Introduction to Aerodynamics
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Module No. # 01
Lecture No. # 06
Forces in Fluids (Contd.)

So, today, we will look for few simple application of the theory that, we did over last two classes. That is, we will try to find out the level surfaces, the static fluid pressure distribution for some simple cases.

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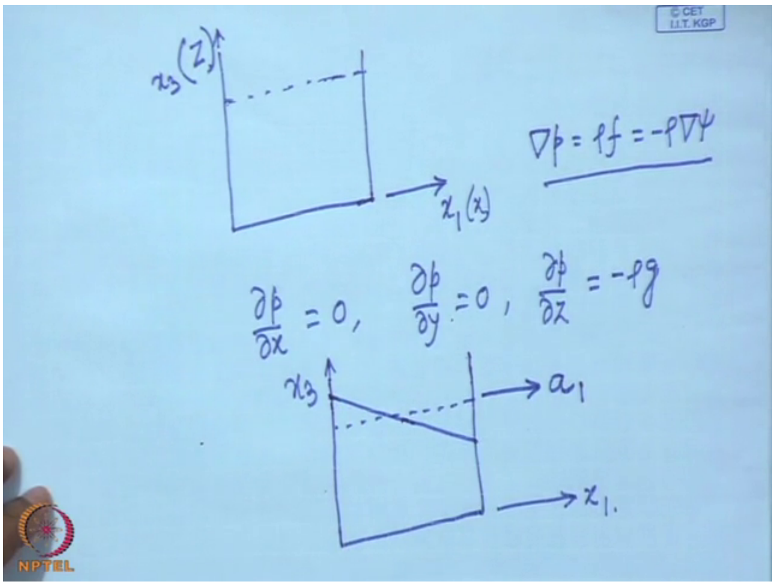

The simplest possible case is quite well known to you, that if we have a fluid say, contained in some container and is completely at rest, subjected to only gravitational force. You can write the three equations that we had, yesterday; that is, grad p equal to rho f, grad p equal to rho f; that is the equation we had, rho f or minus rho grad psi. So, when the body force is simply the gravitational force, you can write this equation grad p will give you three component d p d x d p d y d p d z or d p d x 1 d p d x 2 d p d x 3. And in this case, only that x 3 has the right hand side. f is simply g, which is along x 3 direction. So, if we consider that example, I think you have already done it. Let us say that this is fluid and we denote this x 1 or if you want, you can now write x also. You call this direction to be z and the direction perpendicular to this plane is the y.

There is only gravitational force and this equation can be simply written as d p d x equal to 0, d p d y equal to 0 and d p d z equal to minus rho g. You can get the pressure very easily, the hydrostatic pressure which is simply rho g z plus a constant. Another constant is actually the pressure at the free surface. So, this is the simplest possible application and as you can see the level surfaces are the horizontal planes. They are the level surfaces for pressure as well as for psi. The level surfaces for pressure and psi are simply the horizontal plane. At any horizontal plane both the pressure as well as the potential energy, they are constant. Of course, also density because, that is what we found that, they will have same level surface, density, pressure and the potential energy will have same level surface.

And in this case, these level surfaces are just plain, horizontal plane. Of course, there is one thing you should note here, that if density is non-uniform, if density is non-uniform then you may not be in a position to integrate these equations. Unless, we know explicitly how density varies, we would not be able to integrate it. So, in this case, we have considered the density to be uniform.

Now, we would like to change this problem a little bit. We will consider that this container is now, moving with a uniform steady acceleration. The container is now moving with a uniform steady acceleration and let us say to the right. Along x 1 direction, let us call it a 1. This a 1 is just to represent that, it is in the direction one, x 1 direction. And this is the x 3 direction as before and this acceleration, uniform steady acceleration in the x 1 direction, we are denoting it by a 1. Now, this container has a steady uniform acceleration.

Now, with respect to a reference frame outside of this container, the fluid is of course, not at rest. But, in case of a steady acceleration, if you take your reference frame, which is also moving with that same uniform acceleration, with respect to that reference frame, the fluid and this container is at rest. And we can apply these equations. In that case, let us say without proceeding further, can you say what type of level surface, we will have?

Inclined? So, in one side, it will go up. In one side, it will come down. So, in this situation, which side will go up? Opposite side of the acceleration, that is, in this case this side will go up; that means, say that the level surface will be something like something like this. Now, let us find out. You might have, as it seems that you perhaps have solved this problem some time by using some other approach. But, in this case, we will use this approach that we have developed. That is, writing the equation grad p equal to rho f or minus rho grad psi.

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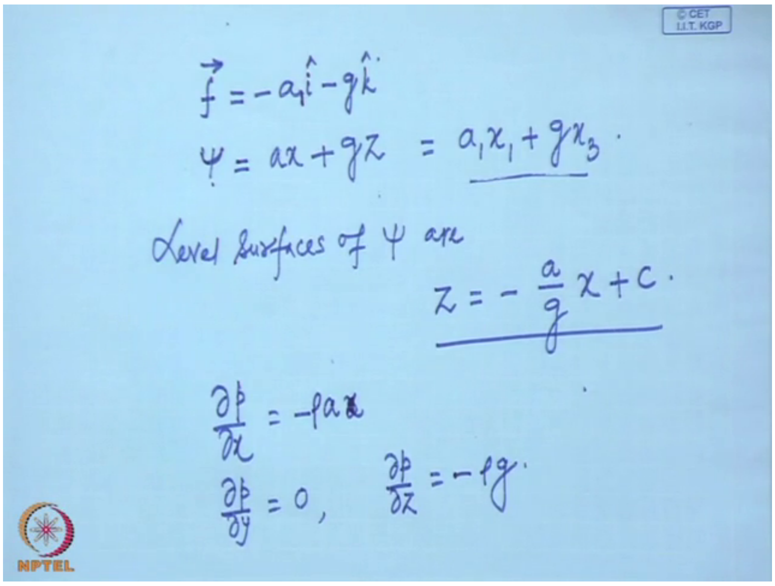

Now, in the vector sense, the body force per unit mass f is what? If we write in your conventional vector notation, what is f? Minus a in the x 1 direction, you are calling it i. Fine. Call it i. Minus g z g k. What is this? That potential function or the potential energy per unit mass, psi. What will be psi, the potential energy function? Remember f is minus grad psi, f is minus grad psi then, what is psi then? Yes. So, what is that psi? (( )) a x plus g z, as you are saying or we will also call it a 1 x 1 plus g x 3.

In this case, perhaps, this will be convenient to write for you a x plus g z. This, I am writing. So, that you keep remembering it and have the practice now. So, you can from here you can find out, what is the level surface of psi? This gives a level surface of psi. That z equal to level surface of psi or what? Level surfaces of psi a x plus g z equal to constant; that is, the level surface and what is that? So, z equal to minus a by g x plus any constant. These are the level surfaces of psi and also these are the level surfaces for pressure and density.

All three psi, p, and rho, all three are same level surfaces and also this, the body force the body force vector will always be normal to that level surface. Now, to find the pressure, we have to integrate certain equations and as we mentioned already, that we can integrate, if we assume that the density is uniform. For non-uniform, we must know the exact explicit variation of density. So, just to avoid that, let us consider uniform density. Then, what are the equations we have now? For pressure, d p d x equal to minus, I was writing a 1, somehow, it has become a. Let us write it a only. Minus a x d p d y equal to zero. There is no force in the y direction and d p d z is, sorry not minus a x, minus a rho, d p d z is minus rho g. Let us integrate the all three equation. The first one gives what?

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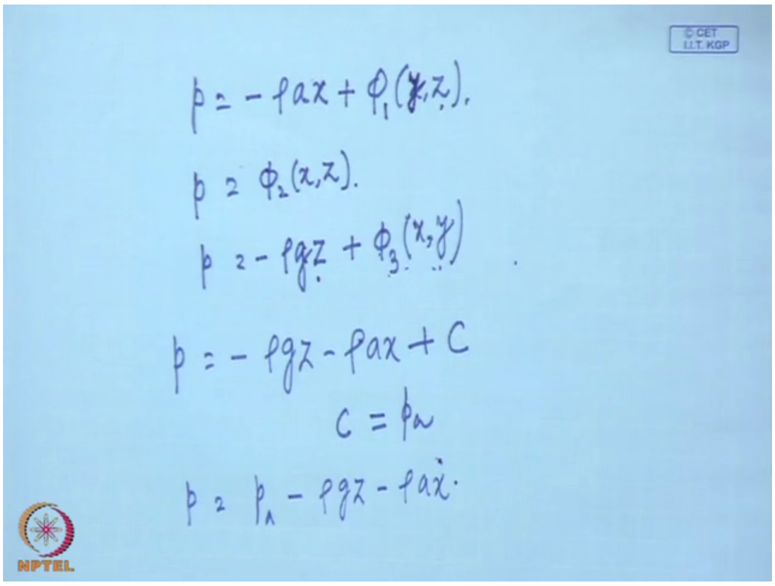

The first equation gives on integration p equal to, that is all, minus rho a x plus? How did you get that plus g by z? Another plus, another function of y z. That you are saying, in the function g. Let us do not use g because, g is already we are using as vibrational acceleration. So, we can write this as say, phi. phi x z. Sorry, y z. The second equation gives p equal to another function. If we want, we can denote these as phi 1 phi 2. So, the first function, let it phi 1 and the second function, in this case is what? It is simply phi 2 x z and the third gives p equal to minus rho a z, sorry, rho g z plus phi 3 x y. Then, if you combine these three, what will you get?

So, looking into these three, we can clearly see that, this pressure is not a function of y. There is no y dependency, anywhere. The third equation contains a function of y but, the second, clearly shows that the pressure is not a function of y. So, in the from the first and third, we can drop the y. This is function of z only and this is function of x only. And then, comparing this first and third relation, we can clearly say that pressure is minus rho g z minus rho a x plus, just a simple constant, let us call it c. This constant can be obtained by satisfying or trying to find the pressure at a fixed location for x and z. So, we can satisfy that, let us say, at the top left corner, in the container at the top left corner. And if we or let us say, still it is here. If we make that point as the origin because, that is what we are actually doing, x and z. the origin is here, at the top left corner, here, which has x equal to zero, z equal to zero. And then, what is pressure at that point? So, pressure of the atmosphere, if we consider atmosphere. Of course, we can take that to be zero also. In that case, we will get what is known as gauge pressure.

So, at if we consider that at x equal to zero, z equal to zero, pressure is simply the atmospheric pressure. Then, from this relation this c becomes the atmospheric pressure. So, and this relation become and again, you can find the level surface of pressure. You will see, that is, the same thing, level surface of pressure coincide with the level surface of psi.

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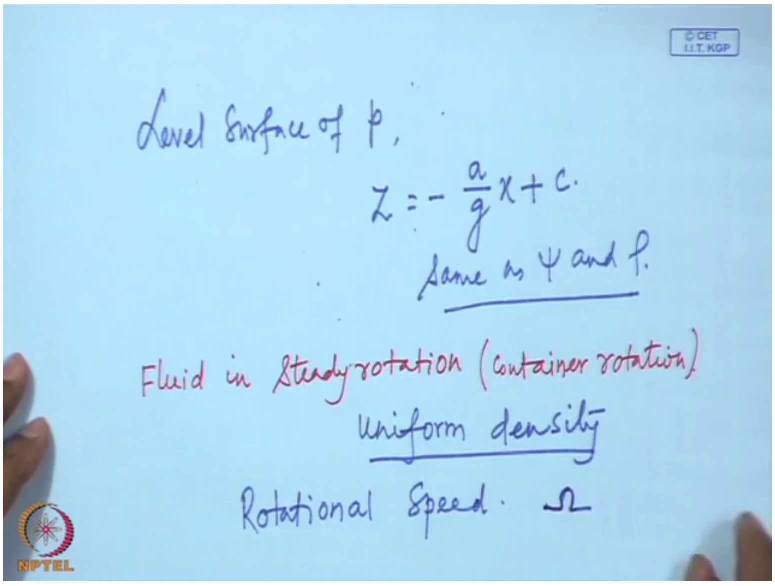

And also you can check, for the free surface, that is, at inclined top surface.

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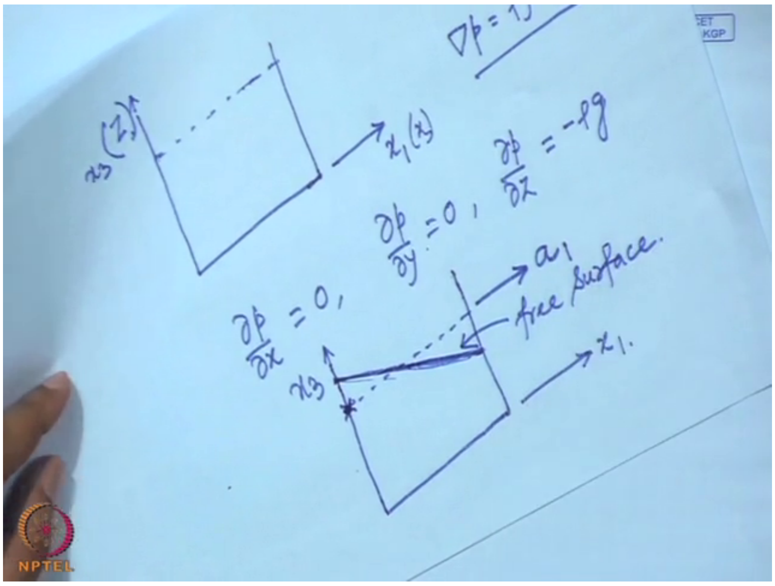

That, you can set that c to be zero, for this, which is called the free surface. For the free surface, you can set c to be zero. The in any other lines which are parallel to it, will have different value of c. And you can also see that the direction of the force vector is perpendicular to that vector surface. Little more to think for you.

Now, let us assume that this acceleration, changing its direction periodically. What do you expect to happen then? Of course, that problem will be little complex. So, we will not, not possible to solve here. But, some idea you can get from this itself. The acceleration is changing its sign. Of course, it is no longer, then a steady state problem like this. It is an unsteady problem. But, what do think will happen to the free surface?

Student: (( ))

No, no. Look to these, your acceleration was to the right. In this problem, that you handled, the acceleration was to the right; the free surface on the top goes up. On the right, it goes down. If we change the direction of acceleration, what will the opposite will happen. Now, if the acceleration direction changes periodically. So, this will also happen periodically, it is likely to happen. And the because of this transience, the free surface no longer remain exactly straight. Well, anyway, we will not be able to go into details of that, what shape the free surface will have or something. But, this much you can understand. That, now, this will the liquid inside, will have an oscillation. Once like this, and again like this. This will continue.

And also think about one thing, that in this process here, the left side has gone up, depending upon the magnitude of this acceleration, the amount that is going up, above the undisturbed free surface. This was the undisturbed free surface, we will call it, the mean surface, let us say.

It has gone this much up because, of this acceleration. For higher acceleration or larger acceleration, this may still go up and if the container has not sufficient amount of height available still, the fluid will spill out. It is possible. So, how much? This is the amount which is still available, that is actually called as free board. So, how much free board should be given? That you can decide. That I, in a real life situation, say, where you are going to construct this type of container or design this type of container and you know that, this is likely to experience or this is this will experience a periodic type of or nearly periodic type of acceleration, which may have this type of magnitude, then, you must know, what type of free board you must provide.

Think about the situation, a dam. Again, another, a container like this. In the actual problem, of course, the cases are little more complex. You will not get a uniform steady acceleration. The acceleration magnitude will change continuously, all those things. But, this gives a fair idea of that type of problem, you are going to face. That you must have sufficient amount of free board, otherwise the water that is contain in the dam, that will spill out over it.

And not only that, that because of these oscillations, it will apply some other type of load on the dam structure. The dam is containing the liquid, so, the liquid is water. So, that water is exerting certain amount of pressure, whether that water is moving or not. But, in case, that water accelerates because of some acceleration like this, then in addition to that pressure, it will apply certain different type of pressure, certain different type of load and that dam must be capable of withstanding that load also. See, that may not be a daily occurrence, may be a frequent occurrence. See, all natural, this type of reservoirs, at least, they are, they do experience a highly unsteady fluctuating load, earthquake. In the earthquake also, the ground accelerates, of course, very rapidly and almost randomly.

We will extend this problem to a little different but, almost similar type of problem where there is no uniform steady acceleration as it is here. But, let us say the container is rotating with a uniform rotational speed. Let us call it omega. The container is rotating at a uniform rotational speed. Say omega. Again, with respect to a rotational frame which is rotating with an angular velocity of omega, the fluid in the container is again steady, with a pseudo body force, with a pseudo body force coming because of the centrifugal acceleration. So, what will be the, say the potential function, in that case, container with a steady rotation. Again, first of all, consider the density is uniform. Consider uniform density, so that, we can carry out the integrations.

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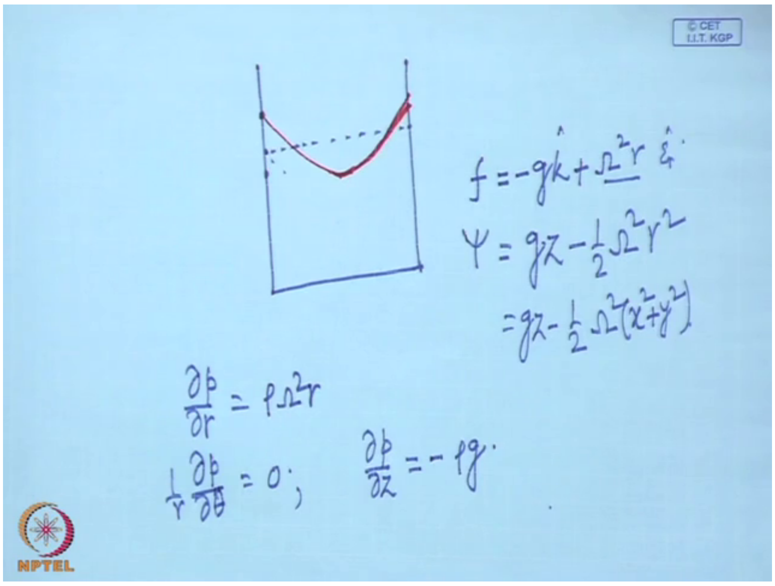

How will the free surface look like? How will the free surface look like?

Student: (( ))

Parabolic in two dimensional case. For a three dimensional, it is a paraboloid of revolution. Will the free surface go up or come down or what? That is the, is the parabola will start from here or from somewhere else? Will be like this or like this? How? Down. That means it will come down somewhere here at the wall it will go up.

Is something like this, you mean to say? Something like this? So, if the free surface is like this, then our theory says that the force that is acting on it, is everywhere normal to it. the force the body force that is acting on it, is everywhere perpendicular to this level surfaces. the free surface is also, one of the level surface.

Now, let us say, how do one to start? Starting with psi, the potential energy per unit mass? So, what is the potential energy per unit mass, in this case? Or if you think, that it is easier or convenient for you to write the force first, instead of the potential energy? And as for as, the coordinate system are concerned, you can use any either you write x y z or x 1 x 2 x 3 or if you prefer x r theta.

What is psi or let say, what is f, whichever is convenient you? The forces, two sources; the gravitational source and the other is the centrifugal. So, how much is f or psi? Tell me, whichever you like, f equal to or psi equal to. f, what is f ? omega square? How can you take r equal to x?

You cannot take r equal to x. Shift, origin you shift at the center, no problem. Wherever you can take origin, only that, now, our reference system is rotating with the angular velocity of omega. That is all. So, that with respect to that reference frame the fluid is at rest. Because, our theory is at this stage is for fluid at rest. So, reference system should satisfy that. Other than that, you can do anything with that reference system. No, no, you tell me something. The centrifugal force should be proportional to r, not x and r is not x. It is x square plus y square to the power half. So, either you keep it in r, you keep it in x y. That is, whichever you prefer. You may remain in x y z coordinate system or you may move to r theta. And of course, the axial coordinate will still remain the cylindrical coordinate system. You may remain in the Cartesian system or you may move to cylindrical system, that, it is up to you. So, the force is minus g k.

Plus or minus? Plus or minus?. Yes, plus or minus? In f, it is minus, minus omega square r. You can you can use either, you can use either. That r can be replaced as x square plus y square plus half and also, you can write in term of i and j using that. They are all fine, anything. But, in this case, it is in the radial direction. So, let us write, that something like this, a unit vector in the radial direction.

Yes, that is what, i was asking, what it is, plus or minus? There is half the reason, why? See, this is a pseudo force, this is a pseudo force. So, there will be a change in sign. So, what it will be? It is plus. Then, what is now, psi? What is now, psi? Psi is again, or f is minus grad psi. So, what will be psi? You are writing in terms of x square. Psi is then g z minus half of omega square r square or (( )).

And this gives of course, the level surfaces, straight away. Without proceeding further, we can say, what are the level surfaces from there. And the level surfaces for pressure and density are the same. We said f, it is a paraboloid of revolution with vertical axis shifted vertically. Now, if we want to find the pressure, the pressure distribution, then, we write that equation for pressure. What is say d p d r? What is d p d r? This is what is d p d r.

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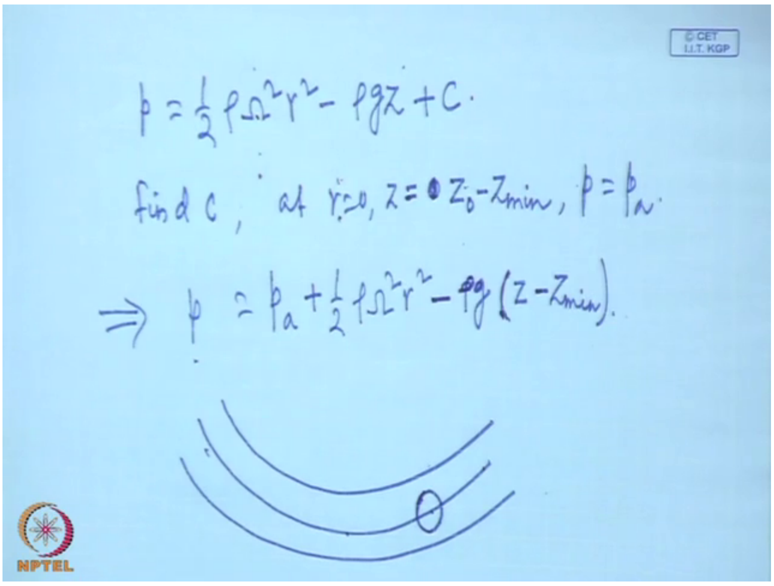

Yes, then what is p? Half rho omega square r square minus rho g z plus c. What is c then? What is c then? Above the free surface, again assume that the, it is atmospheric pressure. But, we see then p a only, atmospheric pressure as before in the last class. Which z? z is a variable. Height of free surface is again not constant. The height of free surface is different at different point then middle one.

Midpoint, which is the say, the minimum height. At the middle, it is a minimum height. At the wall, it is a height is maximum. At the middle, height is minimum. So, we can call z minimum or z min. Then, what is final form of, so, evaluate c? Find c, at r equal to zero, z equal to zero. Or z equal to not zero z equal to, the say, if the you have taken your axis system here know? This is what, is you are taking over axis system. This is what, you are taking over axis system at the middle, undisturbed midpoint? With respect to that, this has a height of, if this height taken as the undisturbed say z zero, z zero and this distance, we call it, z min. So, when you satisfy your equation at this point? Is it z zero minus z min. So, it is z zero, minus z min. Then, what you see or p in final form? Yes, what is the p, in final form? P a, only atmospheric pressure? (( )).

P a plus that half rho r square minus rho g, rho g z naught is zero. rho g z. z naught minus z min minus z plus z. I am writing that z first. z minus z min plus z minus. And what is the pseudo static pressure at z naught? At z naught, the liquid depth is zero. The depth is zero there. In this case, it has become above the liquid surface. So, there is no question of that coming into the picture. Z plus the second is also plus. (( )). Or minus. In bracket also, it should come minus.

No, that is, I think, what you are doing, you are not considering that, it is the depth below the free surface, below the undisturbed free surface. That we have considered and still it is coming? No this z, all this z are negative. All this z are negative, that you have not taken into account. All this z, because your origin, what you are considering, only negative z. What you are considering is only negative z. Even that, see z, z zero, what we are calling which is the distance between the undisturbed free surface to the bottom, undisturbed free surface. So, the total liquid depth, that itself is a negative.

Because your reference system is above it. And I think, that part, you have not considered, that your reference is on the above. So, even z zero what we are calling. In actual coordinate, when we substitute, you should consider it, in the negative sense. Now, you can now find out, how much it has risen with respect to the undisturbed free surface? That means, this height and even both this heights, this height as well as this height. We may consider the volume to find how much is this z min or how much is this this distance above the undisturbed free surface that can be very easily obtained by considering the volume. The total volume of the fluid is unchanged. So, before this disturbance, it was just, in the shape of a cylinder. You can consider a radius for the entire content at to be a, some r and that height is z zero. So, pi r square z zero is the total volume of the fluid.

Now, we can find out how much will be the volume for this, with this free surface. And that will give you this z min and if you denote again, that this is z max, so, z min and z max, that can be obtained from there. And this distance will come as, this distance will come as, you say that omega square r square by 4 g. You can check that. You can think about an alternative variation of this problem. Another variation of this problem, that you may try, yourself. That is, in this case, let us say the container was rotating about a vertical axis.

Think now, the container is rotating about a horizontal axis. You can arrange your container accordingly. So, that it does not look odd. The container is rotating about horizontal axis then, say how will be the level surfaces? Whether they are still paraboloid of revolution or something else? That of course, you can try yourself and see what you get. We will think about another variation of this problem, what we what we are discussing, uniformly rotating fluid. Let us say the fluid density is not uniform, it is non-uniform. Then, of course, we cannot integrate and find this pressure in this analytical expression form but, even if the density of the fluid is non-uniform, you can see that the psi and f, they do not change. Psi do not change at all, which is potential energy per unit mass. So, the level surfaces will remain same. A fluid which is rotating about a vertical axis whether the density is uniform or not the level surfaces will be again paraboloid of revolution and the same level surfaces for pressure, potential energy and density. Only, when the density is uniform, on all level surfaces, density has same value. But, if density is non-uniform then, on different level surfaces like pressure and potential energy, density will also have different value.

Now, think that, you place a spherical sphere. A small sphere made up of some metal and you place it in. And then, think where this, if the sphere is uniform where will this sphere rest? Will it remain anywhere, wherever you place. Assuming, that there is a level surface, there is some, first of all, it will try to locate a level surface, where the weight of that displaced liquid is same as the weight of the sphere.

Let us assume that, we have found a level surface that there is some rho, it is not of everywhere, it will not be same. Because, rho is since rho is changing the weight of the displaced or mass of the displaced liquid will change at different depth. Let say some level surface, the density has appropriate value which will give this amount of mass, then in that level surface, will the sphere remain wherever it is, wherever you put it? Well, that is, what the laws of flotation, says know? That, it will remain at any location in that depth, if a body floats either completely submerged, that is also floating. When the body is completely submerged or when the body is partially submerged, they are all floating. That wherever you keep it, it will remain there. But, will that happen in this case also? Let us say, as an example, that we have found different level surfaces and at this level surface, the mass of this displaced volume as same as the mass of the sphere. Then, will it remain here or here or here, here, anywhere you put it will remain there?

Why? The energy is same, it is a level surface for that energy know. Why should it rotate? See, in this situation, what you know, that the mass of the displaced water and mass of the sphere should balance each other. That alone is not sufficient, also the centrifugal force experienced by that displaced fluid need to be same as the centrifugal force experienced by this sphere. So, for a uniform sphere, such a position will not be found. It will be only possible at the lowest most position, that is, at the middle, it will come to the middle; however, if you make the cylinder non-uniform, if you make the cylinder non-uniform, then it can remain somewhere else, even possibly it may go to the wall.