Introduction to Aerodynamics

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Equations of Motion (Contd.)

So, last time we expressed the energy equation in the form of its integration, and called that as the generalized Bernoulli's equation which showed that the internal energy, the pressure, kinetic energy and potential energy some of these on a material path line is constant, when the flow is inviscid irrotational. Then we considered steady case when material path line and streamline coincides, and we said that this term is constant on a streamline, then we went on to the assumption that if the flow is irrotational and isentropic then this is constant throughout; that there is only one constant for the entire flow field, the generalized Bernoulli's equation which is an integral form of the energy equation.

Now, one thing you should remember that whenever you try to use these equations, you must remember the underlying assumptions that under what condition this is valid, like this is an integral form of the energy equation. So, the basic assumptions involved in energy equation which is say like isotropic fluid, linear stress relationship when those condition or the fluid is (( )) when these conditions are valid only then this energy equation in this particular form is valid. And further integration integrating that equation we also of consider that the body force field is potential field.

We also consider that the pressure field is steady, and under those assumptions only the equation is valid. And then if you want to apply it that is a single constant for the entire flow field, then in addition the flow field must be isentropic. Only then you can apply it if not you cannot apply it in this form. Today, we will be doing almost the similar thing but with the conservation of momentum equation and which we will see will give the original Bernoulli's equation particularly in the form that Bernoulli originally derived many, many years ago.

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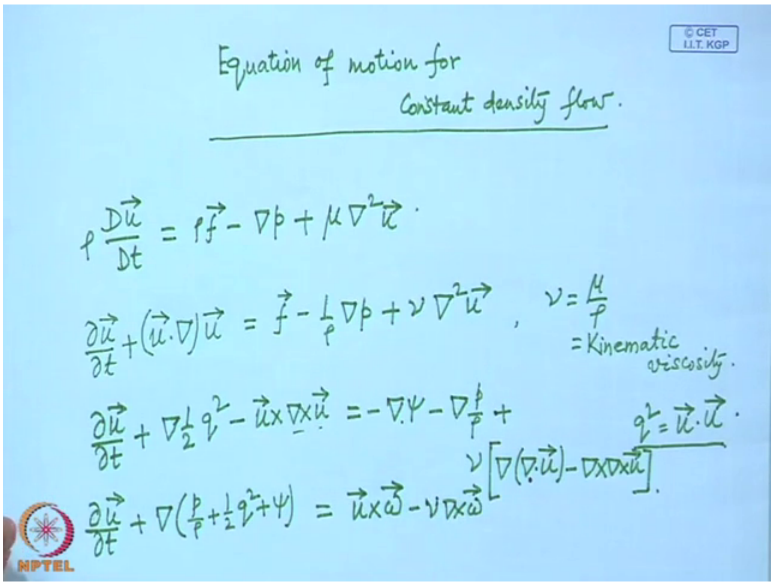

So, we will write this equation of motion for incompressible flow or flow with constant density. By the way when we say equation of motion we mean actually the all three equation the mass conservation equation, the momentum conservation equation, as well as the energy conservation equation but for the time being we will call only take the momentum conservation equation. The momentum conservation equation for constant density flow, we can write at in the standard vector form, let us divide it by this density so that we can write this equation and this d u, dt we will write it in the full form, expanded form which is d u, dt plus this mu by rho is usually denoted by nu is called kinematic viscosity.

We will now use some vector identity, this you have already used in different form. This can be written as gradient of half q square which you remember that q square is u dot u. Once again, we assume that the body force per unit mass or body force is a potential force and if we assume that the body force is a potential force. This can be written as the gradient of the potential energy per unit mass. And here this rho being constant we can take it inside the gradient and this Laplacian of u also, we will write using another vector identity curl of curl of u this is same as Laplacian.

Now, in this equation you see there are three gradient terms gradient of half q square here, gradient of psi and gradient of p by rho. We combined all this gradient terms together on the left hand side. So, we have this term we take to the right hand side, this curl of u will once again denote by that vorticity u cross omega and here you see this first term is 0 because of the incompressible continuity equation the incompressible continuity equation says divergence of u is 0 so the first term is 0. So, you have only the second term which is curl of omega.

Now, if curl omega that is the vorticity 0, that is if the flow is irrotational then the right hand side is identically 0 irrespective of the body viscosity. Whatever the viscosity is the right hand side is 0, if vorticity is 0.

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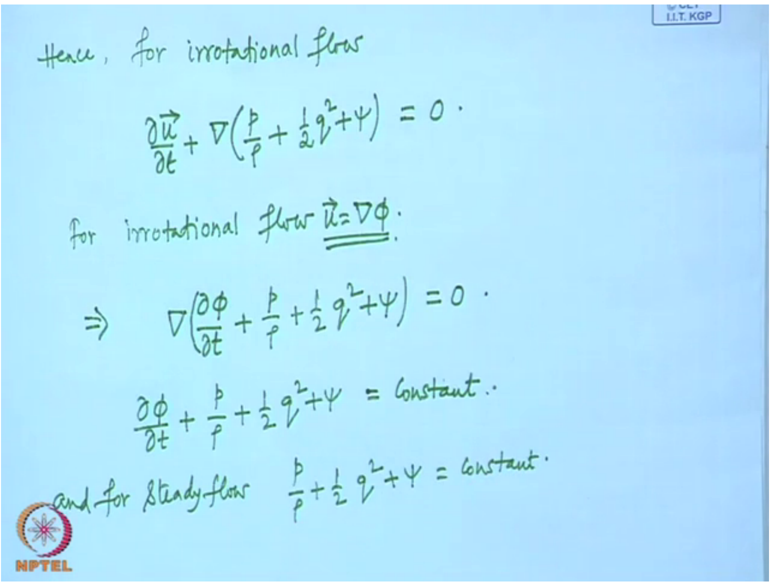

Now, we can write that for irrotational flow if the flow is steady the first term becomes 0 and only this gradient remains and from there we can see this term is constant. Anyway even if the flow is not steady for an irrotational flow we know that if the flow is irrotational the velocity field can be expressed as the gradient of a potential. So, for irrotational flow and this now we can substitute here and gradient of phi is derivative with respect to space, derivative with respect to x y z and we already have here derivative with respect to t but when we differentiate with say respect to x and t it is immaterial which one we first derive first. Whether we differentiate first with t or differentiate first with x it is immaterial.

So, this d this terms become now d dt of gradient of phi which also you can write as gradient of d phi d t. So, writing that this now becomes we have gradient of d phi dt and since this is also a gradient we can put all of them together gradient of d phi dt plus p by rho plus half q square by. So, we see that for irrotational flow this term is now constant and for steady flow the first term is does not existwhich happens to be the original form of the Bernoulli's equation. So, that original Bernoulli's equation is just an integration integral form of the momentum conservation equation or Navier stokes equation.

Of course, it is not a new equation this equation is already there within the equation of motion. So, it is a special form of the Navier stokes equation however this form holds only when the flow is irrotational and see in particularly in many aerodynamical application this psi which represents the potential force and in most aerodynamical flow problem this body force comes from the gravitational force and in this case this psi is the gravitational potential energy per unit mass that is simply that g z where g is gravitational acceleration.

And most often this term need not be written separately explicitly. Look at, see the application of these equation will be always for finding some difference. We can apply this equation to two points in the flow or in some cases say two points on a particular streamline and you see the potential energy difference between two points in a streamline or in general two points in a flow in many cases will be negligible. And even if it is the small quantity that can always be absorbed in the pressure. A difference in pressure will be will take care of the difference in potential energy in many cases, so this term even need not be written separately.

So, we will see that in most aerodynamical application this is usually not written. We will consider another special form of these Navier stokes equation and once again we will derive it for incompressible flow which is a very very important equation and for that we will start from say here the first two three steps are same we will start from here.

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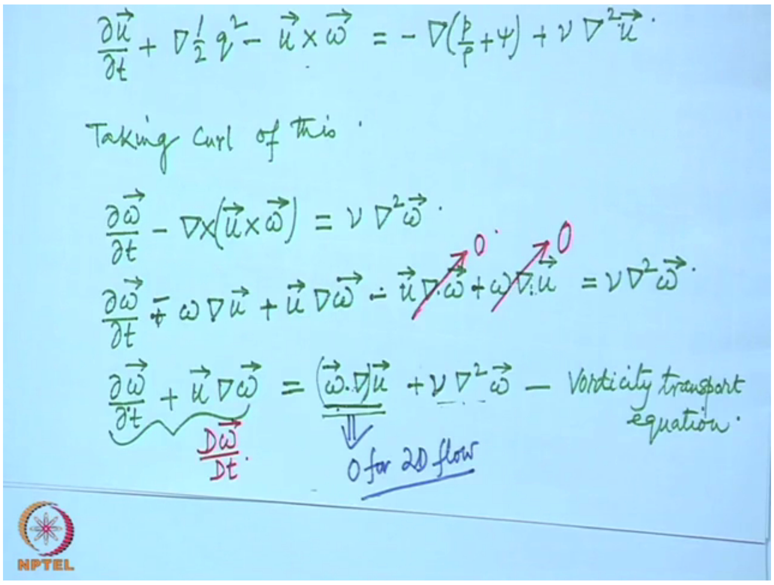

Here, we have what we have the incompressible Navier stokes equation in this form. This is of course, the incompressible Navier stokes equation and from here we have obtained this equation. So, we will start from this. These two step we will not write again. We have applying that vector identity, we have d u dt plus and this last term we will write as it is, this last term instead of writing this we will write the first one. Only this second term that convective acceleration part we are writing using this vector identity.

Now, let us take a curl. Curl of this equation, if we take a curl of this equation what do we get? If you find inconvenient to take curl of this equation the alternative is you have to write the three component of this equation then see the curl is basically some differentiation like the z component of the curl operator is d d y minus, d d x. Similarly, the x component is d d y minus, d d z, x component of the curl operation is d d y minus, d d z. So, if you want to take that in this scalar form first of all you have to write this in three component equation the x component, y component and z component. Then the x component of the equation you have to differentiate with respect to y and with respect to z and subtract the two, which will give you the x component of the curl but anyway we can do it by simple.

So if we take a curl. See, curl is again a special differential operator. By taking curl means you are differentiating with respect to x y z in different combination. So, here the first term look to the first term, we have already a time derivative, but the x derivative and y derivative are independent of this time derivative. So, we can think that we have taken the derivative first and then taken this time derivative, so what it becomes? It is d dt of curl of u. So, the first term will become d dt of curl of u and curl of u is omega. So, the first term will become second term, curl of grad of something, curl of a gradient is always 0, curl of a gradient is always 0.

So, the second term gives 0 or this term becomes 0, the third term becomes minus curl of here also the first term is gradient term and curl of it is 0 and this second term curl of Laplacian of u also again can be rewritten as Laplacian of curl of u. So, the last term will become. This second term as you can see is basically a vector table product type of thing product of three vector curl is a differential operator but a vector differential operator and u and omega are three vectors, this can be written as sorry this is minus this is plus.

Now, since you are considering incompressible flow this divergence of u is 0. This term is 0 and so is this divergence of omega is also 0, omega is always a solenoidal vector, omega is a curl, curl of u so its divergence is always 0. So, divergence of omega is 0 and this for incompressible flow this is also 0 because divergence of u is 0. So, we have only this. So, we will write it like this d omega dt see this is also the Navier stokes equation, this equation then the Navier stokes equation written in terms of u in the same equation. This is not an separate or new equation it is only an alternative form of the Navier stokes equation written in terms of vorticity as the primary or independent unknown.

In many cases later on you will find that this form is perhaps more useful for practical use and equation expressed in this form is called the vorticity transport equation like the momentum conservation equation can also be called as a momentum transport equation, it says how the momentum at one point changes. Similarly, this equation tells how the vorticity at one point changes with time rate of change of vorticity the equation is commonly known as vorticity transport equation, but essentially this is the same Navier stokes equation or momentum conservation equation expressed in the form of vorticity.

Now, if you look at the two terms on the left hand side these two together can be written as these two terms together can be written as that material derivative of vorticity. You can see that this is the material derivative. The specialty of this equation here is that in the original form of the Navier stokes equation there is no term like this, this term, there is no term like this, this term is there in the original Navier stokes equation of course, in terms of u, this is also there in the original Navier stokes equation, omega is being replaced by u it is there, the last term is also there the viscous term the pressure and potential energy part that has vanishes because of that curl operation.

Now, look to this each of this term the first term is of course, simply how the vorticity changes at a point with time. How the vorticity at a point in the space changes with time that is what and how it changes it says the what are the different cause of change. The first term we have already many a time called this is the convective term so it says that the vorticity at a point can change because the vorticity can be convected with the flow with flow velocity u that if there is some vorticity, this vorticity can be taken away from this point by the fluid with this flow the flow itself as it take carries momentum along with it it may carry vorticity also along with it it will carry.

Leave this term first, this last term viscous action, it is always usually diffusive and dissipative action that both viscosity can diffuse vorticity if there is vorticity at one point this what viscosity can spread it to some other part if there is a vorticity in a one portion of the fluid one region of the fluid this vorticity can be carried away with the fluid by the flow velocity. Also, it can be diffused to some other part of the fluid by viscosity these are all right this last term or this term particularly this omega gradient u what is this? This is of course, another cause of change of momentum vorticity but which has no analogous in the momentum equation that is the momentum cannot be changed by this mechanism.

However, vorticity can be changed by this mechanism. This change by or transport by convection and transport by diffusion that are valid for both vorticity as well as momentum. However, this region of change this this term this type of change does not occur in momentum. Momentum cannot be changed by this process while vorticity can be changed by this process. Now, what is this? If you look to this term you can see that in two dimensional case this is 0 identically, this is 0 for two dimensional flow, because this omega gradient u if you write it in this form that omega in this form then what happened, this is in a two dimensional flow vorticity has only one non 0 component that is in the z direction.

The velocity if we consider the velocity in two dimensional case has only x and y component then the vorticity has only one component in the z direction ok. Then this omega dot gradient it will leave only a z derivative and since u does not have any z dependence this term is 0 by default. It is 0 for two dimensional case. This term is 0 for 2D flow, but not so forth three dimensional flow, for three dimensional case of course, it is non zero and what is told is that this term actually represents a deformation that is a vorticity of a vortex tube or vortex filament can change because of the deformation of the vortex filament that is the vortex filament like any material element can deform a vortex element can deform like any material element.

The way one of your string can deform, it can be twisted, it can be extended, it can be shortened similar way a vortex filament that also can be deformed and because of that vortex filament which is this term is often named as vortex stretching term that a vortex filament can be stretched, and because of this stretching the rate of change of vorticity at a point can change. So, this is a special mechanism which does not is which is not true for momentum transport but with which happens in case of vorticity transport, a vorticity at a point can change just because of the stretching or deformation of the vortex filament or vortex element but momentum cannot be changed in that way.

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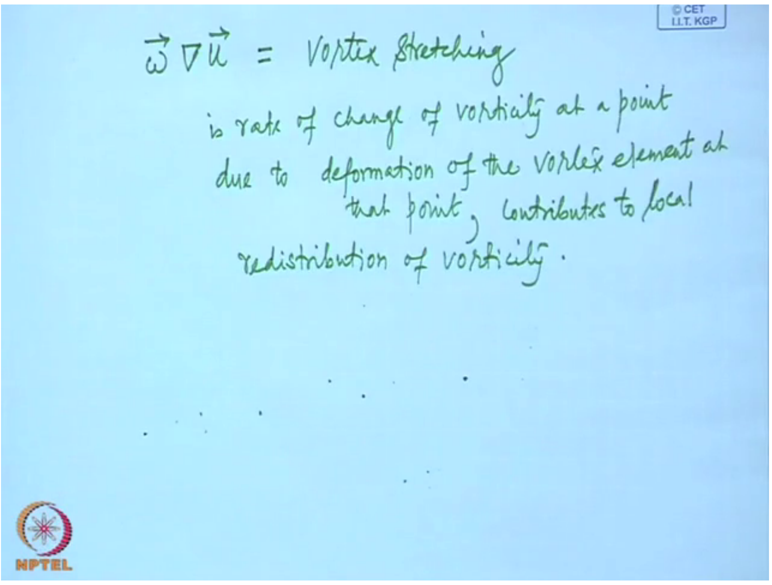

So this term is known as vortex stretching term and is very, very important later on when we will consider vorticity dynamics perhaps in another course. The effect of this term is local basically, while the effect of the other two terms convection and diffusion is global, the effect of this term is local for the time being that is all about the equations of motion in general.

So, we have derived all equations of motion that is mass conservation, momentum conservation and energy conservation and also we have derived alternative forms of those equations for some particular cases, then we will look to a few solutions of these equations of motion of course, as we have mentioned earlier the equations of motion in their general form or also in the simplified form for incompressible flow are non-linear set of equations and their non-linearity come, because of that convective term present in the acceleration and hence all inertial force terms in the energy equation as well as in the momentum equation.

And solving non-linear equations is extremely difficult except for some special cases there are hardly any exact solution available for this equation. Only for a very few limited flow problems there are exact solution available for this equation, and mostly they are for incompressible flows the analytical solution.

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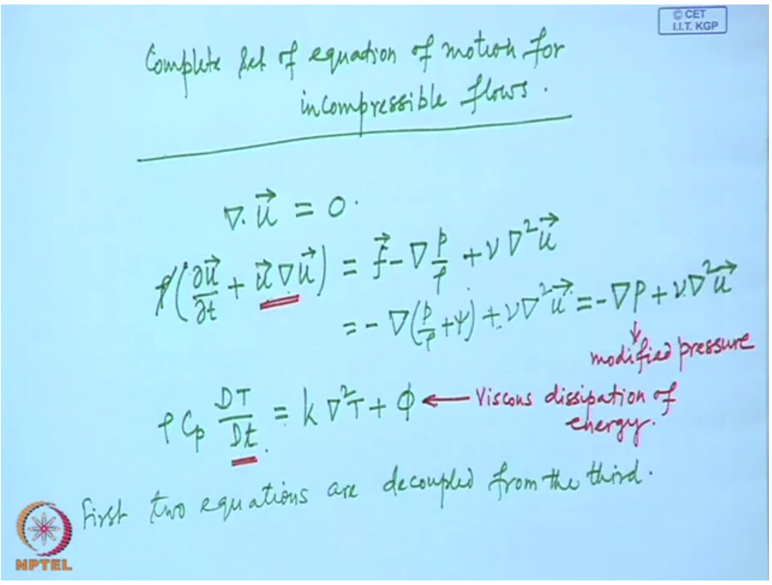

So, before we go to that let us write the complete set of complete set of equation of motion for incompressible flows. Once again that by incompressible flow we mean that the flow in which density is independent of pressure, density is not changed because of pressure. When a flow can be incompressible that we have not yet explained we will come to that little later, but for the time being assume that there are some flows in which density is independent of pressure, density does not change because of change in pressure. And however still density can change because of change in temperature, so if we assume that the change in temperature is not large a small change of temperature may occur and as a consequence the change in density is really negligible, the change in density due to temperature is also negligible.

If this happens that the density does not change because of pressure and the change in temperature is small, so density does not change due to temperature as well, then density is constant and the mass conservation equation or the continuity equation becomes simply divergence of u with 0. The momentum conservation or the Navier stokes equations become or take the rho from this side and make it and once again if we consider this to be the potential body force, this can be written as, this is little restricted from this because in this case the force field may need not be potential for this equation but in this case the force field is only potential.

So, as far as applicability is concern this is little restricted. Now, you see this is both gradient and so you together these we can thought of as a modified pressure and can write as minus gradient p and forget about the body force. This pressure is then called the modified pressure which takes care of the potential body force, and if you look to the energy equation for incompressible flow, the energy equation will become the energy equation for the incompressible flow will become rho, do not confuse it with the velocity potential, this is that viscous dissipation of energy per unit volume and if you look back to your the derivation of the energy equation you see that this is that part of the work done by the viscous traces due to rate of strain. Now, for an incompressible flow with constant density there is as you can see this temperature does not appear anywhere in this equation.

Since, already we have assumed that the change in temperature is small, so density is not changing due to temperature and for that case the if the change in temperature is small then this coefficient of viscosity will also not change much due to temperature. So, there is nothing in these first two equations which depends on temperature for this case so the first two equations has become decoupled from the third equation meaning that these two equations can be solved without this equation, however this equation cannot be solved without the solution of these because this contains u this dt d t this when you write in expanded form it contains u gradient ok.

So, this needs u, but these two equations does not need t. So, the first two equations are decoupled, first two equations are decoupled from the third meaning we can solve this continuity and momentum equation together to find the velocity and pressure field and once the velocity field is known then we can solve this energy equation to find the temperature if we are interested. In many problem we are not even interested to find the temperature, so we do would not even bother. Now, though the equations are much simple then the original equation still these equations are practically unsolvable because of these non-linear term present in this equation, in this term and also if we expand this, this also contains another term like this u grad[ient]- d because of that the equations are non-linear and usually cannot be solved straight away for any given boundary condition.

Until now of course, we have not said boundary about boundary condition that is simply because that boundary conditions are of course, problem specific. Since, the boundary will be changing from problem to problem the boundary conditions will also change and we will talk about boundary conditions for a practical particular problem, but just a few word about the boundary condition. See most often, we are interested flow over a rigid body or a solid body, so that will a rigid body or a solid body will be a boundary of the flow most often or all most in all practical cases.

We are interested as the flow over or within some rigid body that is flow through a pipe, it is within a rigid body or a solid body most often that there will be no deformation of the solid so we can treat it even rigid body. If you have there is a flow over the building then again building is can be treated as either a solid body or rigid body depending upon whether you are interested to consider the deformation of the building or not. If you are thinking flow over an aircraft again it is flow over a rigid body or elastic body depending upon whether you will consider the deformation of the aircraft in your analysis or not.

But anyway it is flow near a solid boundary and what you usually expect of the flow, if there is a flow over a solid body. Can you say what condition it is likely to satisfy? What it will be? What is pretty obvious, that flow will definitely not penetrate that solid body. If there is a flow over this say this table, it is obvious that the flow will not penetrate this table and how can you express it in terms of say these a equations or these unknowns that is the flow velocity normal to this table or normal to this solid body will be 0. Non 0 velocity means it is going inside. So, obviously it will not, so the normal component of the velocity will be 0 is very obvious condition when there is a flow over a solid body.

However, it is also shows that in most cases or all most in all cases even the tangential velocity or the relative tangential velocity of course, when you are talking about velocity we are talking about relative velocity, that the relative velocity, tangential component of the relative velocity is also 0 on the solid body. So, both the tangential and normal component of the relative velocity is 0 on a solid surface or on a rigid boundary so that is the most important boundary condition and this boundary condition is no slip that is there will be no slippage between the flow and the rigid body at that at their interface that is on the surface of a rigid body there will be no slippage.

So, the most important boundary condition is the no slip. Any other boundary when you come across we will a talk about it but this is usually the most important boundary condition and all most in all problem this boundary condition will be required that on a rigid surface or on a solid surface. Let us call it rigid surface, on a rigid surface there will be no slip that that is there will be no relative velocity between the rigid body and the flow.

So, if the rigid body is at rest the flow on the surface of the rigid body will also be at rest. So, we have the most important of the boundary condition, no slip condition that is as far as the boundary condition for this equation is concerned. However, if you want to solve this energy equation as you can see the temperature here is the basic unknown. So, you must have some condition or something to express temperature on a rigid body or on a solid surface. Of course, there is nothing obvious like this usually the conditions will be like this that if you have solid body and there is some flow over it, you will try to see that some of the things like you may want that the temperature of the solid surface should not change, that is what you may want and you may set it that the temperature on the solid surface will not change.

So, the flow which is in contact with it will also have the same temperature which is usually called as the isothermal boundary condition that you are fixing the temperature and keeping it fixed. You are not allowing change the temperature. Sometime, you may have in another form that no heat will go from the solid surface to the flow or vice versa. You can call that as adiabatic boundary condition adiabatic wall, the wall is such that no heat is being transported from the wall to the flow.

So, of course, these will be problem to problem. Now, given these boundary conditions of course, now the boundary can be very complex. The boundary may be a very complex or may be simple some cases. So, if the boundary is very complex even you see this you are saying no slip but you have to satisfy that no slip and if the boundary is very complex satisfying that itself is quiet a difficult task. So, as we mentioned that these equations first of all they are non-linear and then if the boundary is complex then the boundary condition also can be very complex and there are no general exact solution available for this equations for under any boundary condition no general solutions are available.

However, for some simple boundary or for some simple flow problem some exact solutions are available and for the time being we will consider just 2 or 3 such simple example problems and then we will keep those for other problems for later course. So, here just for example, we will consider 2 or 3 very simple cases for which exact solutions are available.