Computational Science in Engineering
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Lecture - 09
Linear Algebra

Let us continue the discussion on determinant. So, what we have looked at so far, I mean, what are the properties of determinant and what is important here is that there are 3 important properties, and then there are multiple derived properties, and all these derive properties can be obtained using those 3 fundamental properties and they are very, very important in the sense.

So, now, using that we can find out particular determinant and I mean there are straight forward where one can find the determinant or there are ways where you can actually use these properties and find the determinant.

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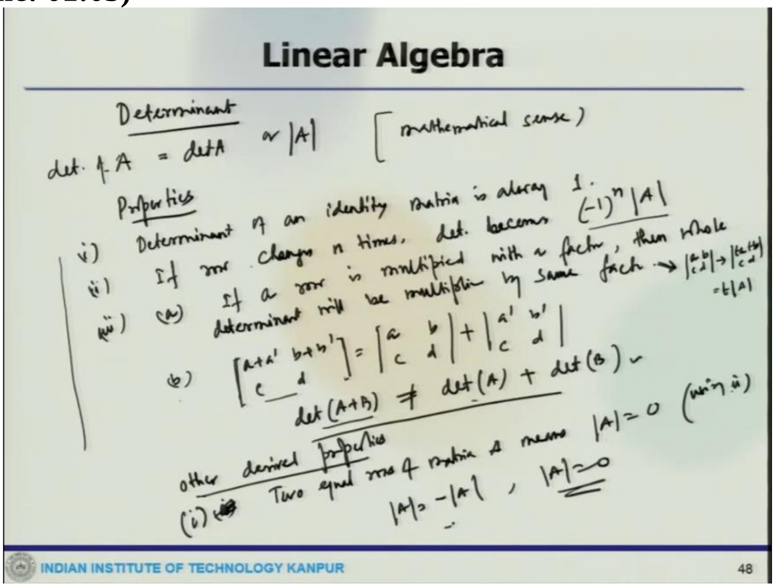

So, if you just recollect what we were talking about, the these are the properties, which we have discussed, just to quickly refresh these things, any identity matrix, the determinant would be 1 if there is a row changes multiple times, then the determinant becomes like this, and then the third property which are splitted into two part one is part (a). So, if a row is multiplied to the factor.

So, for example, one can so, the whole determinant will be multiplied a factor that means, let us say if you have a system

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \to \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t|A|$$

So, this is what three here talks about. So, I mean, we can always take some 2 × 2 systems, which are simple and easy to visualize, and then we can actually go to higher order system.

And the 3 (b) talks about that, if you have det(A+B) which is like this one actually, we can split like that, det(A+B) ≠ det(A)+ det(B). So, these you can decompose like that, but the determinant of these are, so, let me put it this way, if these are the matrix, I mean, you can decompose. But this is not going to be true then there are derive properties like if 2 rows are changes.

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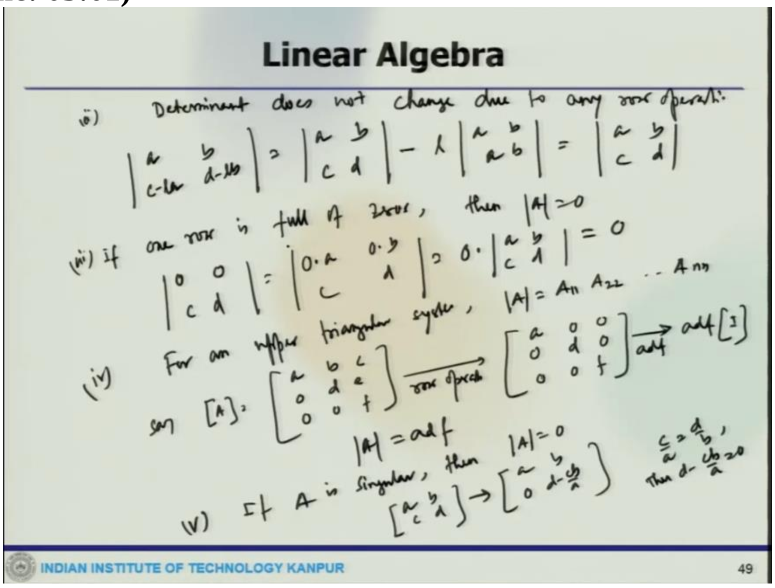

So, determinant does not change due to any row operations like if you have these, so, this is an another property where if you do a row operate, that does not change, then if one row is fully 0, then this is going to be 0 then these are other some of the singular than determinant is going to be 0.

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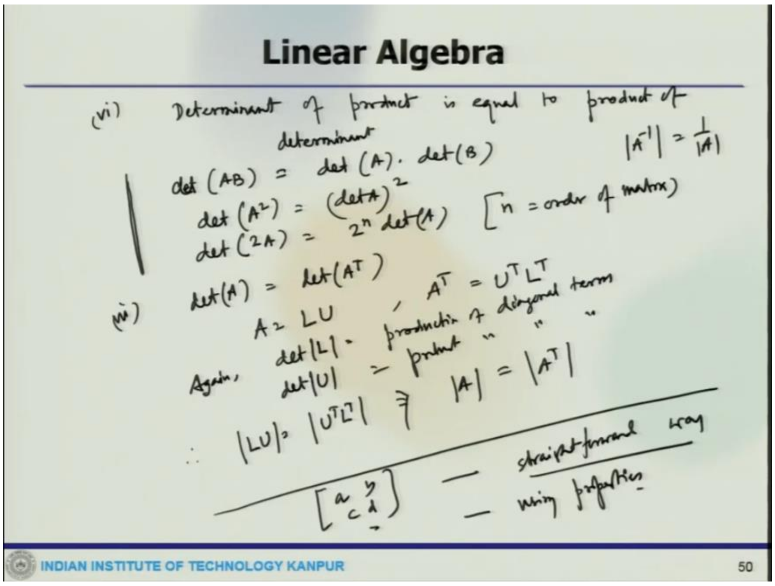

Then you can have determinants like this is true, if you have A and B the matrices which are multiplied again one can look at using a 2 × 2 system, which is quite straightforward. And see all these sorts of proofs or whether these statements are holding good or not one can check. So, as I said, I mean you can take a 2 × 2 system, who just straight forward everyone can find out or you can use these properties to find out that.

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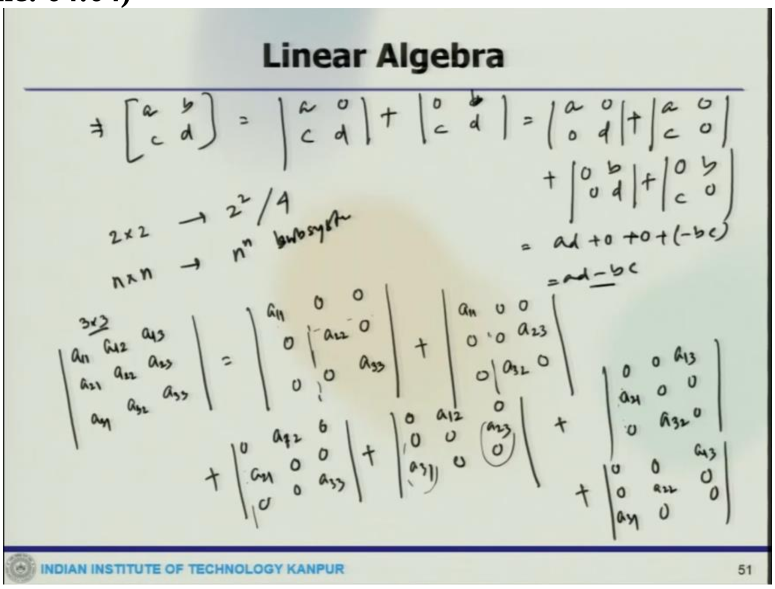

So, that is what and now we have actually looked at a 2 × 2 system and then 2 × 2 system, if you look at that, then this is you can like this, and we have talked about the 2 × 2 system this is exactly using the property. So, it is not directly finding the things like multiplied these and multiplied that. Now, if you have n × n system then you will have sub systems like and just to extrapolate the same we can write a 3 × 3 systems and you can have these like

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

Now, we can for this particular system, these three elements in so we have taken or considered three elements in the first two individually. And then a component while, the subsystems.

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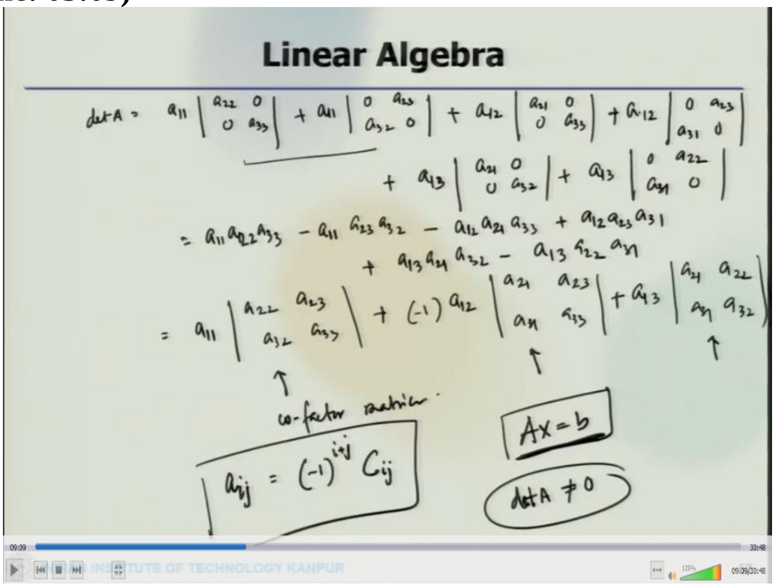

So if you expand that, so this determinant would be

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & 0 \\ 0 & a_{23} \end{vmatrix} + a_{11} \begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix} + a_{12} \begin{vmatrix} a_{23} & 0 \\ 0 & a_{31} \end{vmatrix} + a_{12} \begin{vmatrix} 0 & a_{21} \\ a_{33} & 0 \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & 0 \\ 0 & a_{32} \end{vmatrix} + a_{13} \begin{vmatrix} 0 & a_{22} \\ a_{31} & 0 \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1) a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

So, these are the subsystems, if one expand that you get. So, these are all using the property here.

This guy is going to be again positive, because there are a couple of times row exchange, so it going to be positive, these guys the same, it is positive. And the last guy again with the negative sign, So, essentially, I can also write in other way around, like using the other property, likes just taking these things together, you can write a sub system right like this

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

So, and this subsystem are called the cofactor matrices. So, these are cofactor matrices. So, when you look at the bigger matrix, the way the cofactor matrices are going to show this could be possibly one can expand 4 × 4 system also, that these cofactor matrices. So, whenever you take one row and then you take that I mean rows and column to be knocked up, and then the rest of the things are going to contribute to these cofactor matrices.

So, if you have a 4 × 4 system, these cofactor matrices are going to be 3 × 3 since it is a 3 × 3 system, we end up getting these cofactor matrices by 2 × 2. So, one can write these cofactor matrices, which are like n formula

$$a_{ij} = (-1)^{i+j} C_{ij}$$

So, this is what one can write. So, now, going back to our system, Ax = b the solution, we have looked at the system when the solution can be there, but now, we are also making one more important statement. If the determinant of A is not equals to 0, then only solution exists. So, that is also another way to look at it that when you talk about a linear system.

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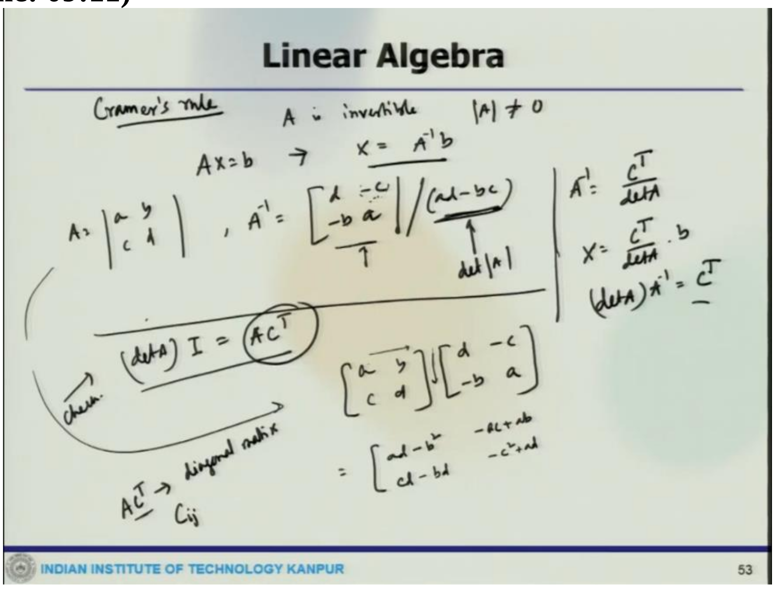

Now, how to find out these things, so, we talked about Cramer's rule. So, so let us say consider A is invertible and also determinant of A is non-zero, then for any linear system will have a solution, Ax = b, where x is $A^{-1}b$. Now, like if you take in 2 × 2 system again, let us say

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

then A inverse would be

$$A^{-1} = \frac{\begin{vmatrix} d & -c \\ -b & a \end{vmatrix}}{ad - bc}$$

So, that is how you are going to get it and if you look at this, this is in the denominator.

This is nothing but the determinant of the A for this particular 2×2 system and this if you look at, these are the component of this is 2×2 system, so, your cofactor matrices would be 1×1 system. So, these are essentially the cofactors but, if you take a and b this is the cofactor then if you take b and d , but with the negative sign that means it got transposed.

So, these are the cofactor matrices which are sort of so, one can say that A inverse would be

$$A^{-1} = \frac{c^T}{\det A}$$

So, you can write that so, the X one can write $X = \frac{c^T}{\det A} b$ and also from here one can write $(\det A)A^{-1} = c^T$. So, one can also check that, if we multiplied with A here.

Then what we get like $(\det A)I = Ac^T$. So, one can check that. So, what is there is that this A into cofactor transpose so, that if you look at it, so, these are even if you take this 2×2 system this is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and this is the transpose of the cofactor matrix. So, $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ and if you look at that, so, this guy is going to multiply to these guys. So, they are going to give you a diagonal matrix and the diagonal matrix are going to be now if you look at that this is

$$\begin{bmatrix} ad - b^2 & -ac + ab \\ cd - bd & -c^2 + ad \end{bmatrix}$$

so, it is again going to give you the so, the question is that why this c^T gives an diagonal matrix. The reason is that this being this C_{ij} this is formed by blocking all elements of i through and get column and that is why this guy is going to give us the diagonal matrix and the diagonal matrix are so let us complete this. So, we will get something like that.

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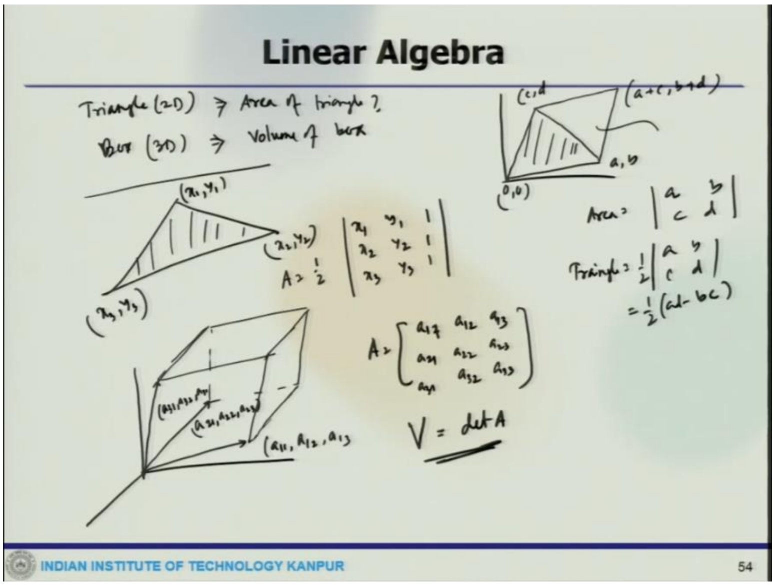

So, let us look at another thing, which is if you have a 2 dimensional system or let us say a triangle or the box to show the for triangle which is 2D, area of triangle, we can see what is that and if it is a box, which is 3D, we can find out the volume of box. Let us look at the first our 2D system, let us say this is the origin (0,0) and then we have like this. So, this is (a, b), this is (c, d), then we get this is (a + c, b + d).

Now, this is an 2D system. Now, this is a parallelogram where the area of this guy this guy would be the determinant of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Now, when you talk about the triangle, let us say this is the triangle. Now, one can find out when there are 3 areas, then one can put the sort of the height perpendicular height and then try to find out so, this is the area of the triangle where if this is the half of this parallelogram.

So that triangle area would be $\frac{1}{2}\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, which is half of $\frac{1}{2}(ad - bc)$. Now, the question is, like, if we have a triangle and which does not have 0 as origin one of the vortex, for example, let us say you get an triangle like this, where you have coordinates like (x_1, y_1), you get (x_2, y_2) (x_3, y_3). So, in that case, the area one can find out also, and how we can use the property of determinant to find out that thing.

I mean, this would be the area would be $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. So, that would be the area of this triangle. Now, similarly, if we take a box, which is 3 dimensional case, let us say one vector goes like this, which is the first row a_{11}, a_{12}, a_{13}, then your second row goes in this direction, which is a_{21}, a_{22}, a_{23} and the third vector, so these are all corresponding to edges of the box, a_{31}, a_{32}, a_{33}.

So, we can complete this box like this. So the volume of this would be so if you see this, this is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. So, the volume would be determinant of A. So for 3D also, you can find out that.

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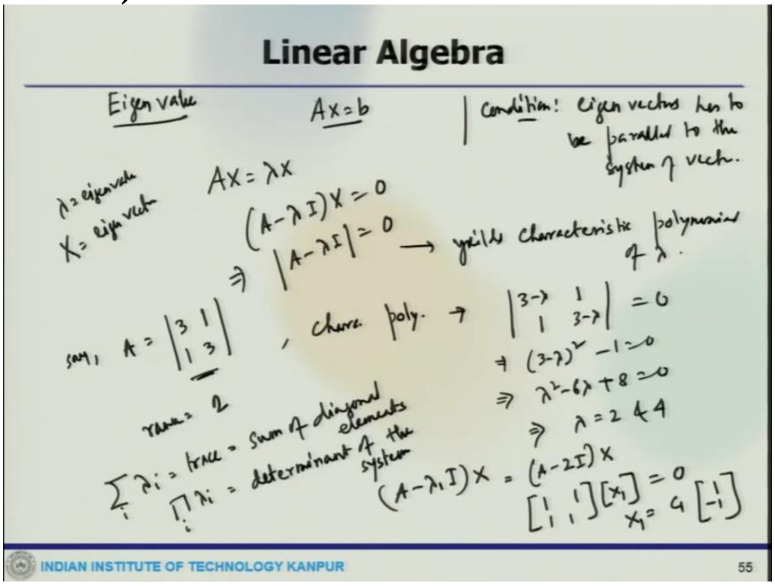

Now, we move to another important thing eigen value. Now, Eigen values and vectors are also related to the linear system like Ax = b. Now the condition of vectors or the vectors which are going to be the Eigen vectors. So, the condition for that is that the Eigen vectors has to be parallel to the system of vectors. So, what do we can then write when we say that Eigen vectors we write that Ax =λx.

Where λ as the Eigen values and X are the Eigen vectors. So, now the null space is orthogonal to the system an Eigen vector is parallel. So, what we can write $(A - \lambda I)X = 0$ and since that is the case, so, $|A - \lambda I| = 0$ and that is because again null space is orthogonal to the system and eigen vector. So, this should gives us the this is characteristics polynomial of λ.

Let us take an example let us say again A 2×2 system which is say $A = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$ then the characteristics polynomial would be $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$. So, that gives us $(3-\lambda)^2 - 1 = 0$

which is $\lambda^2 - 6\lambda + 8 = 0$, which gives us 2 roots one is $\lambda = 2\ and\ 4$. So, here the λ different. So, that means, there are distinct Eigen values for each of the system.

So, which also tells us that all the columns of this particular matrix they are linearly independent and also rank is full rank. So, here the rank would be 2 for this example. Now, also what we can say that some of this λ would be called as stress which is sum of diagonal elements and the product would be λ_i which is determinant of the system. So, what it gives is that here we have $(A - \lambda_i I)X$. So, what we which is like $(A - 2I)X$. So, that is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} [X_1] = 0$. So, from solving here we will get $X_1 = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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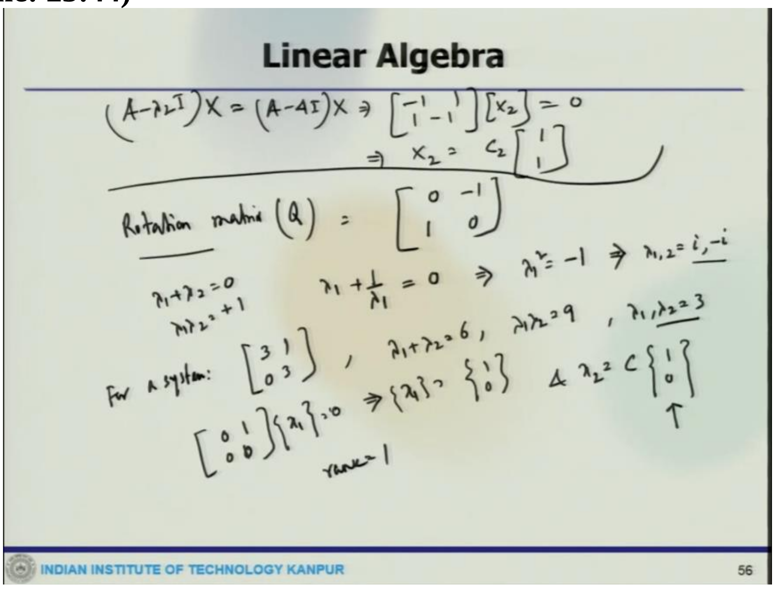

Similarly, we can write $(A - \lambda_2 I)X$ which is $(A - 4I)X$= $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} [X_2] = 0$ which gives us $X_2 = C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, the rotation matrix that we get Q is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and our, so, this is how for this particular example, you find the vectors also two different Eigen values, two different lambdas. Now, we take an example of the rotation matrix, here $\lambda_1 + \lambda_2 = 0$.

Because we have already said that some of the diagonal elements are going to be the trace and $\lambda_1\lambda_2 = 1$. So, what we get not here $\lambda_1 + \frac{1}{\lambda_1} = 0$. So, which gives us that so we take that determinant of the system is a product of the, so, this should be determinant would be 1,

Now, if you look at this example, here the both the eigenvalues are real and distinct, but in this case they are imaginary and distinct. So, as soon as the system changes this is what you are going to get it. Now, similarly, if you take an for a system like this where $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \lambda_1 + \lambda_2 = 6, \lambda_1 \lambda_2 = 9$. So, we get $\lambda_1, \lambda_2 = 3$. So, here both are I mean real but they are same. So, the Eigen vector would be $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \{x_1\}$ = 0 which gives us $\{x_1\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ and again $\{x_2\} = C \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$. So, here in this system you get only one basis vector of Eigen space and the rank of the system here would be 1.

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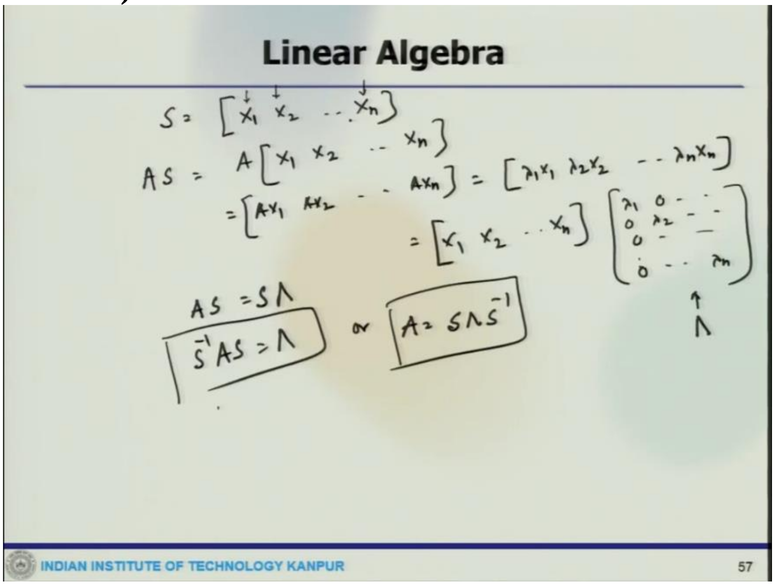

So, now, let us consider the matrix considering the Eigen vectors let us say S is a matrix which considers all these Eigen vectors. So, then if we multiply to the matrix to this we get

$$S = [X_1 \quad X_2 \quad \dots \quad X_n]$$

which is

$$AS = A[X_1 \quad X_2 \quad \dots \quad X_n]$$
$$AS = [AX_1 \quad AX_2 \quad \dots \quad AX_n]$$

again we can write

$$SA = [\lambda_1 X_1 \quad \lambda_2 X_2 \quad \dots \quad \lambda_n X_n]$$

or one can write

$$SA = [X_1 \quad X_2 \quad \dots \quad X_n] \begin{bmatrix} \lambda_1 & 0 & \dots & \dots \\ 0 & \lambda_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

and so, it is just write it.

So, if this one we call it and matrix of the Eigen values then we write AS=S λ this is again the eigenvector matrix S and so, what we can say $S^{-1}AS = \lambda$. So, this is our eigenvalue matrix or other way one can say or A= $S\lambda S^{-1}$ there are one important assumption is there that all λ are distinct corresponding to all distinct Eigen vectors then only this was all independent columns.

So, this Eigen vector matrix of S which containing all the Eigen vectors they are independent and that would only possible when you have all distinct Eigen values like this particular example is a 2×2 system when we have 2 distinct Eigen values, you get 2 different vectors, Eigen vectors and they are independent. So that is an important thing. So, we will stop the discussion here and we will look at the detail thing in the next session.