Satellite Attitude Dynamics and Control
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Lecture – 10
Rotation (Contd.)

Welcome to the 10th lecture. So, today we are already we have discussed about the eigenaxis and this Euler theorem. Today we are going to do the Euler parameters. So, from where the Euler parameters are arising let us look into this.

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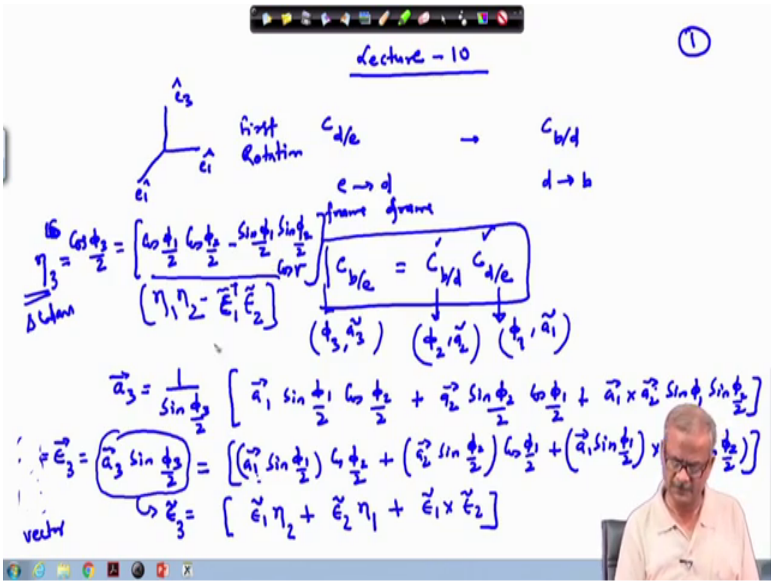

So, if you remember in the 5th lecture we have with derived this if we have two rotations say we have this e frame e 1, e 2 e 3 and from there we give one rotation and go to the d frame. So, first rotation let us say this is the first rotation we indicate that it goes from e to d.

So, this rotation is from e to d, this is e frame and it goes to d frame next rotation we give is from d to b frame I have chosen notation like this. So, here d to b frame so; that means, we are operating like this, the first rotation we have given like this and on that we are operating like this and this we have shown to be equal to c b e and in this connection we have derived from results and that two.

So, the first rotation this one we have indicated by phi 1 and a 1 tilde, this rotation we have indicated by phi 2 and a 2 tilde. So, composite of this two we can be shown by phi 3 a 3 tilde and for this we have done the derivation, but for that the detailed derivation we have not done as I told you that this will be done as part of the tutorial. So, a 3 tilde or a 3 this we have written as 1 by sin phi 3 divided by 2 times a 1 sin phi 1 by 2 and cos phi 2 by 2 plus a 2 sin this is the derivation of our 8th lecture plus a 1 cross a 2 sin phi 1 by 2 sin phi 2 divided by 2 ok.

So, what if we write this quantity as a 3 sin phi 3 by 2 ok? This equal to a 1 sin phi 1 by 2 cos phi 2 by 2 cos phi 1 by 2 and plus a 1 sin phi 1 by 2 cross a 2 sin phi 2 by 2 within write it in this format ok. And here one more result that we have written it was cos phi 3 divided by 2 this equal to cos phi 1 by 2 cos phi 2 divided by 2 minus sin phi 1 by 2 sin phi 2 divided by 2 times this is not cos product and here this cos gamma ok.

Now, we will utilize this also ok. So, let us write this quantity has eta and this quantity let us write this as epsilon. So, in the matrix notation the same thing we can write as epsilon 1 epsilon 2, epsilon 3 this is a scalar and this is a vector. So, if we look into this particular vector. So, we can write this as E tilde if we write in matrix notation then E tilde this is equal to E tilde 3 this is 3 here tag here. So, we see that this is common throughout; this quantity is common throughout ok. So, E tilde 3 we are writing this quantity is E tilde 3.

So, here we can write as E tilde 3, if E 1, E 2, E 3 these are the components of this it is not the same thing like E 3 and they are not the same. So, it may create confusions. So, I will rub it out for the time being so, I will rub this part ok. Now, this part we are writing as E 3 tilde ok. So, following the same line this can be written as E 1 tilde because a 1 is present and this is a vector. So, this is a 1 tilde cos phi by 2. So, from here if we use this notation so, this becomes cos phi by 3. So, we can write this as eta 3.

So, here this we can write as eta 2 and plus these quantity then along the same line this can be written as epsilon tilde 2 and this one as eta 1 plus the quantity which is present here this is epsilon 1 cross epsilon 2.

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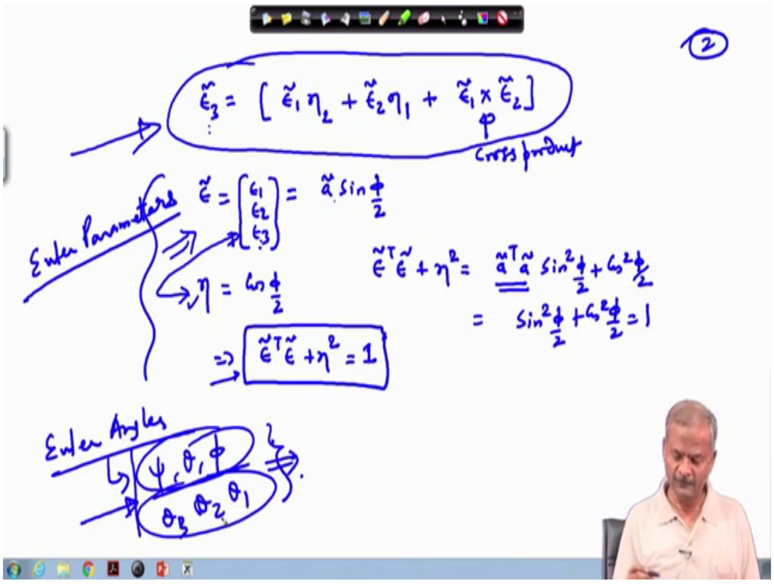

So, what we see that epsilon 3 tilde it is being written in the format epsilon 1 tilde times eta 2 epsilon 2 tilde times eta 1 plus epsilon 1 tilde cross epsilon 2 tilde, this is the cross product ok. So, here as I told you that this epsilon whose components we can write here in the format of epsilon 1 epsilon 2 epsilon 3 remember this and this they are not the same, here prime indicating this has the component I do not see any other way of doing this right now.

So, otherwise we will have to change this tag ok, we are we have to write this as a and this as b and then we can write this is as c. So, using this notation where this quantity; this quantity is nothing, but this is epsilon tilde is equal to E 1, E 2, E 3 equal to a tilde sin phi by 2 and similarly eta this we have defined as cos phi by 2. Now, if you look into the properties of this, if we write it this way and plus eta square just check how much it is. So, this will be a tilde transpose a tilde sin a square phi by 2 and this quantity then becomes cos square phi by 2, this quantity is 1 and therefore, this gets reduced to sin a square phi by 2 plus this equal to 1; that means, E tilde transpose times E tilde plus eta E square this quantity will be equal to 1.

So, this is the constant those the Euler parameters which are epsilon 1, epsilon 3, epsilon 3 and this eta these are the 4 parameters, but they are related together by this equation and therefore, it is a constant means only 3 independent 1, 1 is dependent 1 ok. However, using this notation has the advantage that it will look into this equation. So, if we are looking into the product of two matrices. So, it is a free from unit trigonometric notation there is no trigonometer notation it is a purely algebraic. So, our rotation which is so complex and often it is a difficult to perceive. So, it gets reduced into a very simple format and which we can handle and especially in the controls in the satellite controls this is very useful because using this four parameters it is a beyond our scope to show it in this course that the fourth parameter system it is a free from any kind of similarity.

While the 3 parameter system which you usually write as the Euler's parameter Euler's angle not Euler's parameters the these are the Euler's parameter. Again reminding you this is the Euler parameter. And Euler angles we have written as psi theta phi or writing as theta 3, theta 2, theta 1. So, any set of notation can be use. So, this is three parameter system while this is four parameter system we face while we use this in the rotation we face similarity in many places.

While here if we use this it will be free from the similarity problem and therefore, this is preferable, it can this is also used in certain circumstances (Refer Time: 11:14) whatever you cannot avoid this notation. So, we use it, but we try to ensure that the domain in which we are working there is no similarity involved; if it is involved then we have to tackle some our other ok. So, this way today what we have done, that we have come to the conclusion that institute of using other angles, we can also use Euler parameters.

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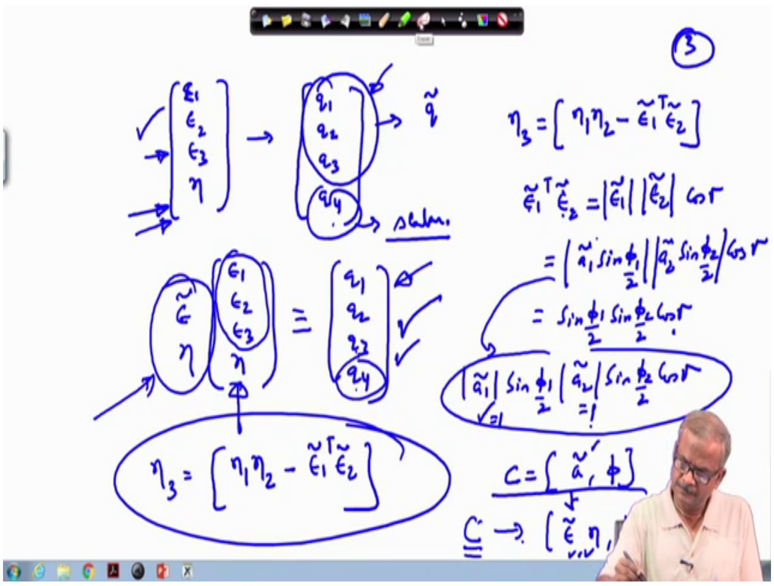

So, Euler parameters epsilon 1, epsilon 2, epsilon 3 and eta instead of using this you can equally you can write in terms of the (Refer Time: 11:59) notation which is written as q 1 q 2 q 3 and q 4 and this part then we will write as q tilde and this is the scalar part. So, we have not followed the quaternion notation, whatever we have done remember here I can reminding you again retelling you that whatever we have done it can be obtained using purely the Hamilton approach which is the quaternion approach, that is the hyper complex number, while we have followed the method of trigonometry and matrix.

So, this is both are equally easy I will not say that one is very easy another one tough, but at certain a stages in the notation that we have followed the expression becomes very complex and derivation especially it becomes very complex, but the matrix notation is always advantages to work with. And therefore, we have a started with this and we will restrict to this rather than going to the working in terms of the quaternion's. However, you can equivalently use this notation; instead of using this equivalently you can use q 1, q 2, q 3 and q 4 ok.

So, of while working with the problems we will use this, while deriving this we have used this notation because, it here writing this as q tilde and q 4 it is a little uncomfortable, here this get separated out this we have written E tilde and this we have written as eta. So, these are the two different notations we have used for this.

While here in this one all are defined by q and this q 4 is especially a stands for this is the scalar part and this is the vector part. So, while working with the problem will prefer to work with this while developing the equation we have used this ok. So, finally, we have the eta 3 which we can write as eta 1, eta 2 if we going the previous page this part ok. So, here we can write this as eta 1 eta 2 minus this part.

So, this part is nothing, but your epsilon tilde 1 times epsilon tilde 2, this is the inner product of these two vectors that will see how it is. So, epsilon 1 tilde transpose epsilon 2 tilde these are the two vectors ok. So, this is the projection of the vector on E 2 or either projection of vector E 2 on E 1 the; obviously, the angle between this let us write this angle is gamma ok.

So, angle between these two vectors it is a gamma. Now, what will be the magnitude of this vector this two vectors? Obviously, we once we are going to develop it we will write in terms of the magnitude. So, this is E 1 tilde magnitude times E 2 tilde magnitude and angle between these two vectors, E 1 is E 1 tilde how much it is that you know this is a 1 tilde sin phi 1 by 2 this magnitude and this is a 2 tilde sin phi 2 divided by 2 times cos gamma. So, if we will look into this and here; obviously, once we have written in this format so, the transpose or things are not required. So, we can take out find phi 1 by 2 sin phi 2 by 2 and times cos gamma.

The magnitude of a 1 and a 2 this is the 1. So, that goes so (Refer Time: 16:04) you need to break it and write it this way a 2 tilde magnitude times sin phi 2 divided by 2 cos gamma. This whole thing can be written in this way and this quantity is equal to 1, this quantity equal to 1 therefore, this gets a (Refer Time: 16:23) so this. So, your eta 3 the composite rotation angle which is the cosine we have expressed in terms of the cosine. So, this is given by eta 1 eta 2 minus epsilon 1 tilde transpose epsilon 2 tilde.

Now, we have till now expressed C in terms of a tilde this we have expressed as a function of a tilde and phi, now we will convert this in terms of epsilon tilde eta and phi. So, if we put here in this format so, we get the rotation matrix in terms of the Euler parameters instead of this a tilde.

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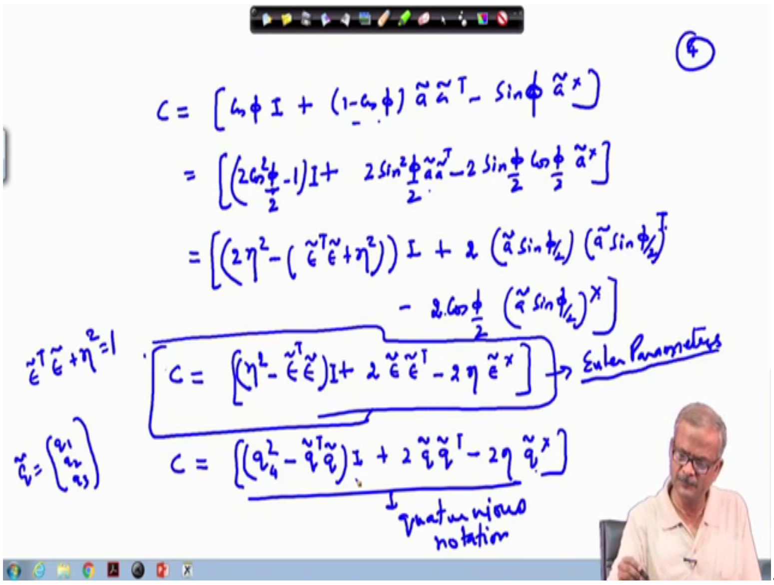

So, C equal to cos phi I plus 1 minus cos phi a tilde a tilde transpose minus sin phi a tilde cross and we need to reset it and write it in a proper way. So, this we can write as see we will use this to write is 2 sin a square phi by 2 we will write it in this format institute of 1 cos this is converted to phi by 2.

Similarly, this one 2 cos a square phi by 2 minus 1 times I and this also you know this will be equal to sin phi by 2 times cos phi by 2 times a tilde cross. And here of course, we are missing the part a tilde times a tilde transpose minus sin here and this can be rearranged in a proper way. So, we know that cos a square phi by 2 we have written cos phi we have written as eta. So, this becomes 2 eta a square minus 1. So, we use this property epsilon tilde transpose epsilon tilde plus eta a square this equal to 1.

So, we will inserted here, now epsilon tilde transpose epsilon tilde and here this can be again rearranged as a tilde sin phi by 2 times a tilde sin phi by 2 transpose minus 2 cos phi by 2 we can replace at this stage or maybe in the next line we will do that. 2 cos phi by two times a tilde sin phi by 2 cross 2 eta square from here we get this as this two will go.

This is eta a square minus epsilon tilde transpose times epsilon tilde plus 2 times this quantity is nothing, but epsilon tilde. So, epsilon tilde times epsilon tilde transpose minus 2 times this quantity is eta. So, we put here as eta and this is epsilon tilde cross and here the we have to put I. Now, this is your rotation matrix in terms of the Euler parameters and it is very useful. So, either you write in terms of Euler parameters or either you write in terms of quaternion notations.

So, if we use the quaternion notations same can be written as here q 2 a square minus this part will write a q tilde. So, q tilde means we are using q 1 q 2 q 3 q tilde this will indicate. So, q tilde transpose q tilde I plus 2 times 1 q tilde times q tilde transpose minus 2 eta times q tilde cross. So, this is in terms of the quaternion notation; quaternion's notation and while this is in terms of the Euler parameters. Both are equivalent, but they are not exactly same as I have earlier also stated, but this notation you can use without any problem either use this or either use this notation your calculation will be free from any or errors, only thing that you have to treat this as a vector ok.

So, once we are dealing with the matrix and the vectors. So, we have to follow their rules. Now, if we break them and merge into a single matrix here they are the different terms ok.

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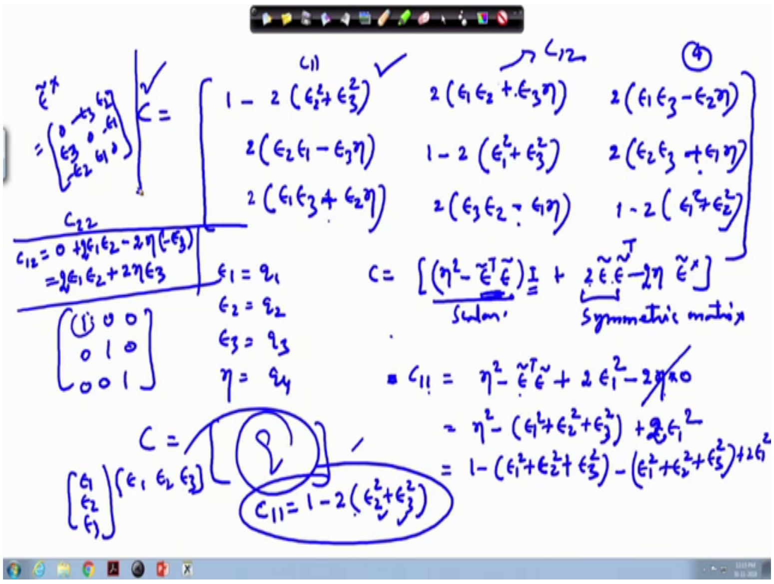

So, if we merge them together so, C will look like 1 minus 2 3 a square, I will show you at least few calculations of this 13 indeed this is true this is plus sin here ok. You can see the minus sin here, then plus sin here, minus sin here plus sin here ok.

So, plus minus here and it is easy to remember it is in a cyclic wave this is the first term. So, here E 1 is absent E 2, E 3 and then E 1 continuous E 1 this is the second term E 1 E 2 and E 3 goes with theta and similarly here the this is the third term. So, E 1 with E 1 it comes the E 3 and the other term then goes to W 2 and eta here only the minus sin as soon as you put it this one is fixed this is the same thing copied here only with the sin changed you have written this.

So, this is also fixed, only thing we have the sin we have to change then this term once we comes to come to this place so 1 minus 2 E 2 is missing here E 1 E 3 this is the 2nd rows 2nd column 2 2 term means this is the C 22. So, C 22 here the second term is missing, you have to fix of this 2, 3 same way epsilon 1 eta here it is plus. So, automatically this gets fixed and this is also the same way. So, here the three is missing E 1 E 2. So, it is a very easy to remember and institute of doing this you can equally you can replace this E 1 by q 1, E 2 by q 2 in this equation e 3 by q 3 and eta by q 4 and you get the same matrix in terms of the quaternion's.

So, I am not going to write in terms of quaternion's whenever required say it can be converted. So, let us check that whatever we have done it is a correct. So, our C matrix in terms of the this notation this is eta a square minus epsilon transpose epsilon times I plus minus eta times epsilon tilde cross.

And two we have missed out perhaps let us see on the previous page, 2 is there so, 2 is here. So, this is two about this is an identity matrix and therefore and this is a symmetric matrix. So, if we try to write this sorry this is scalar one. So, this a inner product is not a symmetric matrix this one here we are missed E times E transpose. So, this is a symmetric matrix; this is symmetric matrix and here this is a scalar quantity and this is identity matrix.

So, if we are start collecting the term. So, let us say I collect the first term C 11. So, how this will look like? This will be eta a square minus epsilon tilde transpose epsilon tilde because the matrix I is 1 1 1 0 0 and this you are multiplying with this quantity which is a scalar term. So, only this term this is related to C 11 component of C 1 2 then the other term which we need to pick up from this place so the this term will be two times. Now, the first term of this will be epsilon 1 x square you just look into this epsilon 1, epsilon 2, epsilon 3 and epsilon 1 epsilon 2 epsilon 3. So, the first term will be epsilon 1 x square.

So, two times epsilon 1 x square minus two times eta times epsilon this is skew symmetric matrix, epsilon tilde cross this is nothing, but your 0 0 0 on the diagonal terms and here the other term will be minus epsilon 3 epsilon 2 minus epsilon 2 epsilon 1 with minus sin epsilon 1 and here epsilon 3. So, the first term here is a diagonal one is 0. So, this you need to multiplied by 0. So, therefore, this goes and what we are left with only this term.

So, eta a square minus epsilon 1 a square plus epsilon 2 a square plus epsilon 3 a square plus 2 epsilon 1 a square. And as we know this eta we can convert in terms of 1 minus epsilon 1 a square plus epsilon 2 a square epsilon 3 a square and here then this particular term epsilon 1 a square 2 a square plus epsilon 3 a square and plus two epsilon 1 a square. So, C 11 then gets reduced to 1 minus two times this quantity and 2 epsilon 1 is there so, two epsilon will get lost and we get mainly this two terms.

So, C 11 is 1 minus 2 epsilon two a square epsilon 3 a square which is what is present here the same way we can look for the order terms. So, here in this matrix if you are looking for the C 12 let us say that you are looking for this is the term which is C 12 this is C 11 ok. So, for C 12 again we look for the high matrix. So, high matrix this quantity is 0 so, it this is scalar gets multiplied by this is 0 so that is 0. So, here we do not get any contribution. So, writing here I do not want to go on the next page because then I will not be able to refer back to this. So, C 12, C 12 the first term is simply 0 ok.

So, this term gets reduced to 0 for C 12 the next term, in the next term if you look into this. So, this is epsilon 1 epsilon 2 so, here we have epsilon 1 epsilon 2. Next we go into the third term which this one; this is minus 2 eta times epsilon cross. So, in the epsilon cross the second term is this. So, minus this will get multiplied by minus 2 eta this is with minus sin here so we have minus 2 eta and then this epsilon cross. So, in the epsilon cross the second term comes with second term we have minus sin, minus epsilon this is epsilon 3 this is the epsilon; epsilon 3 minus epsilon 3.

So, this comes with minus minus epsilon 3 it is ok. So, if we now add it epsilon 1 epsilon 2 and this minus minus gets into plus sign. So, this is 2 and we have missed the 2 term here. So, we need to put the 2 term also for this, for the second term this is getting multiplied by epsilon 1, epsilon 2 and this is getting multiplied by 2. So, we need to put it two here in the place. So, we put a 2 here and then carry this part. So, this is 2 eta times epsilon 3 ok.

So, you can see that this is two times epsilon 1, 2 we can take it outside the bracket. So, this is two times epsilon 1 epsilon 2 times eta times epsilon 3. So, this is what is written here the same way you can expand and you can get this matrix ok. So, finally, we have got this form. So, and this is very useful ok. So, if your Euler parameters are available you can get the C matrix a in this way and so, vice versa if you are matrix is given your C matrix is given and you get back to the Euler parameters so, both can be done.

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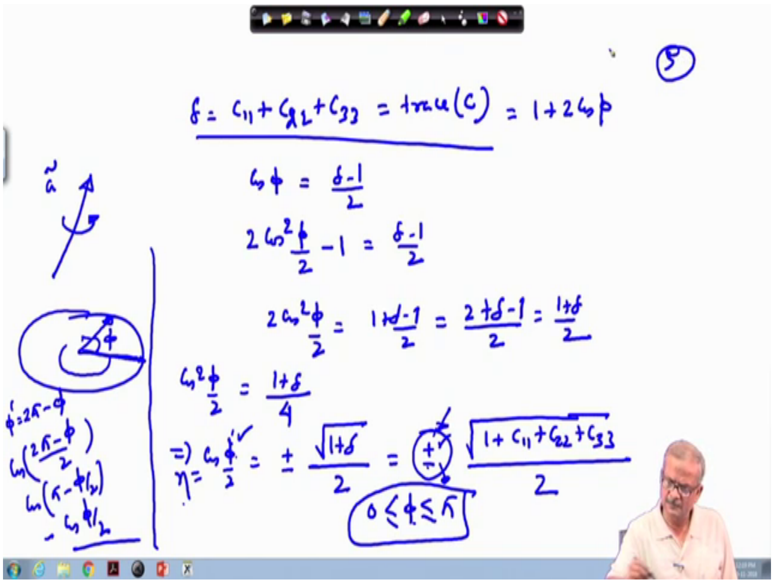

So, already we know that this delta we have written as C 11 this is the trace of the C 2 matrix C 1, C 22 plus C 33, this is a trace of C matrix and this quantity must be equal to 1 plus 2 cos pi. So, from here we get the phi value.

So, this is cos phi equal to delta minus 1 divided by 2 and this we writing has to a square cos a square phi by 2 minus 1 by 2. So, this gives us phi by 2 equal to 1 plus delta minus 1 divided by 2. So, cos a square phi by 2 this equal to 1 plus delta divided by 4 and this implies cos phi by 2 equal to plus minus 1 plus delta under root divided by 2 means plus minus, 1 plus C 11 plus C 22 plus C 33 under root divided by 2. So, if this is known you get the Euler angles from there if this phi the rotation angle and rest what remains, so this is your eta.

So, one of the Euler parameter, now you have to get the epsilon 1, epsilon 2, epsilon 3 and here plus minus sign as we have discussed earlier that if you are choosing this, you choose the sin accordingly that your representation becomes unique ok. So, we will come to this how to look into this again say or maybe we can look at this place itself, we have chosen that if we are rotating about a anticlockwise by phi angle ok.

So, we will choose plus sin means you are doing something like this is your initial position from here you are rotated this vector by phi ok. So, in this position can be achieved from here by giving rotation by 2 pi minus phi and if you insert instead of this here say I write this as phi prime ok so, if this I write as phi prime ok.

So, this will be phi prime equal to 2 pi minus pi, so 2 pi minus phi by 2 and if I take cos of this so this is cos pi minus phi by 2 this is minus cos phi by 2. So, this gives a minus sign. So, these are the you have to choose it in a proper way where to choose plus and where to choose minus because, here we are indicating in general we will choose the plus sign by restricting phi to the range 0 to pi. If you do this so phi by 2 will be restricted to 0 to pi by 2 and then the sin and liberty will not appear. If minus sign is to be taken into accounts so it is depends on the problem we will look for the problem and then accordingly workout.

So, we do not go into those details only through the problems it will be more clear. And the last part what is remaining this is the last part remaining is getting the epsilon 1, epsilon 2, epsilon 3.

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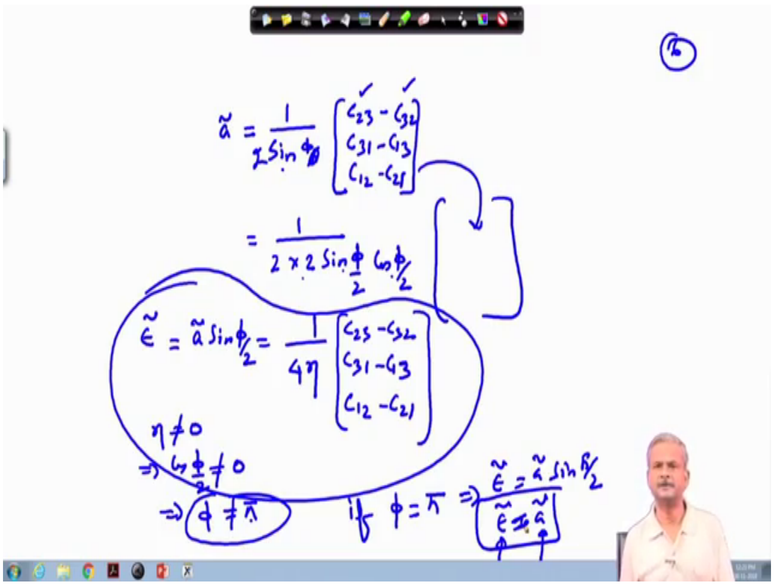

So, we have already we know that a tilde this we can write as 1 by sin phi 2 sin phi and then we have written in terms of the C 1, C 2 etcetera. So, we write that this we have earlier derived ok. So, these quantities are given and from here we have to go back to this. So, this part it can be broken like 2 2 times 2 sin phi by 2 times cos phi by 2 and multiplied by the quantity here. So, what we see that this can be written as 4 cos phi. So, we can write as eta and this sin phi by 2 we can take it here on this side and this we can write as E tilde. So, 1 by 4 eta times the quantity what we have written here C 23 minus C 32, C 31 minus c 13 and C 12 minus C 21. So, this is your epsilon tilde.

So, this way we can go back and forth using these derivations provided this is valid provided eta is not equal to 0. So, this implies that your cos phi by 2 should not be equal to 0 and this implies that phi should not be equal to pi phi by 2 then will be pi by 2 and therefore, the cos will vanish. So, this con this should not be there, if this happens if phi equal to pi. So, this implies that epsilon tilde this will be a tilde sin pi by 2 and epsilon tilde is simply equal to this is the case where the Euler parameter is directly equal to the a vector ok.

So, by taking proper care if you understand the physics properly and if you have the background in the matrix method. So, it is a very easy to do and it can be derived from time to time and if you remember few things then it is a very easy to work with ok. So, we stop here and the next lecture we will go into the rotational kinematics means where the frame is really it is a rotating it is not just oriented, but it is continuously it is orientation it changing with time. So, if it is changing with time then the time factor we have to bring in. So, it may be like one lecture or the next two lectures it may be required, while I planned only 10 lectures for this, but during course of the lecture it has a stretched.

So, what I planned at the next 1 or may be 2 lecture whatever it takes. So, we will finish this rotational kinematics and then we will settle, go into the satellite attitude dynamics. So, that will be our the start this is the basic requirement which is needed for understanding the satellite motion and without this we cannot do. So, we have covered all these things and only part remaining is the rotational kinematics which will cover in the next maybe next two lecture.

Thank you very much for listening, we welcome you to the next lecture.