Satellite Attitude Dynamics and Control
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Lecture – 16
Rotational Dynamics – Rigid Body Dynamics (Contd)

Welcome to the 16 lecture. We have been the discussing about the Rotational Dynamics. So, we will continue with that.

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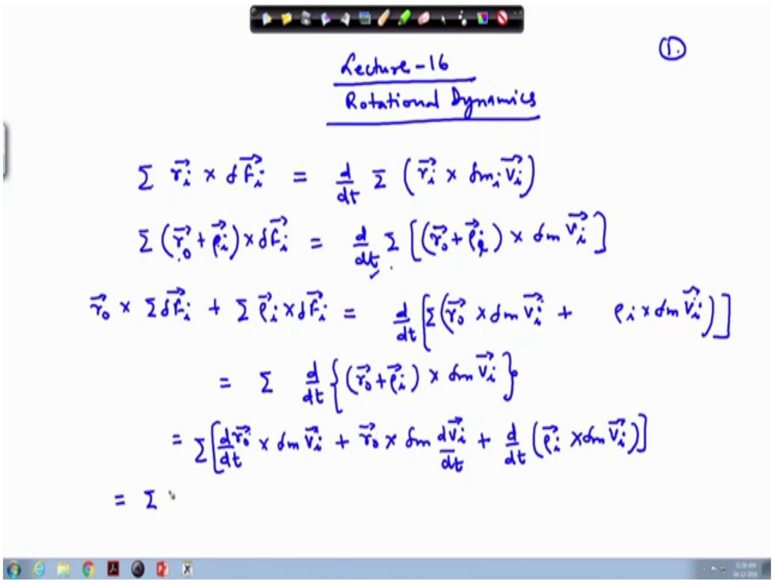

So, already we have derived this equation last time ok. So, here in this equation we first expanded so this is not depending on the summation operator. Therefore, we can take it outside and write this as r 0 cross delta F i plus ok. So, if this rho i, I cannot take it outside the summation sign because as we saw in this graph this figure that this rho i it is a refer to this point and this is part of this body ok.

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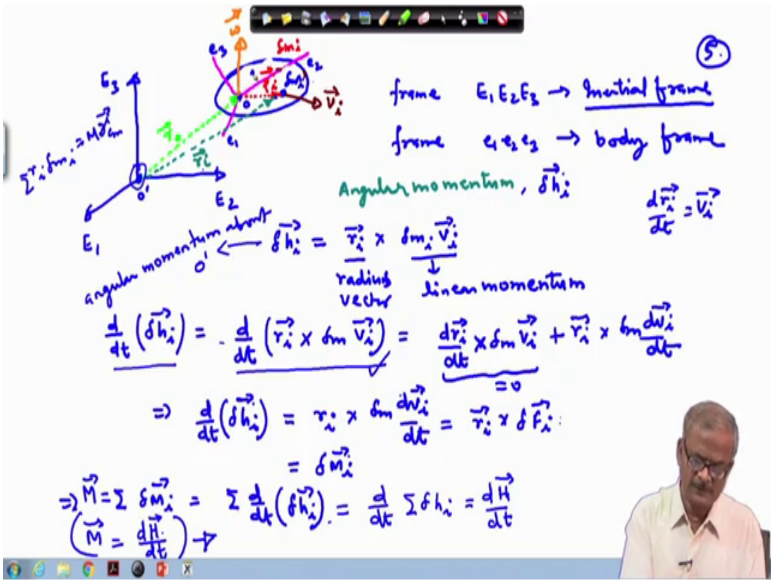

So, here and this r 0 is a fixed point here while we have to sum over the body means this point will be varying. There will be number of points in this body which over which we will take the summation, but this is only one point and it this does not depend on the summation. Therefore, we have taken it outside the summation sign. On the right hand side, in the same way we can write this as d by dt r 0 cross delta m V i plus and there is a summation sign here or either we say that we can take the differentiations inside and then work with it that will be much better.

So, if we look into this particular expression, we can take this is outside and write this as d by dt r 0 plus rho i where summation extends over all the points in the body. So, there therefore, this can be reduce to, we differentiate this first d r 0 by dt cross delta m V i and plus r 0 cross delta m d V i by dt. And the then taking this term we write this term as d by dt rho i cross delta m V i and then we have to do the summation over all the points ok. So, of this is now we expand we work on this particular one. So, this quantity is here nothing, but so we will remove this part.

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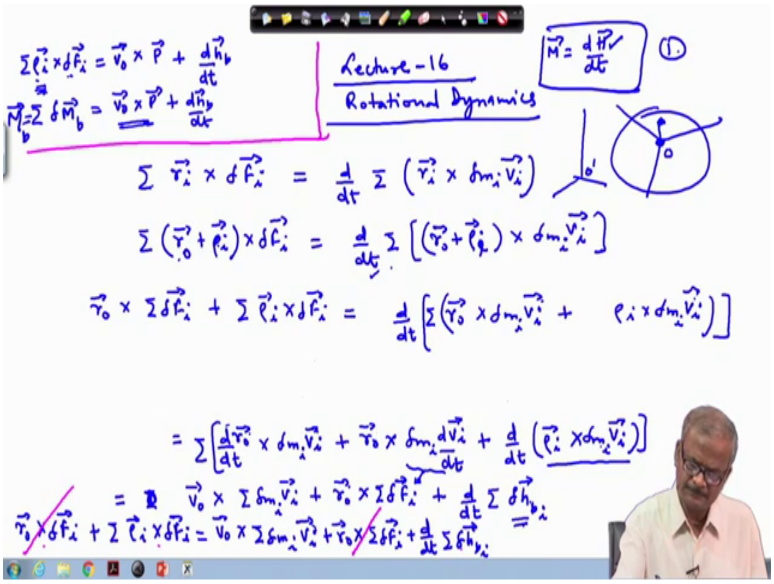

So, what we have what we have done that we have expanded this here in this place. So, taking those a differentiation sign inside and taking the summation side outside. Now taking this particular equation so, this can be written as because again this is not depending on the summation so, we can take it outside. We can write this as the V 0 dr 0 by dt equal to 0 cross delta m V i plus similarly here in this place r 0 cross summation delta m. And, this quantity as we see this quantity is nothing, but delta m times the acceleration of that particle.

This is mi and these are all m i, we are summing over the particle. Here we are not using the integration sign rather we are working with the summation sign. So, the quantity which is present here, this quantity will write as delta F i. So, this is delta F i. And obviously, we have to write the summation also. So, we need to put the summation sign before this. So, taking this summation here this will be out of summation and then cross which is not depending on the summation and this quantity we have written as this one. So, this is delta F i and plus now look here into this quantity what this quantity is rho i time delta m times V i.

So, going back into the previous lecture so, this is your rho i from this place to this place this is rho i and rho i times now we have delta m times V i. So, delta m is the mass of this and V i. So, this is the linear momentum. So, this distance time is the linear momentum that gives me the angular momentum about which point about this point O. So, this quantity can be written as d by dt and; obviously, the summation sign we have to take into account.

So, summation we can write this as delta h b i; b stands for that we are dealing with the body frame. This is with respect to the body frame or either point O ok. There is a difference the here in this expression just look at this and the expression that we derived here. This h is about this point about point O prime, this capital H ok. This is the angular momentum about O prime while h b, this is the angular momentum about O about this point ok. So, this is the difference and you must be very careful about this

So, what we get from this place that this quantity can be written as r 0 cross delta F i; this equal to plus d by dt summation delta h V i. Now if we look into this particular equation so, this term and this term here this term and this term both are same. So, this term and this term they will cancel out and what remains is this particular term on the left hand side and this two terms on the right hand side.

So, of I will write it for convenience here in this place itself. So, if you look here in this point, this is rho i cross delta F i. This is the summation and this is equal to V 0 cross delta m times V i. This is the summation over the whole body, this is the linear momentum of one particle. So, summation over all the particles this will be linear momentum of the whole body ok.

So, linear momentum of the whole body; this is the p and we will come to this further so, what does this mean. And then we have the last term here, this is d by dt and summation we can write this as h b ok. So, you can see the difference that we have developed earlier and here earlier we have written m equal to d H by d t where H was about the origin of the inertial frame. Here in this case this is about the origin of the body frame and what this quantity is rho i times delta F i.

So, rho i times delta F i now this quantity is the torque, here this is related to rho i remember. Going back into the figure, this is the rho i and here the first delta F is acting on this ok. It is in the same direction to let us say that this is the V direction and we changes in the next instant here in this direction. So, F will be given by this change final V minus initial V ok.

So, V direct F direction for simplicity, I am not showing it here in this place. So, here this is your torque equation. So, we can write this as delta m and remember this is about the point O of the body frame. This rho i is measured from the origin of the body frame therefore, we will write here delta M b ok. So, this is quantity delta M b and on the right hand side this is V 0 P plus dh b by dt and this we can sum and write here as M b ok.

So, now it becomes easy to look into that if I have a frame here another frame fix to this point and if I have to write the say the torque; basically if I need to calculate the torque about this point which is we have written as o, this point we have written as O prime. So, I will calculate the torque about this point ok. What is the torque acting on this point O for this body and in the right hand side we will write this expression. So, this is completing my discussion for this angular momentum competition and then the torque competition. But something more we can expand this and have a look into this also. So, we have next.

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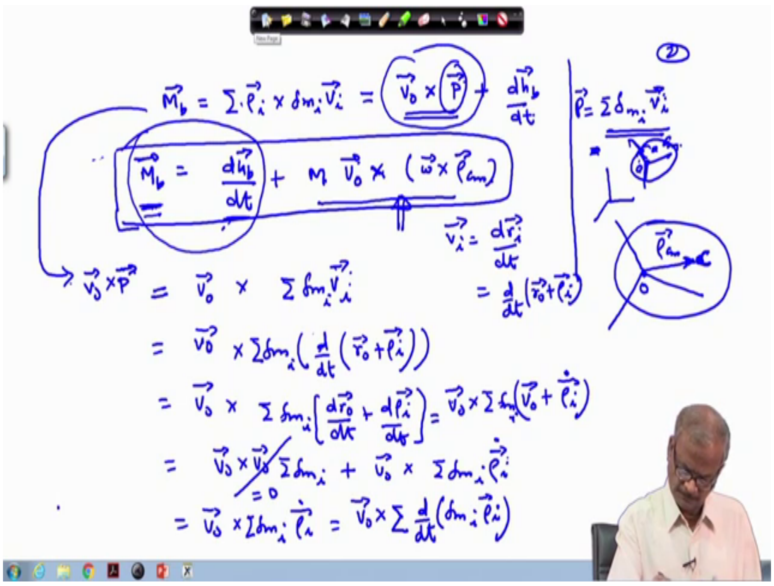

So, what we have got? This is M b equal to rho i cross delta mi times V i summation over this. This is nothing, but equal to V 0 cross P. This is the linear momentum of the body and plus dh b by dt. This particular term can be further expanded. So, we have V 0 cross P. Now the quantity P that we have written, this is delta m i times V i delta mi times V i and summation over the whole body, this is what we have written.

So, now P we rewrite this whole thing and we will get a very interesting equation which will look like this finally. So, of let me write that first that equation first which we are going to derive. So, this will V M b dh b by dt which is this term and plus we will get one term as M is the mass of the body V o cross omega cross rho cm. This is the final form we are going to get and from where we can get the number of insides.

So, of before looking into mathematics, we discuss little bit of the physics what is important here. So, as you understand from our previous discussion that if you are point O which is at which the body frame is fixed ok. If this point is not the centre of mass so, what you generally know that M b equal to this quantity dh b by dt this is what you have been writing all along your education till now. So, there is a in addition one term is there and this term here rho cm appears. So, that implies that if I have this is the inertial frame and there is the body frame and if point O itself coincide with the centre of mass, let us say this point is this is centre of mass; this is centre of mass and here if this point is your origin of the body frame.

In this body this is suppose the centre of mass and here this is of the body frame and this is a rho cm. So, if rho cm is 0, then this term drops out and this is what you get exactly. So, until unless your point, this point O and we will call this is the point C which is the centre of mass. So, until unless your point O coincide with the centre of mass you cannot write this equation ok. So, this equation will be in error, if you try to shift those point O away from the centre of mass and use this equation ok.

We need an extra term to complete this that if my point O is away from the centre of mass so, this extra term must be added and then the whole calculation should be done ok. Now going back again to this point so, this is the basic physics involved in this must be you take care of. So, M b equal to V o cross summation this particular part we are working with. Now this V i can be written as d by dt basically your V i is dr i by dt ok. So, and ri is nothing, but r 0 plus rho i. So, we need to take the derivative of this.

And if we differentiate this so, this gets reduce to dr 0 by dt plus and this can be written as V 0 cross delta m and here then look into this part this is V 0 plus rho i dot and then break the bracket. So, V 0 is a quantity which is not depending on the summation sign. Therefore, this can be taken outside and the first term then you will have here delta m i this is delta m i and the other term V 0 cross delta m i times rho i dot. Now this term gets reduce to 0 because it is other cross product of the same vector and therefore, this is not M b here, again we should modify this where this is the only one part we are writing. This part this is V 0 cross P. So, of this particular part then gets reduce to V 0 cross delta m i summation rho i dot and this mass is constant ok. Therefore, we can put here in a special form, we can write this as d by dt delta m i times rho i.

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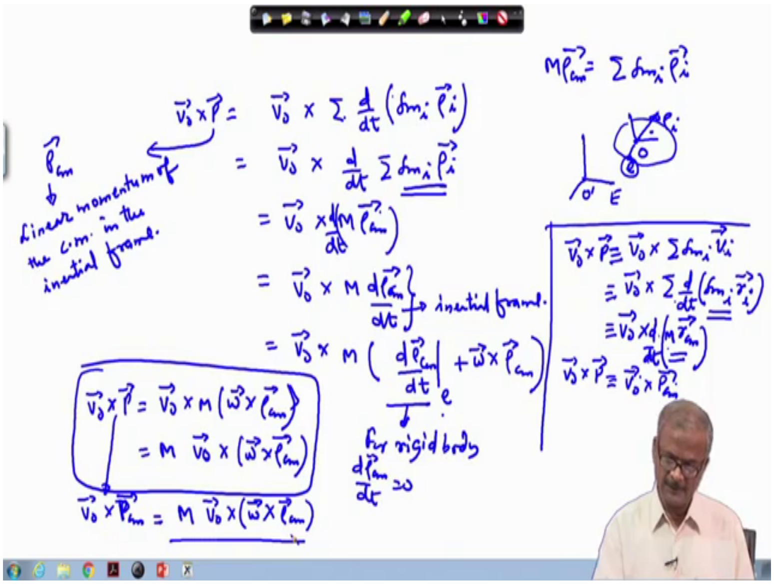

So, we have V 0 cross P this is equal to V 0 cross summation d by dt summation d by dt and delta m i times rho i. Exchange this operator and you know that this quantity is nothing, but the location of the centre of mass. It is relate to the centre of mass. So, this is M times rho c m and then we have d by dt. So this gets reduce to V 0 cross M times d rho cm by dt.

So, V 0 cross m and this is with respect to now remember that we are taking about the point this rho i is about the point O. This is the inertial frame, this is the body frame ok. This is rho i has been taken about this point O, but a still this describes in the inertial frame ok. And if we have rigid body then for that case say if for anybody, we can write it something like this d rho cm by dt with respect to the body frame which we have indicated by e plus omega cross rho c m. This already we have discuss, this is the transport theorem in mechanics.

So, if you have the rigid body, for rigid body this quantity is 0 d rho cm by dt. This will be 0 means the centre of mass location does not change with time with respect to this point O this what it the states. So, this point goes to 0 and here therefore, V 0 cross P this becomes V 0 cross M times omega cross rho c or we can write this as V 0 cross omega cross rho c. So, this is the particular expansion that we wanted to derive and we have done it here ok.

This particular part V 0 cross delta M times V i, this also we will discuss or. So, if which we have written as V V 0 times P. So, we can discuss it here itself V 0 cross P, we have written as V 0 cross summation delta mi times V i. So, this we can write as d by dt delta mi times ri because V i is nothing, but ri here ri dri by dt and delta m is constant. So, therefore, we have written it like this. So, what this quantity is V 0 this is this quantity is nothing, but the M times rcm d by dt ok, this particular quantity ok.

If we take the summation inside and this differential operator outside so, we take it outside. So, the summation inside indicates M times rcm where rcm rcm is the say the centre of mass if it is located here. Let us say here in this figure, it is located here. So, rcm is measured from this place to this place from here to here because ri is measured from here. So, ri times delta mi summation this will indicate your M times rcm where rcm is measured from this point. So, somewhere here your centre of mass is located.

So, therefore, this V 0 cross P that we expanded here. This is the same thing here written this should you can write here this is identically something like this. So, V 0 cross and what this quantity is, this is P cm. So, the P which is appearing here, this is nothing, but this is P cm that is the linear momentum of the centre of mass in the inertial frame. This is the linear momentum of the centre of mass in the inertial frame. So, we get rid off by doing this what we have done that V 0 cross P cm ok. This can be written like this gets rid this is P cm remember, this is the P cm this can be written using this.

So, we get rid of the reference to any reference to the inertial reference frame and that is the great that is of great help because if you have to all the time keep measuring this distance. It will be very troublesome say in the case of the year craft, we do not have to measure the distance from this point to this point; if this is an aircraft it’s a where so it is. So, convenient and therefore, our final equation this gets reduce to we will use this part, this part and together with this part. So, this is our final conclusion M 0 V times omega cross rho cm.

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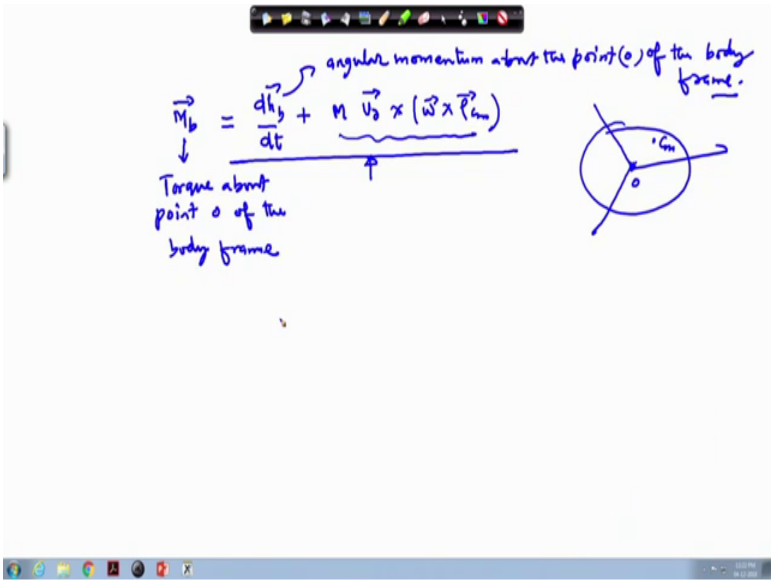

So, the final conclusion what we get is dh b by dt plus M times V 0 cross omega cross rho cm and this quantity is nothing, but equal to M b where M b is the torque about the body frame. So, this is the torque about point O of the body frame and h b, this is the angular momentum about the point O of the body frame ok.

And this equation whenever you are dealing with a system where the your reference point is not coinciding with the, this point O is not coinciding with the centre of mass centre of mass is located somewhere here. Then you must add this extra term to your equation of motion or the equation of the equation of rotational motion in order to get correct answer ok.

So, thank you very much and we will continue again in the next lecture.