are different vectors. So, let us say a and b these are initial vectors. So, we need to convert these two vectors to so, from a we have to convert to capital A and b vector to capital B to satisfy the orthogonality conditions which is $A^T B = 0$. So, we have to do that now, in order to do that also we can find out the e which would be b -p and e would be orthogonal to A. Now, here one important thing to be note here that one vector is to be assumed to be constant and equivalent to this. So, like what we have said here a should be equivalent to capital A.

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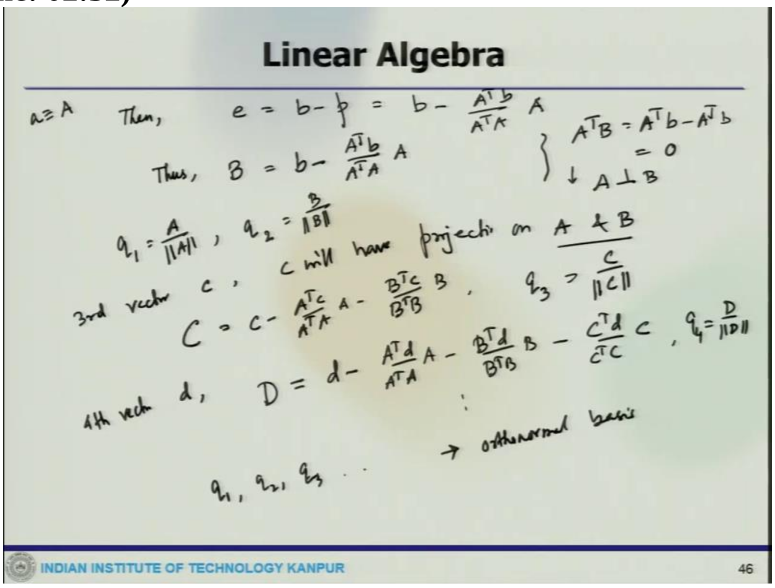

So, then what we can write so, A is equivalent to this then we can write that error for b minus projection vector is b - $\frac{A^T B}{A^T A} A$. So, these are all coming from the projection and all these. So, that is why we have discussed these projections and how to project a vector into another vector because these are the things what we are going to use thus we get capital B would be

$$B = b - \frac{A^T b}{A^T A} A$$

So, we can show $A^T B = A^T b - A^T b = 0$.

So, that means they are orthogonal. So, which means that A is orthogonal to B. Now, we find out $q_1 = \frac{A}{||A||}$ and $q_2 = \frac{B}{||B||}$. Now, we consider a third vector which is C, so, the third vector is C, now C will have projections on both A and B. Now, when I move to one to another vector. So, we start with the vector as we said one vector has to be equivalent or is the same then second vector will have a projection on A and then the third vector will have projection on both B vector and A.

How do you find that? So, here capital C would be

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T B}{B^T B} B$$

So, $q_3 = \frac{C}{||C||}$. Now, similarly, we can have a 4th vector, which is let us say d. So, d will have projections on all three A B and C. So, the D would be

$$D = d - \frac{A^T d}{A^T A} A - \frac{B^T d}{B^T B} B - \frac{C^T d}{C^T C} C$$

and like this we can go on if it is a field vector six Vector and so on we can find out this.

So, this is what you call the; I mean any vectors which are n number of vectors you can have these and these q_1. So, here q_3 this would be $q_4 = \frac{D}{||D||}$. So, all these q_1 q_2 q_3 these are the orthonormal basis and they form that orthogonal matrix.

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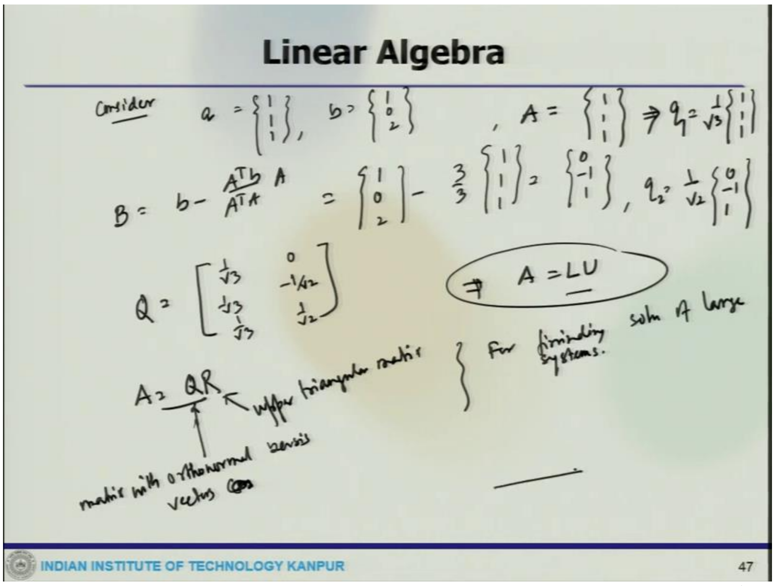

Let us have a look at some examples, which might let us consider $a = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ and $b = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$. So, $A = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ and this is b. So, $B = b - \frac{A^T b}{A^T A} A$, which is

$$B = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix} - \frac{3}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

So, from here we get $q_1 = \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ and here $q_2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$. So, these q_1 and q_2 are orthonormal.

So, the Q would be

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now, since we decompose a matrix A we can decompose A matrix let us say A into L and U lower triangular upper triangular similarly, we can decompose. So, this is what we have already done any matrix m, m × n we can decompose into lower triangular and upper triangular. Similarly, you can decompose the matrix A to the Q and R where Q is the orthonormal basis vector consisting the matrix and R is the upper triangular matrix this would be upper triangular matrix and Q is orthonormal basis vectors.

So, this is a matrix with orthonormal basis vectors it is an orthogonal matrix. So, this helps in solution of extremely large matrices or linear systems. So, now, you see, we started with the linear system of Ax = b and we have been talking about when we can find solution when the solution would exist or not. But then there are different decompositions that we have done like LU and now we are looking at QR and these would help for finding solution of large systems.

Now, one important thing to note here is that the given vectors must be independent and if not, then orthogonality condition will not be satisfied? So, the first vector needs to be independent that is very, very important.

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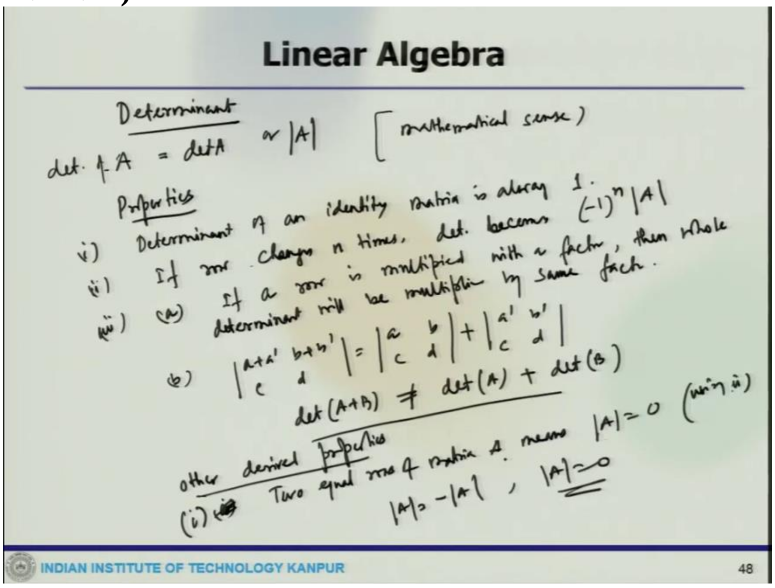

Now that is from here, now we will go to the another set of things that we will discuss is the determinant. So, mathematically one can write that determinant of A is like det A or A. So, this is what one writes in a mathematical sense, but there are certain properties of the determinant which are very important to be satisfied.

i) Determinant of an identity matrix is always 1

ii) if row changes n times. So, the determinant becomes $(-1)^n|A|$

iii) there are two part:

a) if a row is multiplied with the factor, then whole determinant will be multiplied by same factor

b) we can write $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$. So, means the det(A+B)≠ det(A)+ det(B). So, this is what is an important criterion now, there are other derived properties.

So, like derived properties like we can say two row equal rows of matrix A means is 0.

i) So, this one can say by the property of using the property of two. So, here so, let us say these are we can say or rather capital A or let us continue. So, this is using two. Now, if equal rows are interchange, then A is minus of A hence, A is 0. So, this is what using property 2 we can prove that if 2 equal rows are of a matrix A. That means, if the matrix has to equal rows then the determinant is going to be 0.

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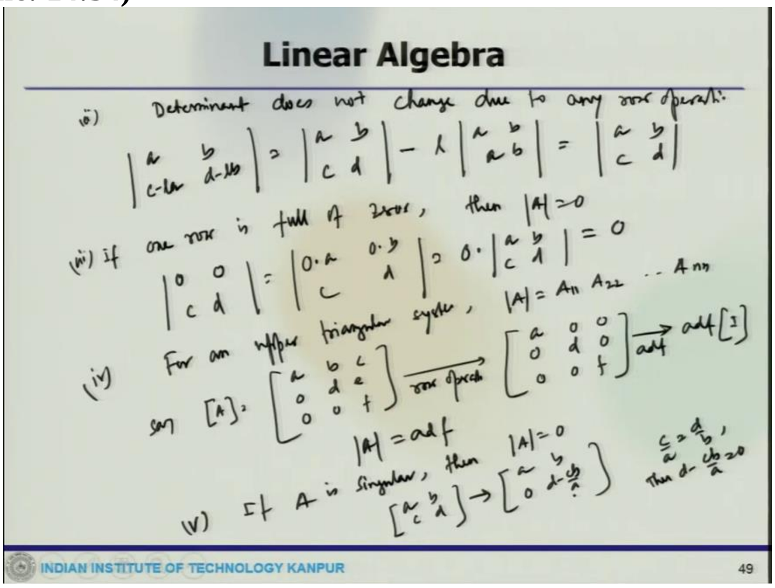

ii) Determinant does not change due to any row operation which means

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

So, determinant does not change due to any row operation.

iii) Now, if one row is full of 0s then we get the determinant is 0 for example, can say

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0-a & 0-b \\ c & d \end{vmatrix} = 0\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

So, when you say that these are derived properties that means, we are using all these properties from these 3 sets of properties.

iv) Now, for an upper triangular system $|A| = A_{11}A_{22} \ldots A_{nn}$ say we get let us say $[A] = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ then we do row operation we get $\begin{bmatrix} a & 0 & 0 \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$. So, we get $adf[I]$. So, the determinant should be $|A| = adf$. Now, if A is singular then $|A| = 0$. We can show that let us say $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and we get $\begin{bmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{bmatrix}$. Now, to a become singular we have to have c/a = d/b then $d - \frac{cb}{a}$ is also 0. So, that one full row 0 means this is going to be given.

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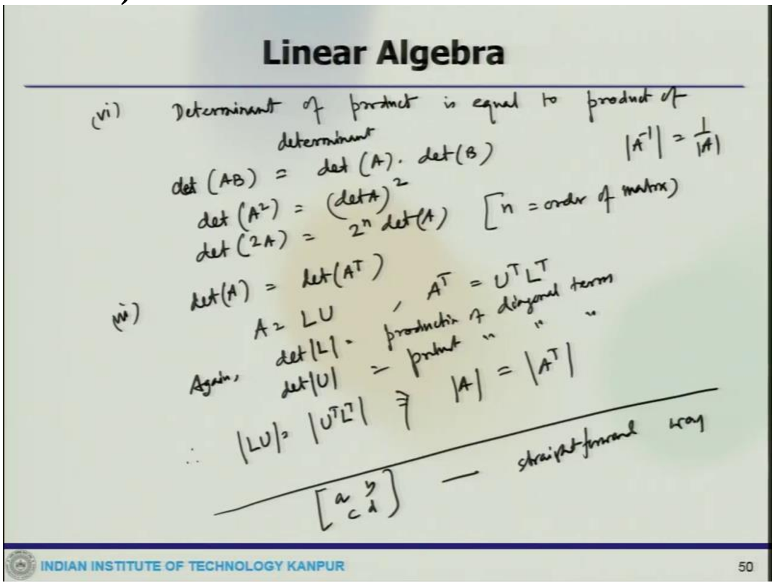

Now determinant of product so, determinant of product is equal to product of determinant. So, what we can say

$$\det(AB) = \det(A) \cdot \det(B)$$

So, we can say

$$\det(A^2) = (\det A)^2$$

So,

$$\det(2A) = 2^n \det(A)$$

where n is order of matrix. So, and also one can say $|A^{-1}| = \frac{1}{|A|}$. Now, $\det(A) = \det(A^T)$. Here we can decompose as lower triangular and upper triangular.

So, transpose these upper triangular into lower triangular transpose. So, again we can say determinant of L is product of diagonal terms and determinant of U, this will be the product of diagonal terms U is this which means determinant is same. Now, how do you find the determinant? Let us take a matrix of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So, we know how to find the determinantin a straightforward fashion. So, I mean, this is a simple 2×2 system, and we can find these out.

But here, what we will try to look at is that, we try to find out the determinant of this matrix we will use the properties of the determinant. So, I mean, one approach is the one say straightforward way, but other approaches that using properties.

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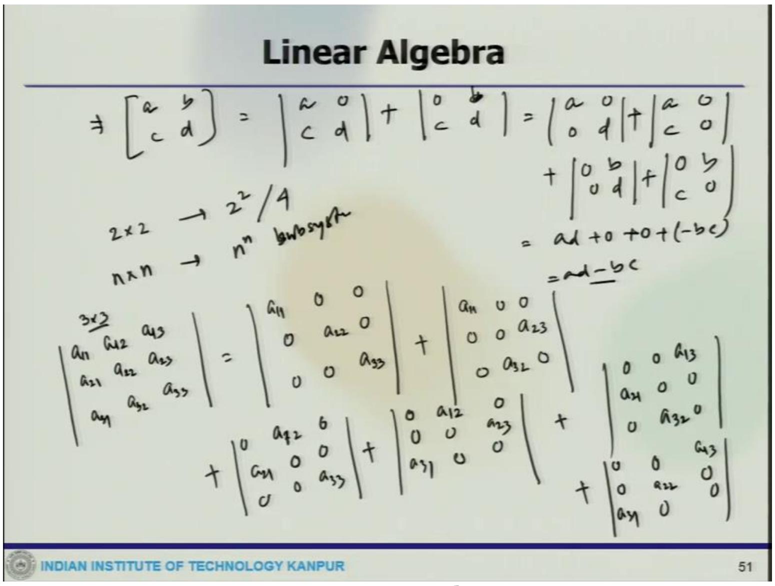

So, let us look at that how we can achieve that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So, we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

So, what we can say here is that ad + 0 + 0 + - bc. So, this would be ad - bc. Now, for a 2 × 2 system here into 2 × 2 system we are getting a 2 square or 4 breakup subsystem. Now, for n×n system will have n to the power n sub system I mean like you can say a 3 × 3 system here.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

So, here we are considering 3 elements in first row individually and accompanied with a 2 × 2 order subsystem.

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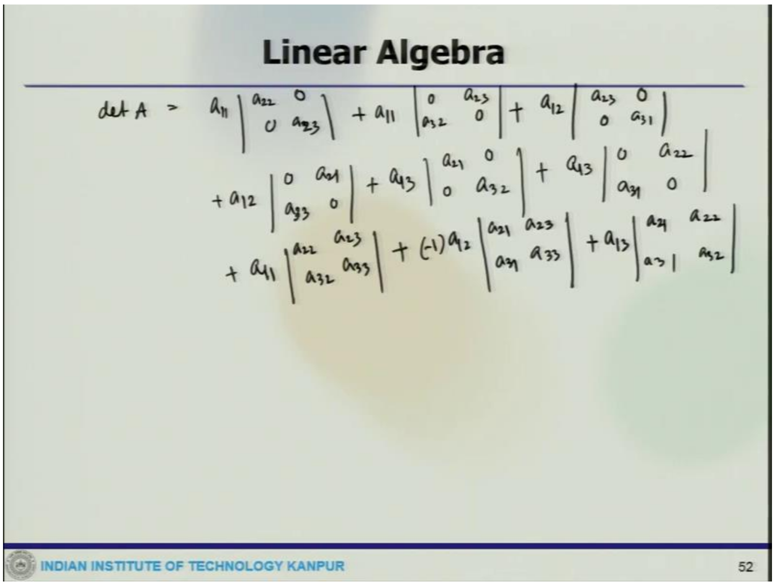

So, now, what we can write like determinant of A would be

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & 0 \\ 0 & a_{23} \end{vmatrix} + a_{11}\begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix} + a_{12}\begin{vmatrix} a_{23} & 0 \\ 0 & a_{31} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & a_{21} \\ a_{33} & 0 \end{vmatrix}$$

$$+ a_{13}\begin{vmatrix} a_{21} & 0 \\ 0 & a_{32} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & a_{22} \\ a_{31} & 0 \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

So, this is how you can see this is a 2 sub systems that we have taken then a well here you go by 2×2 sub system here.

So, if you break the subsystem like that, then you can find out so, and all these matrices which are going to be there or these are called the so called co factor matrices. So, we can actually, when you go to 3×3 system, you can have these 2×2 sub systems and can look at the things so we can finalize that calculation in the next session and see how you opt in these 2×2 subsystems and find our determinant.