Space Flight Mechanics

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Model No. 01
Lecture No. 26
Trajectory Transfer (Contd)

The last time we have been discussing about the Hohmann, Bielliptical transfer, so we will try to complete it this time.

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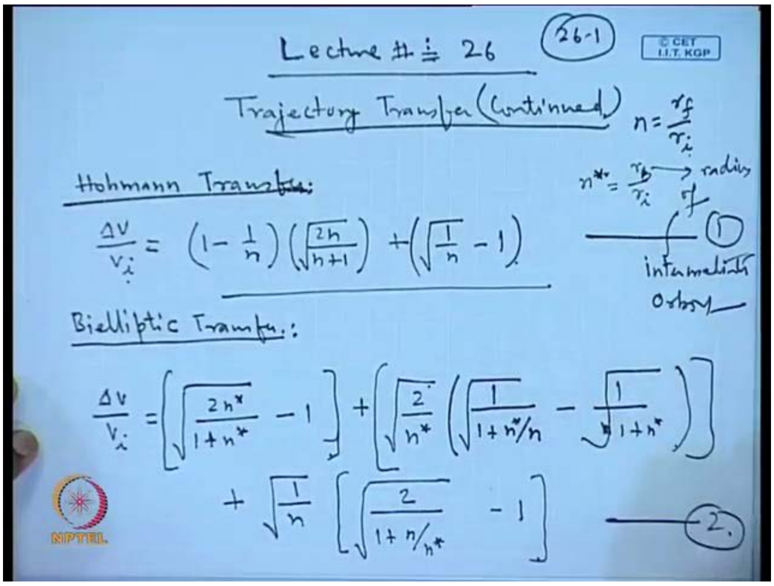

So, last time as we developed for the Hohmann transfer, the change in velocity delta v divided by vi, is given by this equation which I have numbered as 1. And for the bi elliptic transfer, similarly I got the equation delta v by v i as this one, where n is the, n we choose as r f by r i, where r f is the radius of the final orbit, and i is the radius of the initial orbit, and n star was written as the r b by r i. Hence, this is the intermediate orbit, where r b was the radius of intermediate orbit. Now, we will utilize these two equations to make the graph.

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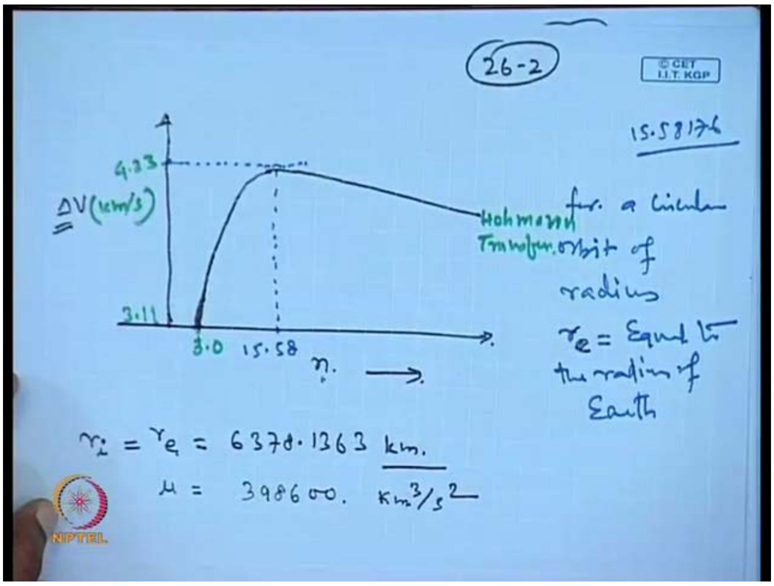

So, if we plot delta v on this axis, and n on this axis. So, for the Hohmann transfer that is using equation1, will get a curve which will look like, and this maximum value also we found it theoretically, and this value was around 15.58. So, exact value for this we calculated, exact on the more precise value, and that value turned out to be 15.58, 15.58176, so we are approximating this is in 15.58. And, for a circular orbit of radius r e; that is equal to the radius of earth. So, suppose a satellite is moving in a circular orbit, whose radius is just as the radius of the earth, and replace earth by a point mass of the same mass as that of the earth. Then, you can get the value of this delta v. So here the r earth value, this will be around 6378.1363 kilometer. The value for mu for the earth is 398600 kilometer cubic second square. So, this constitutes here initial radius.

Now, with this initial radius r i., we can calculate v i equal to mu by r i under root, so 398600 divided by 6378.1363 and this gives you velocity in kilometer per second. So, this will turn out to be around 7.9 0536 kilometers per second. So, if you choose n which is nothing, but equal to the radius of the final orbit, divided by radius of the initial orbit is equal to 15.58176. Then you can calculate delta v according to equation 1, delta v calculated. So, delta v by v i what you get it is 0.536258 by inserting this value of n, so insert in the equation number 1, insert it in equation 1, and therefore you get this quantity. So, if delta v is known, v i is known from this place. Therefore, delta v will be 0.536258 times 7.90536, and this will turn out to be around 4.23 kilometer per second.

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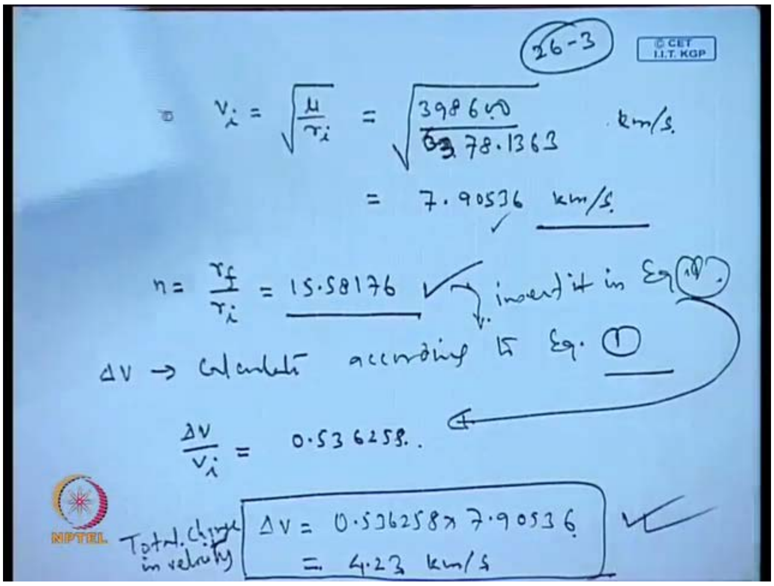

So, this is the change in velocity required, total change in velocity or the total impulse required. And this we have calculated earlier, and connected this with the amount of propellant that will be consumed, and theoretically we develop that equation. Therefore, corresponding to this value, this value turns out to be 4.23 and this we are plotting in kilometer per second. Similarly, we can put here, say the value of n to be 3, so this will turn out to be around 3.11, and this we are doing for Hohmann transfer, this is for Hohmann transfer.

So, you can see that once these equations are available, so without much problem we are able to draw this curve, and this can be easily done on a computer. So remember that we are choosing, the radius of the orbit to be the same as the radius of the earth. So you have the earth here, so you can assume that the earth is here, but suppose this is the radius of the earth, this is r e, so obviously if earth is there then you cannot move it, but the this earth can be assume to be a point mass here of n, and then from satellite is moving in the same radius. So, this is the m mass of the earth, and mu is given to you, this is value of mu for the earth, this radius is available here. Now, based on this we have got this result. So, from here onwards we can work further to calculate various values. The next question is what will happen to this bi elliptic transfer. Now, bi elliptic transfer we have another parameter appearing, which is n star, this is our n star. So obviously, we will get multiple curves in the case of bi elliptic transfer, because n star is present. We can choose different values of n star and then we can vary the value of n and plot the curve, so for different values of n star obviously then we get a number of curves, as we change the value of n star.

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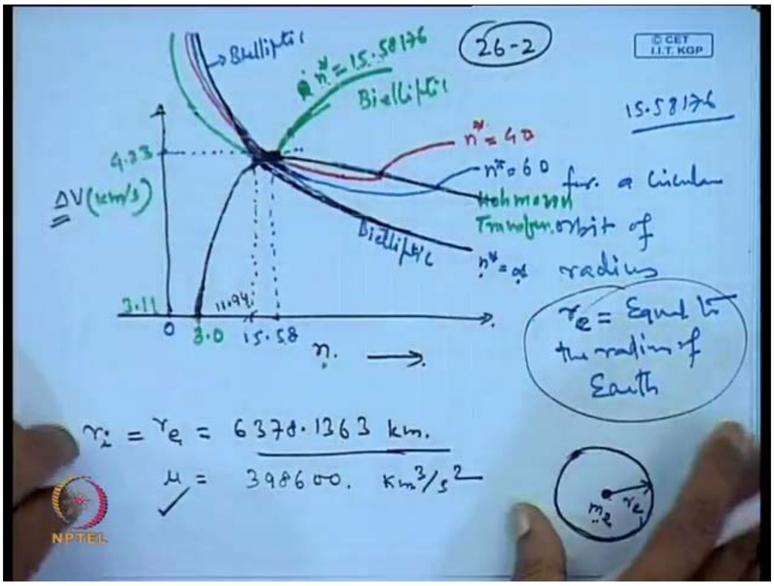

So, using equation 2, that is this equation, again we can work out for the bi elliptic transfer case. So, plotting this, this is n star equal to 15.58176, so this green line is referring to the bi elliptic transfer. So, n star means the intermediate orbit, so radius of the intermediate orbit if it is 15.58176 times the initial orbit, radius of the initial orbit, then we get this curve. And on this side this is starts from here, and then goes into this direction. If we increase the value of in a star further, so we get a different curve. Similarly, if we take the value of n star to be 60, so we get another curve, and if we choose n star to be infinity, this curve results. This is little shifted this not exactly in the same point, so if here the 15.58 point is there.

So, earlier we have found out the crossing point for the Hohmann transfer curve and the bi elliptic transfer curve, and this point it turned out to be a 11.94. So, for n equal to infinity this curve cuts here in this point, and thereafter the other curve with other parameters, as for example this is for 60 n star equal to 60, this also it does not pass through this point, rather it goes through right of the this black line, which is coming from here to here. So, this is our bi elliptic transfer, this bi elliptic transfer and this is the black one which is shown here, this is the Hohmann transfer. So, the Hohmann transfer starts from here and it goes into this direction. While the bi elliptic transfer all of them for the smaller value of n, it starts here for as small. As we take the value of n star equal to infinity. So it is a starting here in this place, and as we change the value of n, so it is a progressing in this direction. Similarly, this one has resulted for n star is equal to 15.58, and as we change the value of n gradually from 0 onwards, so this point is 0. So, as we change it from 0 onwards, so similarly we will get the result for this curve. So, it comes here, it cuts near 15.58 and then goes in this direction.

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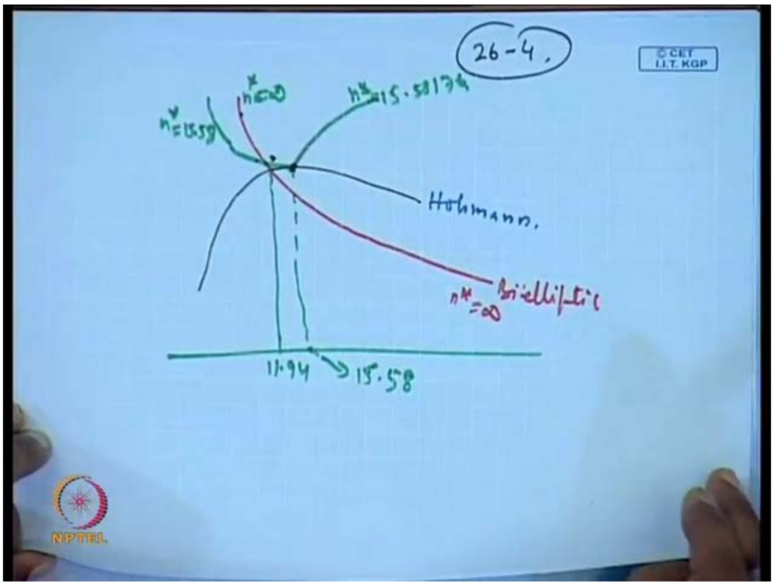

So over all, this cutting points we can show like this is our Hohmann transfer, and this is the bi elliptic transfer, for n star equal to infinity. This point where it cut on the x axis, this point was our 11.94, and where it is cutting this point is 15.58. You, can see the difference in these 2 point; this green line is crossing the red line and cutting it here in this point, and thereafter it is moving in this direction, so for all other values of n star. So, the lines will lies between this n star equal to infinity, this is for n star equal to infinity and this is for n star equals to 15.58. So, all other lines must lie between these two ranges.

So, wherever the 2 curves are cutting each other, that can be easily solve by using this two equations. We have the first Hohmann transfer equation and this is the second bi elliptic transfer. So this curve, the cutting point we are defining for the Hohmann transfer and n star equal to infinity. So in that case, in the bi elliptic transfer equation we have put n star equal to infinity, and this equation can get simplified, and then this 2 must be equated, to solve for this cutting point which I have shown here in this place.

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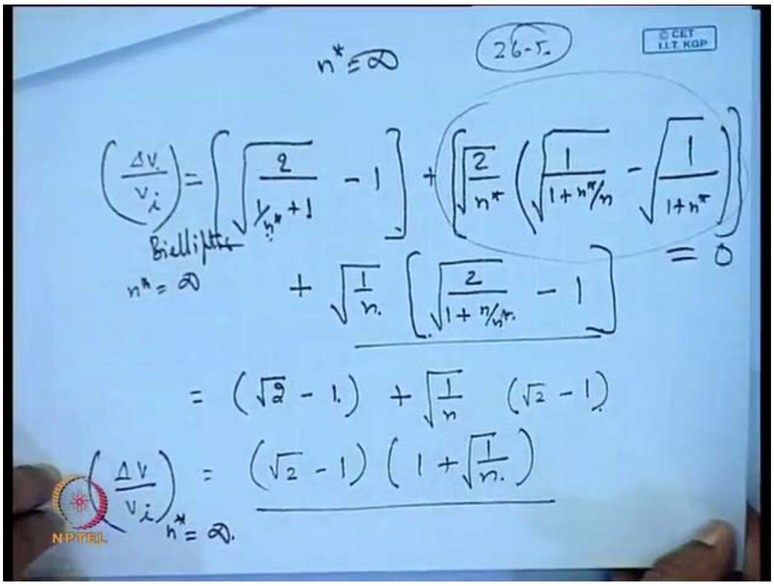

So, from this bi elliptic transfer if we put n star is equal to infinity, so this will result in delta v by v i. This, quantity you can see from here, this is 2 divided by 1 by n star plus 1 under root minus 1, this is 2 divided by n star times 1 by. Now, in this equation put n star equal to infinity, so as we put this term becomes equal to 0. Here 1 by n star this will become equal to 0, so put n star equal to infinity. So, this becomes, the first term becomes 2 under root minus 1. This, whole term will turn out to be 0 and here we have 1 by n under root, and here n as putting n star equal to infinity, this goes to 0 and what you get here under root 2 minus 1. So this becomes under root 2 minus 1 times 1 plus 1 by n under root. So, delta v by v i, and this is for bi elliptic, bi elliptic n star equal to infinity. So, what we are trying to find it out, that when this bi elliptic transfer and the Hohmann transfer they will equalize.

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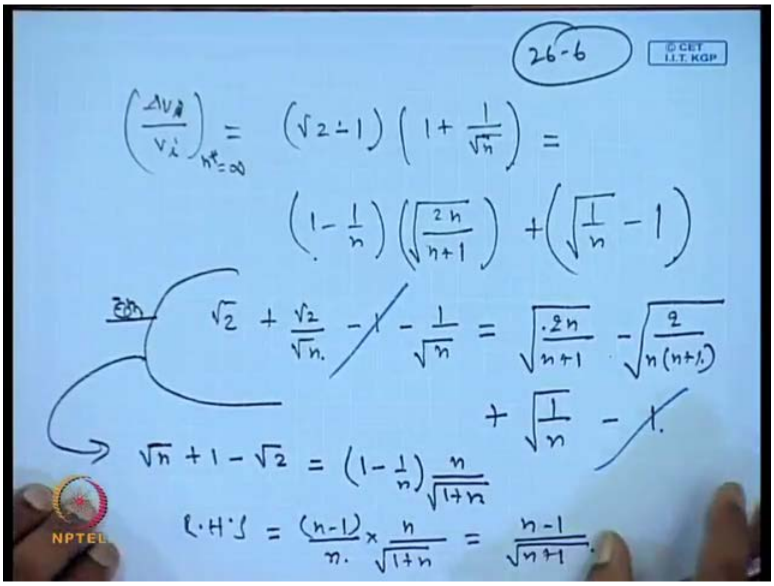

And therefore, we need to solve this two equations; delta v by v i root 2 minus 1, this is equal to 1 minus 1 by n times 2 n by n plus 1 under root, how to get this equations alternatively we will look into that later on. Right now just concentrate on finding out the cutting point, which is the point here or in this figure we have shown this point we are trying to find it as right now. So, we can solve it, the simplification is little longer, but some of the steps can be carried out here. So, what we can do, we can expand this term here, on the right hand side, so this I will write on the left hand side, this particular term which is appearing on the right hand side, so this is 2 n by. This, term also we need to expand, this is root 2 plus root 2 by root n minus 1 minus 1 by under root. Now, expand this, this become 2 n by n plus 1 under root minus 2 by n times n plus 1 under root, plus 1 by n under root minus 1. So, this two terms this and this they cancel out, and after little bit of simplification, you can write them as under root n plus 1 minus root 2 equal to. So, this equation, can we reduce to this equation, it is a easier doing it, but I am just skipping those steps here. This can be further simplified as you can see from this place, the r h s this is nothing, but n minus 1, divided by n. so, this gets reduce to n minus 1 divide by n plus 1 under root.

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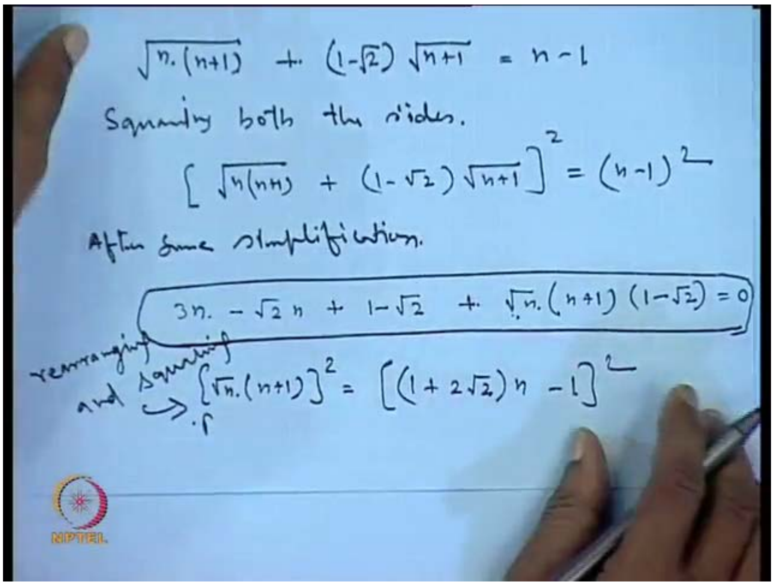

So therefore, this equation then you can write as, this particular equation this can be written as; n times n plus 1 under root. So what we are doing, this term we are the term in the denominator, we are taking on the left hand side, so this term under root n let us multiplied by this 1, 1 minus under root 2 times n plus 1 under root equal to n minus 1. And because this is containing the square root sign here, so we need to eliminate it, so we need to square both the sides. So, squaring both the sides, and minus 1 square, so this is n minus 1 square. Now, this can be further simplified and after some simplification. This, can be reduce to c n minus under root 2 n plus 1 minus under root 2, but still this equation, it contains the square root sign for n. So, we need to eliminate this square root sign again, so again we do the same process of a squaring by rearranging the terms. So, after this rearrangement you can find it out that the terms will appear as; rearranging and squaring, this leads to this equation and simplified further.

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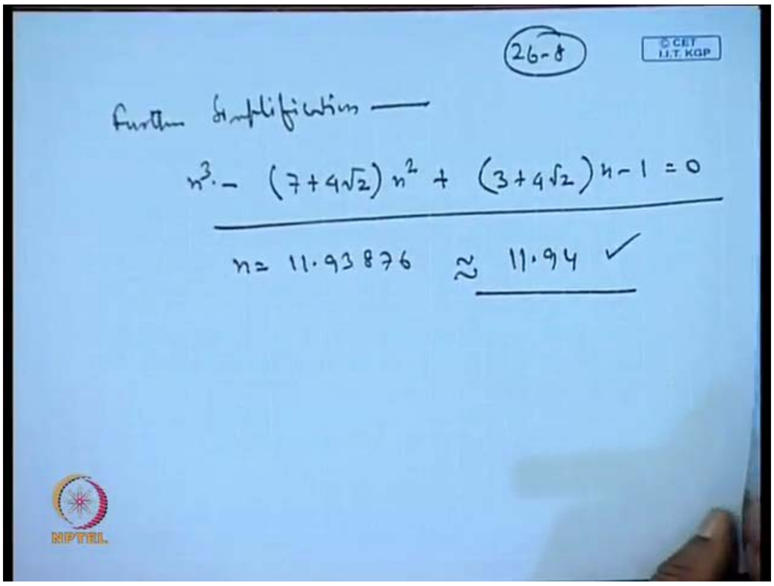

So, further simplification, this is result in a cubic equation, and you can solve it on mat lab, and this will give you n equal to 11.93876 which I have approximated as a 11.94. So, this is the solution that we got last time. We have used this last time we are in this place, this is the cutting point for the Hohmann transfer and the bi elliptic transfer for the limiting case. This is the limiting case, where n star equal to infinity, so the intermediate orbit is at infinity.

So, in this case we are the bi elliptic transfer and the Hohmann transfer, they get equalized for the value of n equal to 11.94, that is the r f by r i equal to n equal to 11.94. This is the final radius of the final orbit by radius of the initial orbit. So, this completes our discussion for the Hohmann and the bi elliptic transfer.

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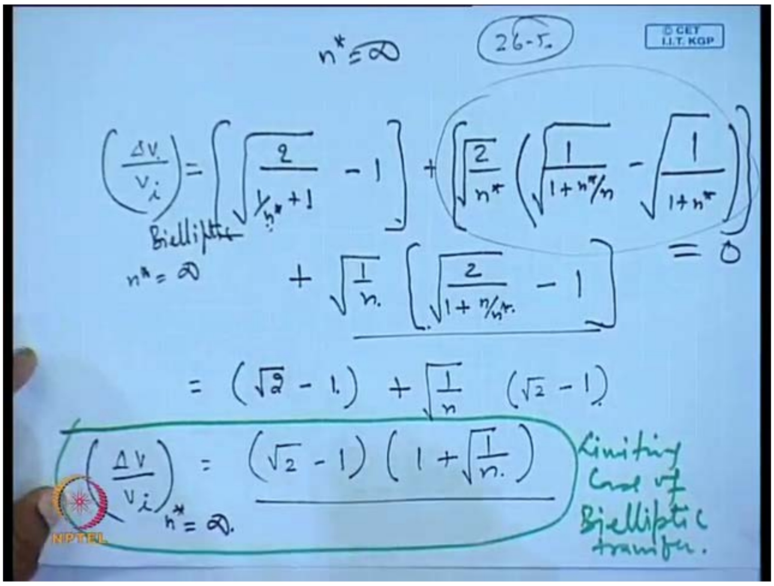

Now, what I told at the relation that we got, for the limiting case of bi elliptic transfer. This is a limiting case of bi elliptic transfer, so this equation can be derived alternatively.

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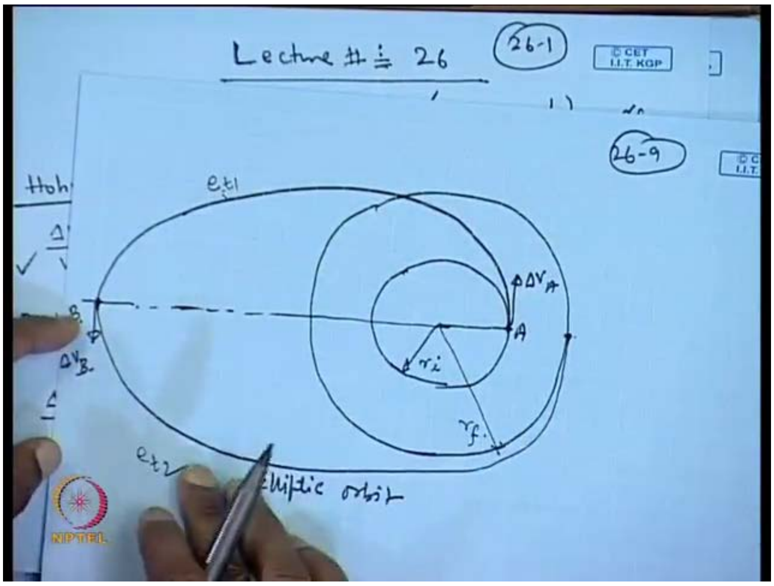

So, in the bi elliptic transfer what we have done, this is the initial orbit given, whose radius is r i, and this is the final orbit given whose is r f. So, bi elliptic transfer, we transfer from point A to point B, and here the impulse delta v A was given and then again the impulse here delta v B was given, and then it is transfer to this point, and in an elliptic orbit, this is the elliptic orbit. So, we can write this as the eccentricity of this orbit to be 81 and a eccentricity of the this orbit to be 82. So, if we are transferring from this point to the infinity. So you can utilize the Hohmann transfer relationship to find out this result, so our equation for the Hohmann transfer is. This is the equation given to us, so using this equation, using this equation number 1. So, if we put in this Hohmann transfer n equal to infinity, suppose we are here in this place and we are trying to transfer it to infinity, so we put n equal to infinity in this point.

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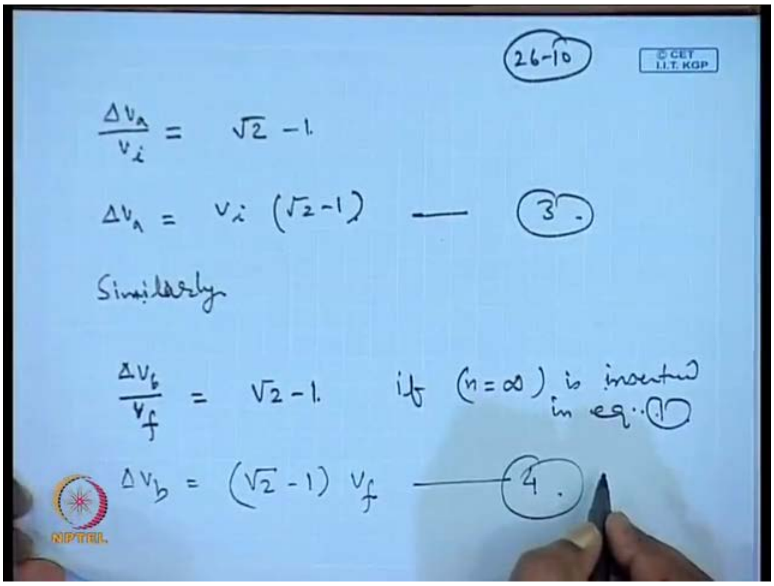

So the result we get will be delta v by v i, this equals ton equal to infinity, so this term vanishes and here if we divide this is, then we get root 2, so this is root 2 and again term vanishes and we get minus 1. Therefore, we can write delta v a is equal to v i times, root 2 minus 1. Now, you need to get back into this point, so if you look back from this point, suppose that you need to transfer from this point to infinity, so still we will utilize the same equation, only thing here instead of v initial it will be v final. So, this is equivalent to that you need to escape from infinity and come to this point, so using this equation again you can write the result. So, delta v b by v final this will be equal to root 2 minus 1, if n equal to infinity is inserted in equation 1, this is our... So delta v b becomes.

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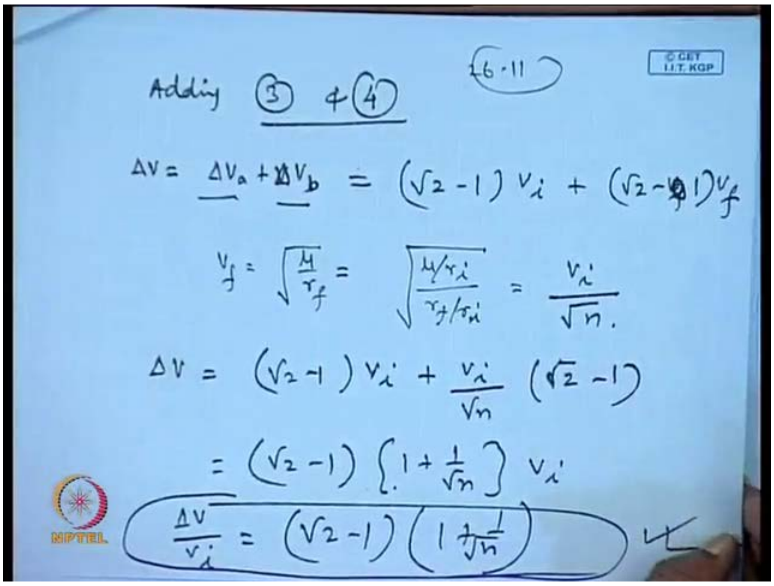

Now, add this two equations, adding 3 and 4 this gives you delta v is equal to v a plus delta v b. So why delta, it is written in terms of delta v, because you need impulse to escape from this, but as a whole we are calculating the energy required, to get into this circular orbit. And, that you can put here another notice and also, but I am just putting here delta v, so delta v a plus delta v b, this turns out to be minus root 2 minus 1 v i plus root 2 minus 1 v f, root 2 minus 1 times v f. Now, v f is mu by r f under root, because it is in a circular orbit and this you can write as r f divided by r i and mu by r i under root is nothing, but v i and r f by r i is nothing, but n.

So, this is what we get, and therefore delta v equal to root 2 minus 1 v i plus under root n times root 2 minus 1, taking root 2 common, root 2 minus 1 common, what we get here 1 plus 1 by n under root times v i or delta v by vi equal to root 2 minus 1. This is what we got during the bi elliptic equation also. So, we see that the same result can be obtain in a multiple of way, but whatever objective was hereto find out how we are getting this point 11.94. We are the, this black curve for the bi elliptic transfer and black curve for the Hohmann transfer; they are intersecting which is the point 11.94.

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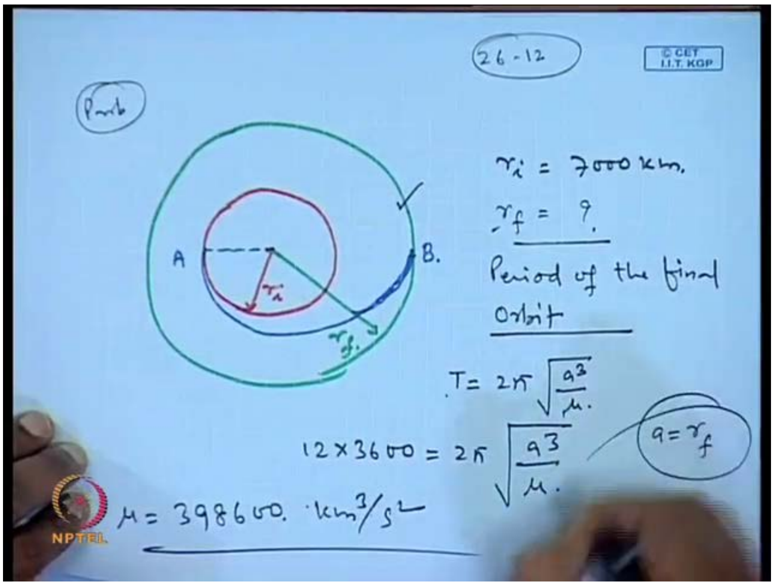

We take a simple problem, but it will effectively demonstrate the concept that we have been developing till now. So, in this problem we are given a initial orbit, whose radius is r i and the satellite is to be transfer from the initial to the final orbit, whose radius is r f, using Hohmann transfer. This is point A and this is point B. Now, all the information are not given, so r i is given to v seven thousand kilometer, r f is not given, but instead of r f what is given, is your period of the final orbit which is circular in this case, so this is the orbit whose radius is r f and its period is given, so we can calculate it. So, period will be given as T equal to 2 pi, a cube by mu under root, and this is 12 hours, so 12 times 3600 equal to 2 pi a cube by mu under root, and mu we know this quantities 398600 kilometer cubic per second square. So, once we insert this value into this equation, it can be solved for a, and a will be nothing, but equal to r f in this case.

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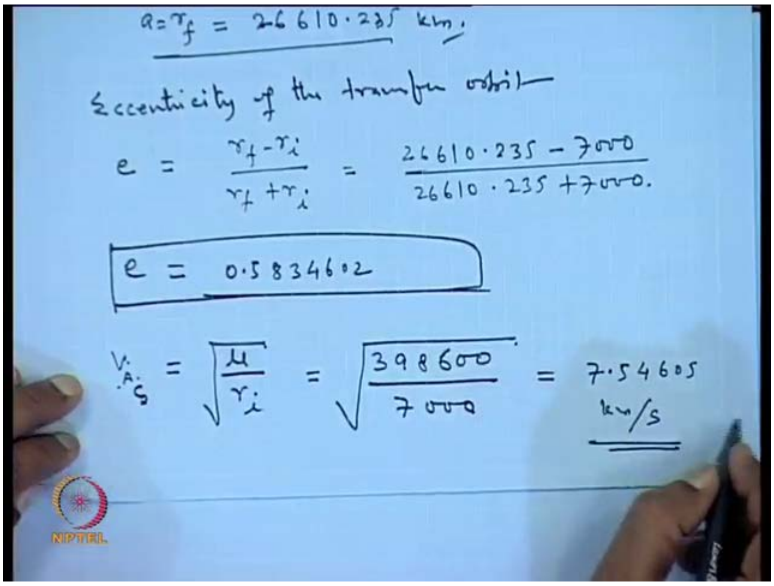

So, a equal to r f if we solve it. Finally, we get 26610.2235 kilometers, now you have the initial or radius is given, and final radius is also known to you. So, this quantity we have just now found out, this is 26610.235 kilometer. And therefore, eccentricity of this transfer orbit can be calculated which is own by the blue line here. So, eccentricity of the transfer orbit, this will be given by r f minus r i divided by r f plus r i, so this is 26610.235 minus 7000, and this value turns around to be 0.5834602.

Now, velocity in the initial circular orbit, so v A in the circular orbit which is shown by C here. To, indicate the circular orbit so this can be written as mu by r i under root.mu is nothing, but 398600 and divided by 7000 under root. Now, we need to send this into the elliptic orbit.

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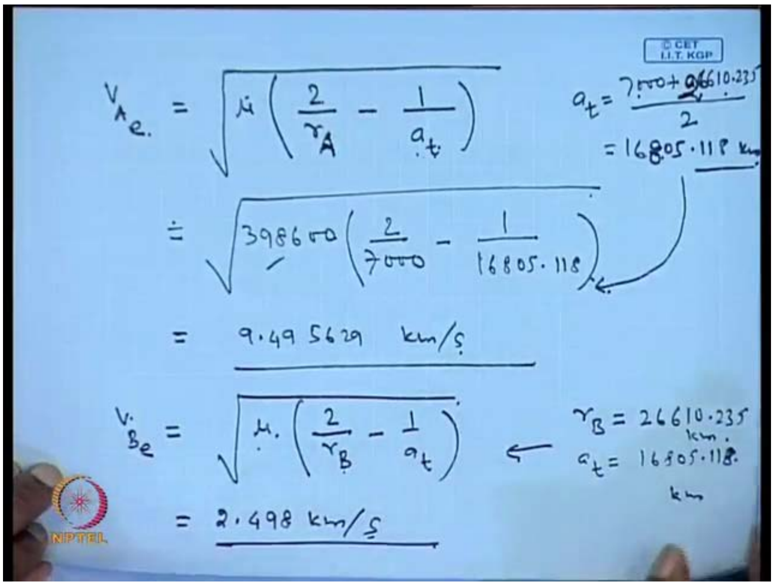

So for that v a in the elliptic orbit, it can be written as 2 by r a minus 1, by a transfer orbit which is the elliptic orbit, semi major axis of the transfer orbit. So, inserting the values and a will turn out to be 16805.118, so a transfer orbit this simply you can write as 7000 plus 26610.235 which is the radius of the initial orbit, radius of the final orbit divided by 2. This will a sixteen point, r i plus what we have taken r f plus r i divided by 2, this was 26. So, this will turn out to be around 16805.118 kilometer, and this is a result we are inserting here. And this gives you the value 9.495629. So, from here how much impulse is required that you can calculate how much velocity change is required. Now, the satellite will go into this position, so in this elliptic orbit you need to calculate the velocity of the satellite and we are doing it from the first principle instead of using the equations. So, v b in elliptical orbit and then this can be written as mu by r b minus 1 by a transfer under root, and r b is here 26610.235 and a t is 16805.118 kilometer, and this also kilometer, mu is same as given here. So, inserting these values you can calculate this and this will turn out to be. So, v b in the elliptical orbit this is 2.498 kilometer per second. Now, at this point here we give the delta v A impulse and here we need to give delta v B impulse. So, you need to change the velocity here in this point.

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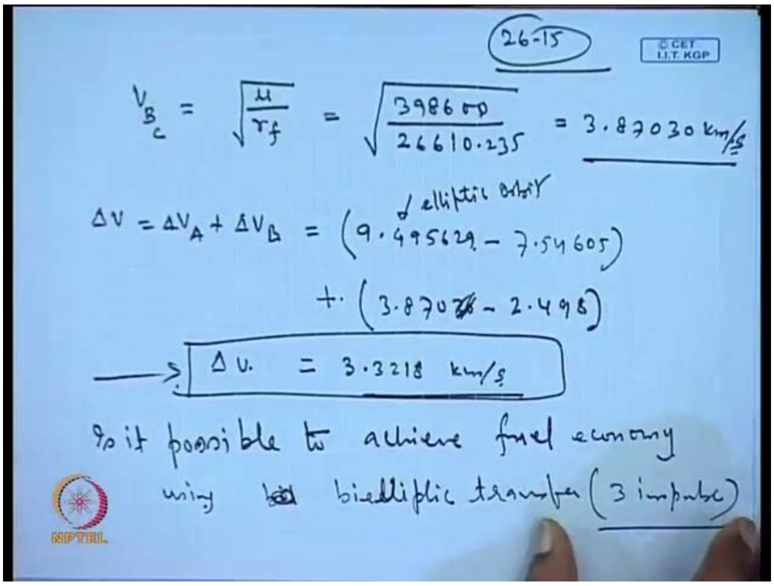

So you need to find the velocity in the circular orbit at B, so v B in the circular orbit, this will be mu by r f under root, so mu is 398600 and radius of the final orbit is 26610.235. So, we have calculated the velocity at these points, in the circular as well as in the elliptic orbit. Now, it is easy to calculate how much impulse will be required. So how much change in velocity is required is given by delta v A plus delta v B. This is 9 point, in the elliptic orbit at point A, the value we have got there is 9.495629 and minus we have to subtract the value that we got in the elliptic orbit at point A, so this was in the elliptic orbit and in the circular orbit we have got as 7.54.

So, this is in elliptic orbit. And the circular orbit we have got which is 7.54605, so insert this value also here, and then delta v similarly, we need to find out subtract from the velocity in the circular orbit, minus the velocity in the elliptical orbit at point B. And this value write now we have derived and this is 3.87 minus 2.498, so we can delete this, so this will give the value this is 3.3218. So, this is the total impulse required or the total velocity change the required that we need to provide. Now, once you know this velocity, so it is easier to calculate the amount of propellant required, by the equations we have developed earlier. So, this very simple problem gives a lot of information, about the discussion we have made earlier, and it clarifies all the points numerically. Now, the question is, is it possible to achieve fuel economy, by elliptic transfer which is 3 impulse transfers.

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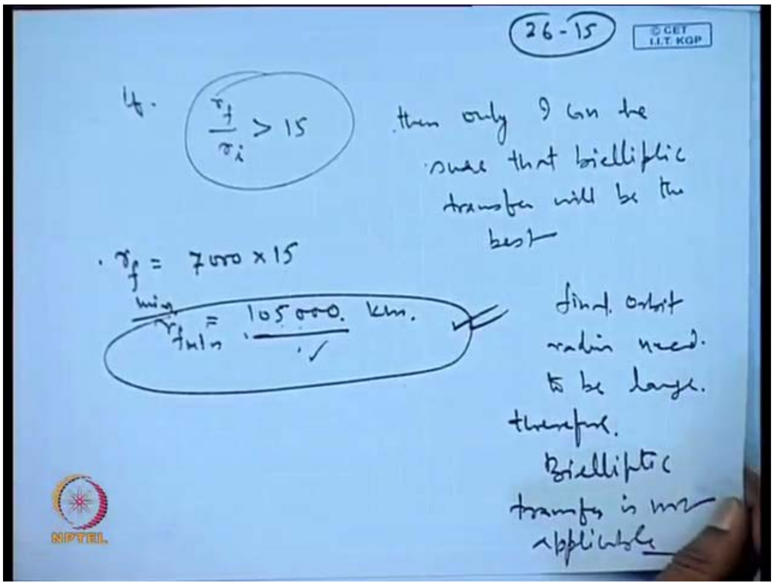

So naturally we have discussed about the bi elliptic transfer just before. So, can we say anything about this from the graph we did. So, in this region as earlier we have discuss, before the point 11 point, this point is 15.58 and this point is 11.94. So before this point Hohmann transfer is based and after 15.58 undoubtedly by bi elliptic transfer is the best. In mid between, it remains doubtful, here the curves are passing that you have to see, and this list can be done using the numerical calculation. So, in this reason I am sure that the Hohmann transfer is based, in this reason the bi elliptic transfer is based. So, what we can state that if r f by r i, this is greater than 15, than only I can be sure that bi elliptic transfer will be the best, but in that case as you can see the r f will be 7000 times 15. So this is the minimum value of r f required, 105000 kilometers, so this is your r f minimum.

For, transferring in such an orbit we will see that the amount of time required will be very high, as compared to the smaller orbit that we have taken for the transfer here. So, for a smaller orbit our Hohmann transfer is good, but if the radius of the orbit increases, and in this case you can see, that how large it is becoming, so then only your bi elliptic transfer is becoming more useful. So here in this case, you have the final orbit, so the final orbit radius need to be, therefore bi elliptic transfer is not applicable. so, if again look into this curve this is not very clear but, still here this has got little blurred, but in this reason definitely you can see that till the point 11.94, the Hohmann transfer curve, it lies below the bi elliptic transfer curve, which is here the bi elliptic transfer curve for the intermediate radius of n star equal to infinity, and this is for n star equal to 15.58176.

So in this reason where the n is less than 11.94, delta v is coming lesser than the value of the bi elliptic transfer, which is corresponding to n star equal to infinity. And obviously if you try to choose for n star equal to infinity, so you know the how much time consuming it will become, if n star is infinity means virtually infinite time we require to transfer, so this is not possible, so we cannot considered of the bi elliptic transfer. So, ultimately the limit remains here in this point; that we go for this transfer which is the Hohmann transfer. And rest others you can consider here, so this is obviously eliminated, this case is eliminated and we can look for the other cases and find out for the intermediate things like n star equal to 15.58 or n star equal to 40, or n star equal to 60, which is going to be more efficient as we go above the 15.58 point, but this is obviously not useful.

So, from here what we conclude that our Hohmann transfer is best for our satellite transfer from one orbit to another orbit, as we do in our visual cases for the earth satellite, where the satellite may be in the lower orbit or may be in the g t o which is called the geo synchronized transfer orbit. It as to be transfer into the geo synchronize orbit. So if that kind of transfer is to be done than in naturally you go for the Hohmann transfer. And this whatever we are looking here into this, so if n star equal to 40 another values, so that may be useful for the inter planetary transfer. Once we are going from this planet to another planet, so at that time this orbits may be useful, otherwise in the present contest were we are dealing with the earth, these orbits the bi elliptic transfer as not useful. So, if bi elliptic transfer will be not useful, until unless we have the final orbit radius much larger than the initial orbit radius.

So in this case we took the 7000 kilometer and then 15 times more means 105000 kilometer. So this type of cases will happen only in the case, once you are going to transfer the satellite from earth to moon. Only it is a possible than, nearest possibilities otherwise it is a not possible. So you want raise the orbit, so how to raise it, so this is solution here. We stop here and we continue with the discussion next time, so will go into the trajectory transfer further and looking into the problems and other things. Thank you very much.