Satellite Attitude Dynamics and Control
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Lecture – 11
Rotational Kinematics

Welcome to the 11th lecture which is on Rotational Kinematics. So, we start with basically in the rotational kinematics matches database derivative is involved, and how safe you are given certain quantity. And at the present position you know the satellite orientation. If you know the angular velocity of the satellite, so you should be able to predict what will the future orientation of the satellite.

And it's very much required, without this we cannot do the control, because what is the present angular velocity of the satellite and towards what orientation we want to go. So, accordingly the angular velocity has to be modified ok. So, this comes into picture while we deal with the satellite rotational motion ok. So, let us start.

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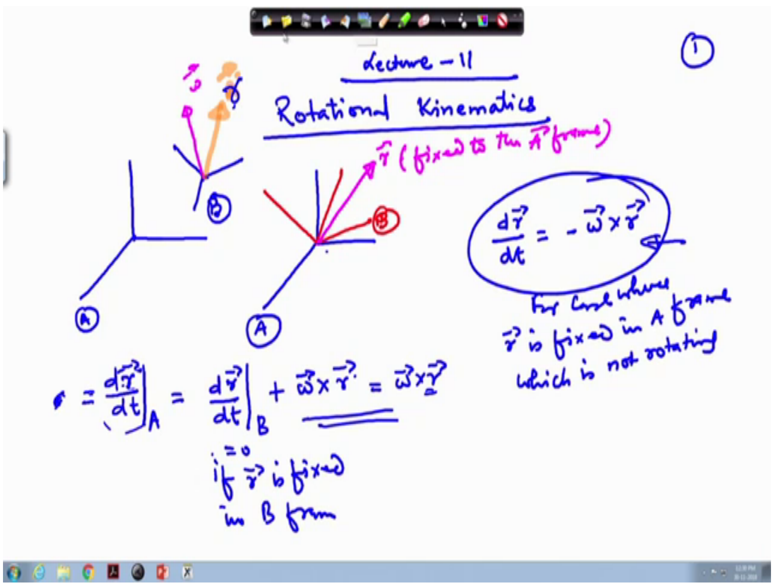

Say I have a frame here which I write as A frame and another frame which is the B frame here. This B frame may be located here at this point the origin of both of them we can locate here and we can show it like, this is the A frame and the B frame it looks like this; this is the B frame. So, A frame has rotated to B position and let us say that we have a vector r which is fix to the through the A frame.

So, later on we will do the derivation its call the transport theorem in mechanics, ok. Say if we have these two frame and this frame is rotating at angular velocity omega ok. So, this is rotating at the angular velocity is omega and let us suppose that there is a vector which is fixed in this frame. This is A vector which is fixed here.

So if this frame is B frame is rotating then, dA by dt with respect to where we will name it as some other thing let us say this is the r vector. So, therefore this dr by dt with respect to A frame this will be written as dr by dt with respect to the B frame plus omega cross r. If this vector r is fixed in B frame, this quantity will be 0 if r is fixed in B frame. So, the rate of change of the r vector as seen from the rotation frame that is frame A will be omega cross r, ok.

So, here if you break it, so you can get the components if you are right. Component there are different ways of doing it, if you are looking here in this body terms itself with this is often called the body frame and this is called a inertial frame. So, whether you are looking in the body frame or the inertial frame components terms, so accordingly we can break it. Now what I want to emphasize here that instead of this vector r being fixed to the B frame it is fixed to the A frame, as shown here in this place.

This r vector remains fixed to the A frame and A frame is not rotating B frame is rotating; means the A frame was there now the A frame has say if another frame B which was coinciding with the A frame which are starts rotating. So, its orientation from A frame, its changes to and goes to this place and which we are referring as the B frame. So, if this happens then in that case we will write as dr by dt this equal to minus omega cross r, here this is for the case where r is fixed in A frame which is not rotating. However, it's a matter of just of perspective if you sit on the B frames.

So, you will see that A frame is rotating if you sit on A frame you will see that the B is rotating. But, once it comes to the dynamics then there is a difference; the rotating frame can never be inertial frame but inertial frame has to be non-rotating. So, in that case there is a proper distinction. Here in this case just be if we are looking for the kinematics either we sit on this or either we sit on this it does not matter. So, for the angular changes it does not matter, but if we are looking for the torque and then the forces so at that time really it matters.

So, this result we are going to obviously, we will derive this how it comes, but its a just opposite of this, here this frame is fixed. So, in the body frame it does not change, so we have set to 0. While here in this case this quantity on the left hand side because its fixed on the A frame; you say the r vector is fixed in the A frame and therefore this will not change with time in the A frame. So, we can set it to 0 and then you can see that dr by dt it will be given by minus omega cross r. So, this is what we have written it. So, depending on the situation we can use it.

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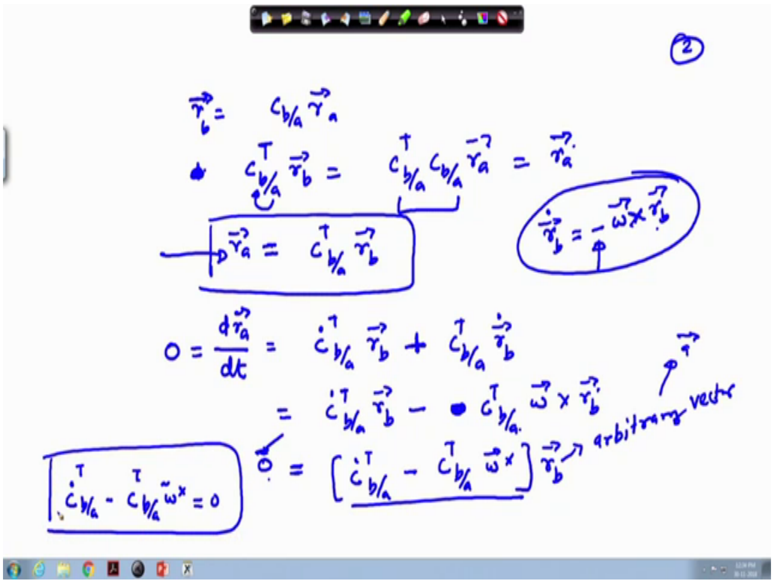

So, let us now write the thing in the previous one in terms of the matrix notation. So this is c b a: now this vector is represented, the components of this vector in the B frame is given by r b. And if we take the derivative of this or we can change do little bit of the changes. So, if we take multiply both side by b a transpose, there are number of ways of working this, another one is in terms of vectors we can do in terms of vectors so but other than indicating this is a vector there you have to show this as the vectors then.

So, if we write it like this multiply both side by c transpose b by a, this is indicating rotation from a to b ok. So, this will be c transpose b slash a. And obviously, we know that this quantity is high and therefore this gets reduced to a and therefore, ra equal to c transpose b slash a r b.

Now, we assume that this vector is fixed in the A frame, and therefore if I take derivative of this vector ok. So, we have to write the derivative of this also c transpose b slash a on the right hand side, and plus c transpose. Because, this is fixed in the A frame so we set it to 0, on the right hand side you have r dot b. Now this quantity has discussed previously; this quantity is minus omega cross r b. Because this vector, your frame is rotating; isn't it? Your frame is rotating and therefore the vector A has been represented like this. Now if we look from the frame B then it will looks like the vector r b ok. And in the because the frame is rotating; so therefore the components of the vector r b it keeps changes and therefore you get this r dot b here.

And why this minus sign is appearing here; that I have already explained you. So, put it here in this place b slash a r b and then minus c transpose b a omega cross r b. You have taken r b outside, this equal to 0. Now you see that, this is an arbitrary vector: arbitrary vector because it corresponds to a. So, this is nonzero, and if the left hand side is 0 and therefore this is the null vector is appearing here just like because we have set it to 0. So, I have not put here tilde or any other thing, but this is indicating a null vector ok.

So therefore, this indicates that c transpose b slash a minus c transpose b a omega cross; now here we can put it in matrix notation this must be equal to 0.

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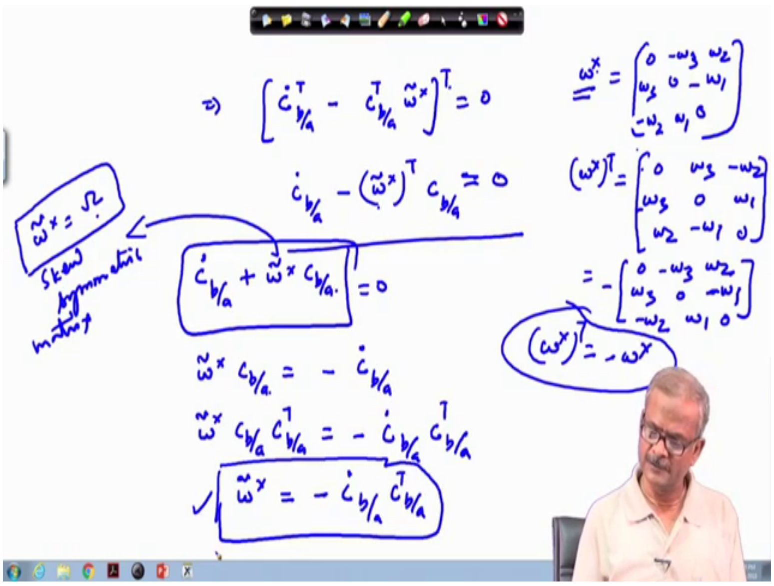

So, this implies if we take the transpose of these whole things. So, c dot transpose b slash a minus c transpose b slash a omega tilde cross take transpose of this, this must be equal to 0. And if we take the transpose, so this gets reduced c dot b slash a minus omega tilde cross transpose times c b slash a; this equal to 0. Now this quantity you know this is the skew symmetric matrix: 0 0 0, minus omega 3 omega 2 omega 1 with minus sign here ok. So, if we keep the transpose of this matrix, this will come here in this place and then and take minus sign outside. So this is here, do not get confused this is here omega minus omega to, here again we have to correct in this place. This is minus omega 2 and this is omega 1.

So, if we take the transpose of this, this is omega 0 minus omega 3 omega 2, then omega 3 0 minus omega 1, minus omega 2 omega 1 and 0. And if we take the minus sign out of this; so, this gives you minus omega 3 omega 2, omega 3 is 0 minus omega 1, and then minus omega 2. So, this is nothing but your minus omega cross. So, omega cross transpose this quantity is equal to this quantity. ok.

Therefore, we can replace this by b slash a plus this minus minus sign that makes it plus omega tilde cross c b slash a, and this equal to 0. Other way we can write this as omega tilde c b slash a this equal to minus c dot B a, and then operate on both side by c b a transpose. So, this quantity is high, so that gives you omega cross.

if you remember this quantity often we have written as omega tilde cross equal to capital omega. It is the basically skew symmetric matrix; when this relation is very useful while working with the attitude kinematics. So, the same thing can be worked out using the vectrix notation; what we have developed earlier. Perhaps, in the very first lecture we have discussed that vectrix, but we are not going to work using that because the time is limited.

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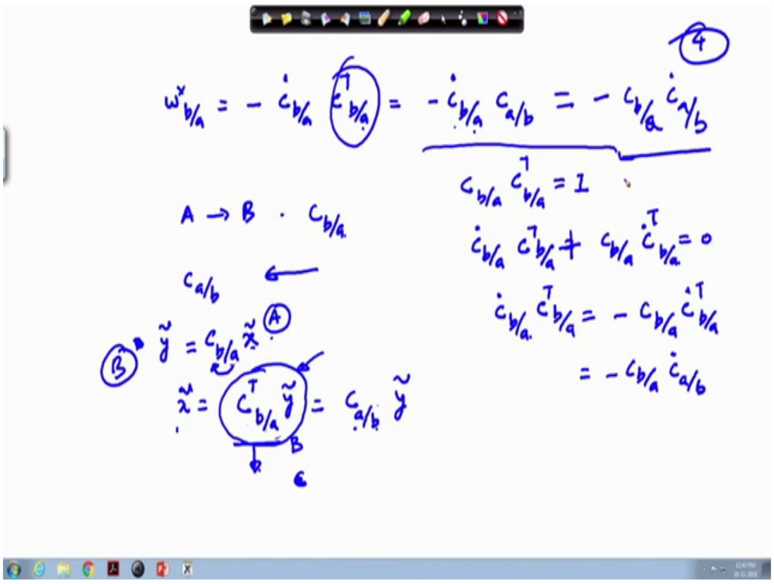

So, once we have got this omega cross b slash a minus c dot. This relationship will be very useful while working, so we will use it quite often whenever require. Now look into this part; this transpose B a ok. So, as you know that going from frame A to frame B we are writing this as c b slash a; and if we go back from this frame to this place. So, we are writing this as c a plus b; means any vector let us say we have A vector x **x** tilde which we are operating by x plus b ok. So, this will be indicated by y tilde, while if we write in the same way here. So, this will be c a plus b we are operating by transpose in. So, this x tilde will be equal to c a transpose c a slash B times y tilde.

Sorry, c operate on both side twice c transpose a b. So, if we operate this will be x tilde and here we will have c transpose a slash b y tilde. And this quantity is nothing but. So, we can write this as c b slash a times y tilde; sorry what we have done b slash a a to b here we have written that; let me correct it I have used the wrong rotation here. So, we are going from A frame to B frame. So, this is in A frame and here this is in B frame so we have written this as A frame to B frame.

This is what the notation we are following. Now if we write it this way, multiplying on both sides so here also we need to do the correction, this notational problem we have to keep tracking so this will be b slash a transpose ok. And this is nothing but; so this is a matrix which takes you from A frame to B frame. And therefore, if you look into this relation, this converts from B frame to A frame. And therefore here we change it and we write this as a b, and see you are converting it from B frame to the A frame.

So therefore, this we can convert as this is a to b frame and this one we will write as b to a frame ok. Now if you remember that c b slash a times c transpose b slash a this is equal to I. And if we differentiate this, this will plus c b slash a times c dot transpose b slash a this will be equal to 0. So that means, c dot b slash a transpose c transpose b slash a this will be equal to minus c transpose b slash a. And this is nothing but your minus c b slash a time c dot a b. So, here then you can write this as minus c b a times ca dot a slash B.

Now, these relations are very useful, this may look trivial, but while working in some of the problems you may require this kind of tricks. So, once we have done this, this can be used to prove that the angular velocity they add vectorially, but we will set this problem as a tutorial problem rather than discussing this in class for the in the view of the limited time available for lectures ok.

So, next we look into conversion from one frame to another frame. Now for the angular velocity vector also needs to be converted frame from either the Euler notation to the other parameter notations just like we have used epsilon tilde phi.

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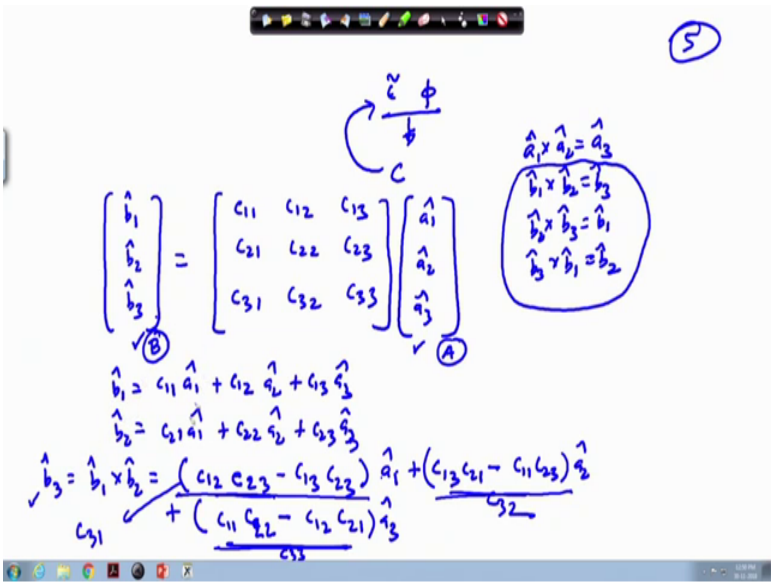

So, you may need to convert from this place to in terms of the c matrix or either go from this place to this place; back and forth of a while, working with the problem this is quite often required. So, c of the things which may be useful while working, I will list it here say that b 1 cap if we have this vectrix b 3 cap, this is the basis vector; c 12, c 32 c 33. And this we are getting from this vectrix ok. So, this is the basis vector, this is another, vector this is in frame A and this is in frame B. So, this is vector for frame B basis vector for frame a.

Now, also we know that a 1 cross a 2 this equal to a 3, b 1 cross b 2 equal to b 3, similarly b 2 cross b 3 equal to b 1, and b 3 cross b 1 equal to b 2. So, this relationship you are aware of because it is a right hand triad ok. So, this can be used adventurously to certain some of the entries here. Let us look into this.

So, we have b 1 cap equal to c 11 a 1 cap plus c 12 a 2 cap plus c 13 a 3 cap. Similarly b 2 cap is equal to c 21 a 1 cap a 3 cap ok. Now if we keep the cross product b 1 cross b 2 this will be equal to b 3. So, if we take the cross product of this. So this will get simplified as c 12 times c 23 minus c 13; you can just check it I am writing it first because there is no point in expanding multiplying and writing here: c 21 minus c 11 c 23 a 2 cap c 22 minus c 12 times c 21 a 3 cap.

So, this way you can form all the things ok. So, if you try to look into this you will get this kind of terms from one place to another place. So, here these are in terms of a 1 a 2 a 3. So, what we have got here? This is described in terms of a 1 a 2 a 3, similarly you can also describe here in terms of b 1 b 2 b 3 rather than writing in terms of a 1 a 2 a 3. And if you do this, so we get some very useful results.

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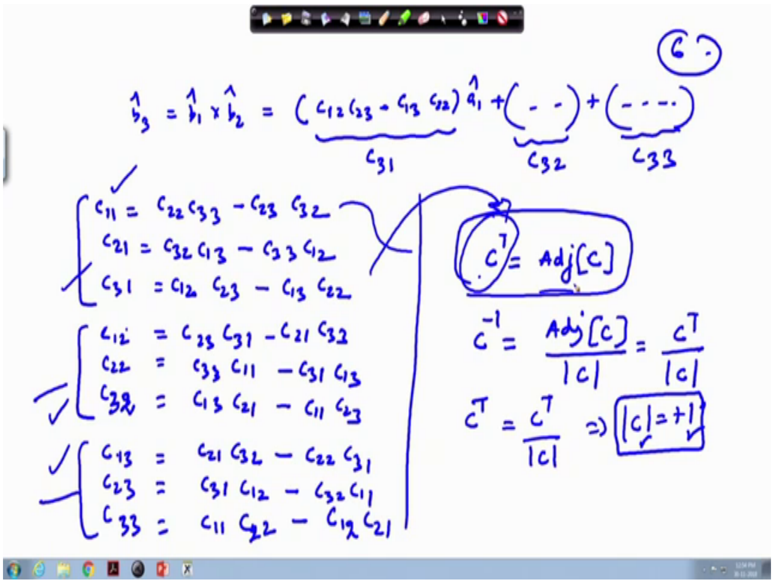

So, from the previous page b 3 cap this we have written as b 1 cap times b 2 cap, then c 12 c 23 minus a 1 cap plus other two terms ok. So, what it gives? So, if you look into this, this whole thing. So, this is basically a conversion from A frame to B frame: c 11 c 22. So, this is indicating your a 1 is here, a 2 is here, and a 3 is here a 1 a 2 and a 3 here: a 1 a 2 and a 3.

And on the left hand side this is your b 3, so b 3 is here. So, where and he will look for this particular part. So, this indicates that this must be this term I want to write it on the next page. This is the first term and then the second term, so this term this must be c 31. Similarly this must be the c 32 term, and this must be the c 33 term the last one So, this is your c 33 term, this is c 32 term and this is c 31 term.

So what we see that, this b is already written here in terms of c 31 c 32 c three c 33. And using this relation we are also getting like this. So, that indicates that c 31 is equal to this, c 32 is equal to this, and c 33 is equal to this. So that means, all the nine components they are not independent ok. So, if you get any six the other three you can get using this relationship.

So, this says that if you form a matrix like this you will get something like this c 22 c 33 minus. So this is one set, similarly we will get the second set and the third set c 11 c 21. And similarly here you get c 12 c 22 and c 32. And here you will get c 13 c 23 and c 33.

So, if you get these two sets directly then you can use these values here in this place, because you can c 12 is present here c 23 is present here.

So, same way the other are present on the right hand side. So, using this six you can calculate this quantity is here, and therefore you can get this ok. So, if this shows that they are not just independent, moreover we can write something more here. Let me complete this to c 33 c 11 minus c 22 minus c 12 times c 21.

Now, using this six you will be able to work out this. Another part which is interesting to look into this; that c transpose the c matrix which is forming here this is nothing but head joint c the way it has been written c 11. So, the first element of the c matrix which is c 22's c 33 minus c 23 times c 32. So, this can be this whole left hand side if you write it in the terms of the matrix ok. So, this is nothing but the c transpose, this is the first column, this is the second column, this is the third column. So, using this it can be written here in this way. And we are aware that c inverse also it's written as ad joint c divided by determinant c.

So, from this place this indicates that this is equal to c transpose divided by c determinant ok. And because c is a rotation matrix and therefore this implies that this quantity will be c transpose. So, this is equal to c transpose divided by c, and this implies c determinant equal to plus 1. So, this is another proof that the determinant of the rotation matrix will be plus 1, which comes from here. And it is not difficult to just look yourself, because you otherwise we lose a lot of time and discussing all these thing; its a quite simple to look that these quantity represented here, it can be written in this format.

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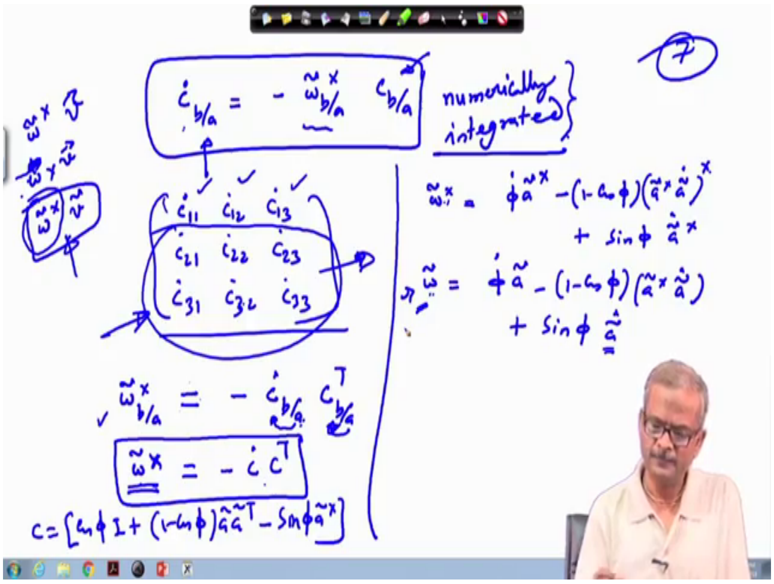

So, once we have got the rotation matrix and c dot b slash a. So therefore, we have written this as minus omega tilde b slash a and this cross times c b a. So, if you look into this matrix at any instant these matrix entries are available to you matrix elements. If omega is you know the omega at a particular instant of time; so, you can get the matrix element at the next instant of time by numerically integrating it. So, this needs to be numerically integrated. This cannot be solved analytically only for few special cases some analytical solution may just, otherwise this has to be done numerically.

So, again if you look from this place here you have different elements, like you will have c 11 dot c 12 dot c 13 dot ok; similarly, c 21 dot 31 dot c 32 dot and c 33 dot. So, this way if you look correspondingly you will have on the right hand side when then used numerical integration method just to integrate it. Now it so happens that if you have determine this 6 ok. So, these are the updated value of the, if you integrate it for this 6 ok.

So, you get the updated value of c 21 c 22 c 23 like c 31 c 32 and c 33. And if this six are available then it is not required that you again integrate these three also rather than doing that you can just use this relationship, because these are the updated value now, so this will also indicate the updated value. Because this c matrix all the entries are not independent. Therefore, getting the derivative of c integrating it we get the updated value of this insert it here, in this place on the right hand side, so you get the updated value of the c 1 c 2; all this things c 11 c 21 c 31. So, rather than integrating for this one also you can directly solve from the previous one.

So, for these are the some of the technologies while you work with the attitude dynamics you may face and you have to you can solve these problems accordingly. So, if this against going back we have indicated as omega tilde b cross a and horizon we can write it like this; equal to c dot b slash a times c b a transpose this we are doing just by multiplying it by c b a transpose on both the side and writing it and with a minus sign here.

In general if you just want to; I do not want to carry this so I will write it in this format so that this gets simplified. So, this just indicates transformation from one frame to another frame for the angular velocity. Sorry, these are the matrix in matrixes which are indicating, this one indicates transformation from A frame to B frame and this is for the A frame to B frame, but the rate of change of the elements of those matrices. And from here the angular velocity can be obtained and vice versa. If the angular velocity is given you can also get to this point, getting the elements of the c matrix.

For that we need to use certain identities but we will skip at those things at this. So, omega tilde cross if we expand it and work it out because c if we know that c is can be written in terms of cos phi I plus 1 minus cos phi a tilde a tilde transpose minus sin phi a tilde cross ok. So from here, we can write c dot in terms phi dot and a dot.

And if we insert here in this place which will again we will take this as a tutorial problem. So, if we insert it, we get this omega cross. So, omega cross then it turns out to be phi dot a tilde a tilde dot cross plus sin phi times a tilde dot cross, here also the cross is there. So, this is a vector omega tilde is basically a vector and this cross means if you are rotating those by this vectors. So, you can write this omega tilde cross v. Or otherwise if you are writing in matrix notations, you can write it like this. So, if you are using this notation this becomes a matrix, if you are using this notation here put it a arrow. So, if you are using these notations, so this becomes a vector.

So, if we use the vector notation and or either the matrix notations of this can be eliminated from both the sides. And this can be written as y tilde a tilde minus 1 minus cross phi a tilde cross and a tilde dot plus; this cross is eliminated from all the places.

And then this can be used to get back; it can be solved for a dot, but for that you need to do certain operation on this vector ok.

We will continue in the next lecture this part itself, and finish this. So hopefully this the next lecture is the last lecture.

Thank you very much.