Satellite Attitude Dynamics and Control
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Lecture – 18
Rigid Body Dynamics

Welcome to the 18th lecture, we are started last time with Newton's law and then went to the Rigid Body Dynamics; we will continue with that.

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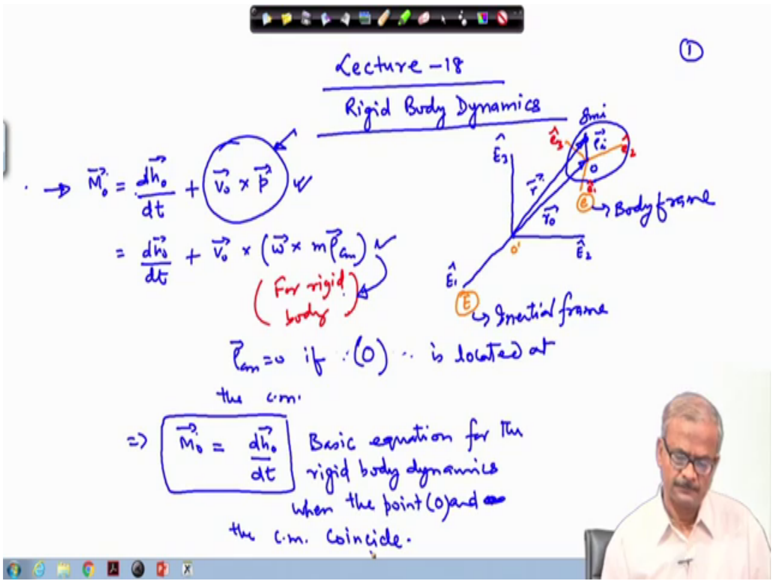

So, if you remember that last time we have derived this relationship. So, of and we saw that the extra term appears if we do not choose the point of reference in the body frame or the body frame if we are, if origin of the body frame is not located at the centre of mass then this extra term it appears.

And this must be added otherwise the whole calculation will be wrong. However, in most of the cases the origin of the this, origin of the body frame it lies at the centre of mass. And, in that case as this can be seen that for the rigid body we reduce it to this equation; this is for the rigid body and if the origin of the body in this body frame which is indicated by e, this is body frame and this we are taking as the inertial frame.

So, if the origin of the body frame if it is located at the centre of mass then rho cm will be 0. So, rho cm this will be equal to 0 if the centre of mass is located at by or if origin of the body frame which we are writing as O; if O is located at the centre of mass. And then this implies that M 0 or perhaps M 0 or M b we have written for this case. So, here M b it refers to the moment about in this point or either we write in terms of M O and M O prime is the moment about this point.

So, this gets reduce to dh 0 by dt which forms our basic equation for the rigid body dynamics when the point O, when the point O and the centre of mass coincide ok. So, there are various cases we can receive of the topic we are presenting here the same thing can be presented in numerous ways and it differs from author to author. However, I will try here to present it in the simplest way so, that you do not face any difficulty.

Many of the books the material presented their it's a very difficult to write away it get into, but in this case I will keep the notations and everything very simple. And, it will definitely get into your mind at the least effort, with the least effort you can get it. So, we continue with this.

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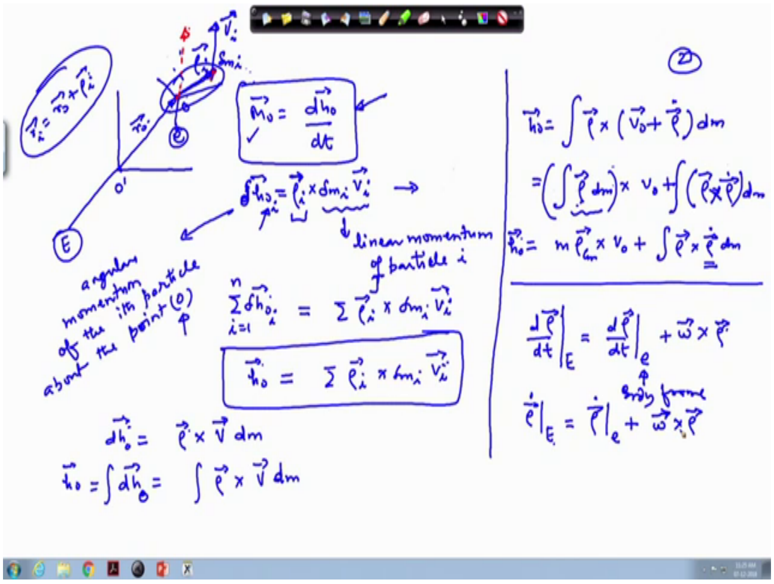

So, as we had in the previous figure if in the previous figure we have taken this. So, using this figure M 0 we are writing as dh 0 by dt. Now if we put here the h 0 then we will get this M 0. Now, h 0 now if this is the point O and this is the point we have written as O prime, this is the e frame and this is the capital E frame inertial frame.

Any point we have indicated this by rho i and these distance from here to here by r 0 and then the h 0 we have written as delta m i which is the mass times v i. So, this constitutes the, this is your linear momentum of particle i and this is h 0. So, we are taking here in this case already we have taken the moment about this point. We took the angular momentum with respect to this point and then we have reduce to this form ok. So, we need not go back to this point again now, we can proceed with this. So, here we have to write this h 0 is about this point.

So, we need to write it this h 0 you can refer back to the previous lectures. So, this h 0 is rho i times delta m i times v i, where this is the distance here your delta m i the i th particle of mass in delta m i is located ok. Thus and maybe this you can write as delta h 0 to indicate this is for the small particle and if you integrate it or sum it over. So, in the differential notation the same thing will be written by because we are not differentiating it, it is just one particle system of ok.

So, we will continue with the summation notation that will be the most easiest one to follow here. So, delta h 0 and taking the summation over all the particles and this is for the i th particle so, we need to put here the tag i also. So, this is the angular momentum of the its particle, of the i th particle about the point O. Here we have to be careful because, we have eliminated this O prime already and we are working in terms of this point.

And if we integrate it so, we get this as the this if this sum it up. So, we get this quantity here h 0 equal to rho i; if we write the same thing in the differential notation purely in differential notation. So, of the same thing we write here as rho, rho is the distance of the particle where delta m instead of writing here delta m will write in terms of dm. So, rho cross v dm where v is the velocity of this i th particle we have taken here.

So, this is v i ok. So, instead of v i we will write it as v and then we need to integrate it and this gives us h 0 because we have taken about the point O. And further this can be expanded as we know that this velocity of the point O will be v o plus rho dot and integrate it over all the points. And this can be broken then, we can write little bit better notation rho dm cross v 0 because, v 0 is not dependent on the mass integration. We are basically integrating it over the whole body.

So, this integration is with respect to the mass. So, this can be taken out or either we can write it like this. So, the other point this will be rho cross rho dot. So, you know this quantity this is nothing, but the location of the centre of mass and this quantity this can be simplified further. So, it's a very easy if you do not need to memorize it, if you understand the physics you will be able to write it. Only thing that we have to carry the notations properly which frame we are working with, this we have arrived from the writing the equation about O prime ok.

So, of at least you need to understand that higher we are start from this place or either from this place, we can reduce it to this point. Some of the books they directly a start from this point write the equation for the h and then keep on deriving the equation of motion, but we have a started from this point. So, the angular this angular momentum of this particle we have written in terms of the absolute angular momentum where, r i we have written as r 0 plus rho i and then we are reducing it to this format ok.

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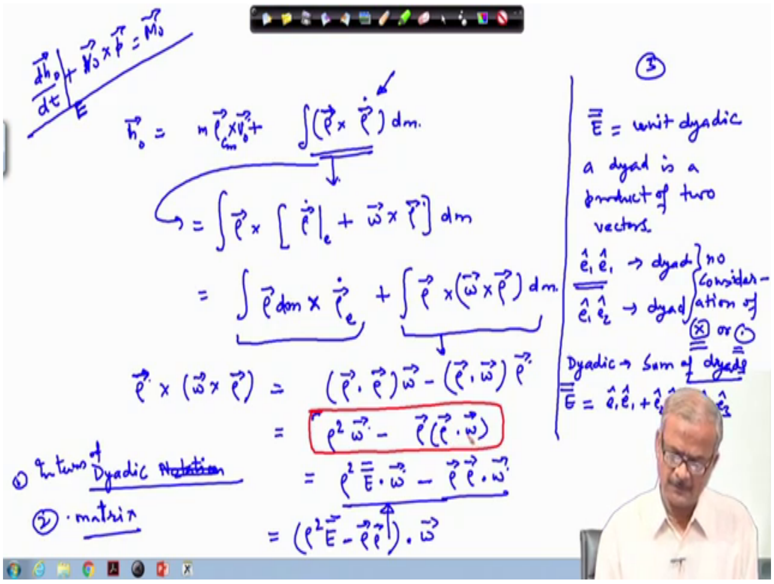

So, we have now h 0 this equal to m times rho cm plus rho cross rho dot dm. And here one term is missing we have here v 0 also cross v 0. Now this term we have to look into this is a rho cross rho dot now this rho dot if you remember that while we have written dh 0 by dt plus v 0 cross p equal to M 0. So, at that time I have told you this derivative with respect to the e frame means of with respect to the inertial frame. So, here also while we are working with as you look here in this place so, though the distance is measured from this point this rho i is from this point to this point this is your rho vector from here to here this is the rho i vector.

But once you take the derivative. So, this rho i variation it has two components one it can be say if this is a rigid body and there is a point here and the rigid body is rotating, but this length is not changing ok. So, the rho i in the inertial frame; obviously, it will change its orientation right now it is here in this direction after sometime it will be along this direction ok. So, of as indicated by red line after sometime this rho i may go here in this direction.

So, without change in length this vector can change together with this if this is a non rigid body. So, in that case this distance from the point o to this point it will keep changing with time ok. So, those two components has we have written earlier this rho dot with respect to the d rho by dt with respect to the E frame this can be written as d rho by dt with respect to the e frame or this is the body frame plus omega cross rho ok.

Means the rho dot which is appearing in this equation here this is nothing, but this quantity rho dot with respect to the E frame and this can be written as rho dot with respect to the small e frame and plus omega cross rho. So therefore, we can write this as rho cross rho dot with respect to the e frame plus omega cross rho and then we have to integrate it. So, we get this quantity here. So, this will be rho d m cross rho dot with respect to the e frame plus rho cross omega cross rho.

So, this quantity while we are working with this; so, we have to do this in a particular way this quantity first we will look into this what this quantity is this is rho cross we can write this quantity as rho dot rho times omega minus ok. So, this quantity is nothing, but rho a square omega minus. Now we have two notations for this one is call the dyadic notation which can we can follow or in terms of dyadic you says in terms of dyadic and in the second one we can write in terms of matrix notation ok.

So, this quantity we can write here as i will come to this point what this quantity is exactly the quantity that E double bar we have written it’s a call, the it’s a call the unit dyadic. First of all a dyad is a product of two vectors right, if we have vector e 1 and if we multiply it with vector e 2 so, this constitutes a dyad. Similarly, e 1 cap e 2 cap this is also a dyad. So, here no consideration of dot or cross product, no consideration of cross or dot we are not considering this is irrespective of the dot or cross.

So, the unit dyadic so, under dyadic is you know dyadic this is some of dyads. So, a some of two or more dyads so, that gets into a dyadic and therefore, this e can be written as unit dyadic its a written as e 1 e 1 cap e 2 e 2 cap plus e 3 cap e 3 cap. So, if you look into this.

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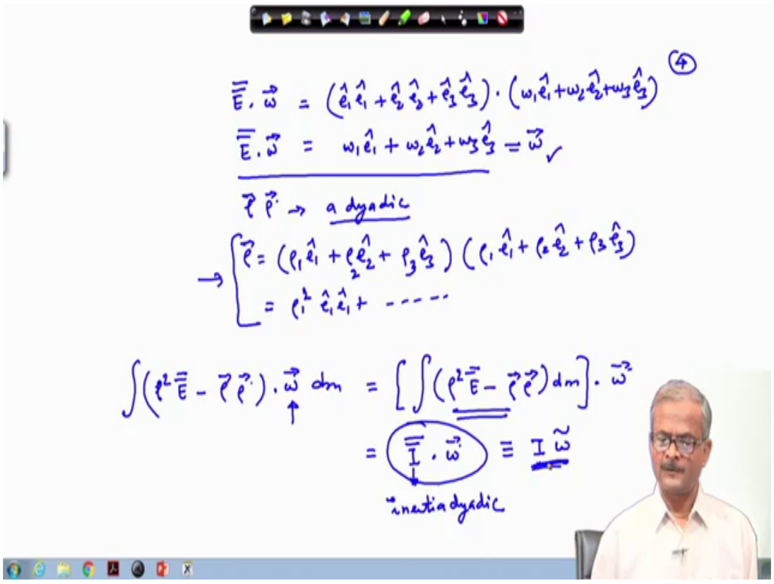

Now, if we take the draught product of this dyadic with this omega vector. So, you can check this here e 1 cap e 1 cap plus ok. So, if we take the dot product only the dot product between the similar vectors we will survive these are the unit vectors and the dot product between e 1 e 2 e 1 e 3 they will vanish. So, we will get here omega 1 e 1 omega 2 e 2 plus omega 3 e 3.

This implies that a unit dyadic lift the vector as it is if you operate take the dot product of a vector with unit dyadic it leaves the vector as it is it does not change it. Similarly, your the vector appearing there this is rho rho this is also a dyadic because its a product of 2 vectors now this rho you can write as rho 1 times e 1 cap these are expressed in the body components rho 2 time c 2 cap rho 3 times e 3 cap and then multiplied with rho 1 e 1 ok. So, here in this case what we see that if we multiply it. So, this will be something like rho 1 a square e 1 cap e 1 cap and so on the other terms.

So, we did not expand it here right now. Now if going back to this equation; so, therefore, this equation can be written as rho a square E double bar minus ok. So, of rho a square E double bar minus rho rho dot omega; so, what we get there rho a square E double bar dot omega and these we are integrating over the term dm. So, this omega it does not depend on the mass ok, this is not over the mass. So, this is the rotation rate of the frame which is fixed in the body. So, what we can do that we can take it outside rho a square integrate it over the body and this is dot omega.

So, you we can represent something like this and the quantity which is present here this is written as I double bar and this quantity its say call the inertia dyadic. Now again picking up this equation we consider this equation rho a square omega minus rho rho dot omega and this is integrated over dm.

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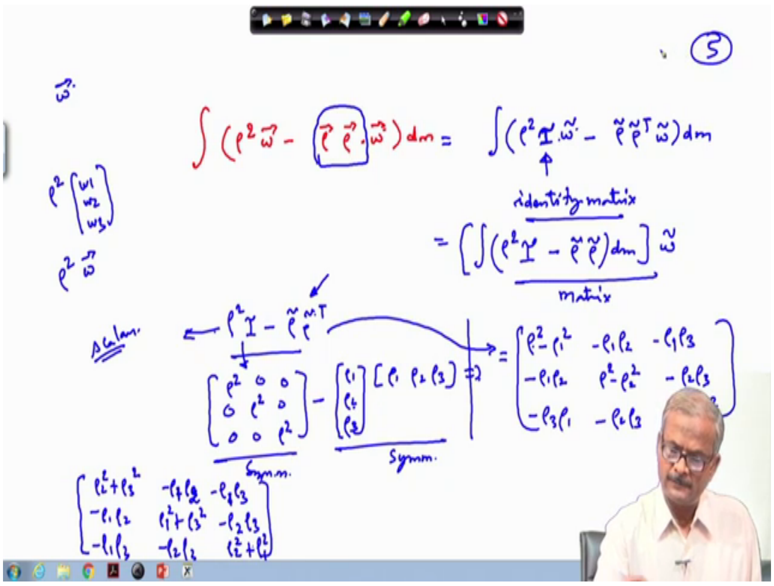

So, here this is rho a square rho a square omega minus rho rho dot omega and dm this is the integration we are looking for. So, this integration can also be represented like this we can write rho a square and this omega we can write here the unit matrix and unit matrix I will represent it like this or the identity matrix ok. So, this is I times omega this does not change.

This say if although a proper way will be to first convert this into the of of this vector into the matrix for converting this vector into the matrix notation we can refer to the matrix, but rather than writing it like this you can because that will just take more time to do the same problem rather than writing it like this you can consider that this omega is nothing, but omega 1 omega 2 omega 3 ok. So, either if you write in matrix format you will write like this and if you write it in the vector format you will write it like this.

So, in the matrix format this is your identity matrix and this is omega minus now you know that the quantity from here to here rho times rho dot there is a dot also. So, if you refer back to our lecture earlier. So, this we have written as rho tilde rho tilde transpose rho times rho dot this we have written as this quantity and we have probe this also. And therefore, what remains there this is omega tilde and this omega also omega which is coming in the vector notation so, that we will change to tilde.

So, this removes the unnecessary work in terms of the matrix and this is the same thing. So, quickly for from your exam point of you or while you are working so, we can quickly go between this and this without considering the matrix. Now, look into this quantity we can take out this omega outside and this is nothing, but a matrix. So, if you look into this quantity rho a square; rho a square is color and what this quantity is here this is transpose.

So, these turns out to be a symmetric matrix. So, here basically you have rho a square rho a square rho a square 0 0 minus and then you have rho 1 rho 2 rho 3 rho 1 rho 2 rho 3. So, and this turns out to be a symmetric matrix this also symmetric matrix this is also symmetric matrix. So, this says that this two together which we have represented here it forms a symmetric matrix and we can write this as; so, what we get from this place.

If we expanded and write it; so, this will be rho a square minus rho 1 a square the first term the second term is here 0 here. So, only think that we will enter from this place which is a rho 1 rho 2 with minus sign then similarly rho 1 rho 3 ok. So, this quantity indicate that this is equal to this value with minus sign similarly here rho 2 times rho 1 with minus sign. So, this is rho 1 rho 2 with minus sign this one again this is rho a square minus the middle term which will come to be rho 2 rho 2 rho rho 2 to a square and then this rho 2 times rho 3 with minus sign ok. Similarly, we have the rho 3 rho 1 with minus sign then rho 2 rho 3 minus sign then rho a square minus rho 3 a square.

And this term if you further simplify. So, rho a square minus rho 1 a square you know this will be equal to rho 2 a square plus rho 3 a square and then the rest of the terms here we can write rho 2 rho 3 rho 1 rho 2 and then minus rho 2 rho 3; this is a matrix format similarly here minus rho 1 rho 2 this will be here rho 2 we are subtracting. So, we get here rho 1 a square plus rho 3 a square and similarly this place this is rho 2 this is a rho 1 times rho 3 rho 2 times rho 3 with minus sign.

Rho 1 rho 3 with minus sign rho 2 rho 3 with minus sign and here it comes to be rho 2 a square plus rho 2 a square plus rho 2 a square plus rho 1 a square. So, this quantity gets reduce to this and next we need to integrated 5 next page we have 6.

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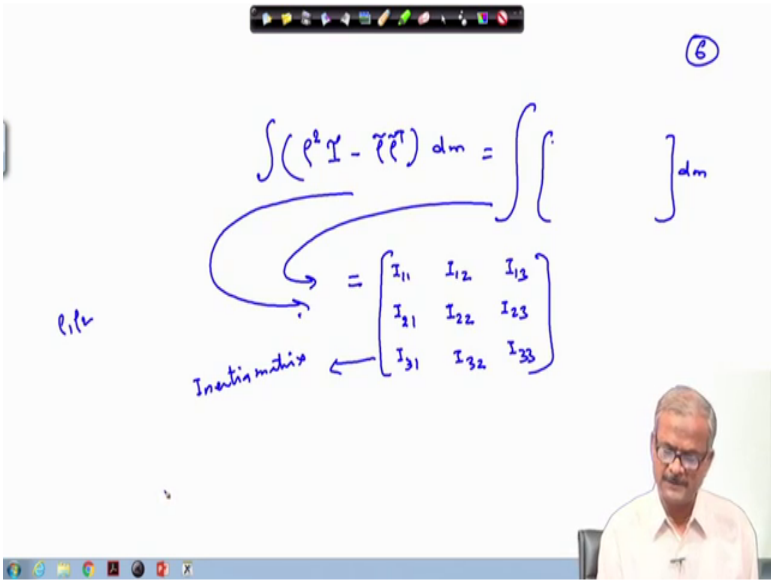

So, of rho a square times I minus rho times rho transpose dm; now if you integrated means the whole matrix we have to copy it here and integrated. So, the first term after integration that will get. So, this will result in I 1 1 the next term we can write as I 1 2 I 1 3 I 1 2 or I 2 1 whatever you want to write because, it is the same both of them involves rho 1 and rho 2.

So, if the same or may be let us write here I 2 1 to be more to be more formal I 3 1 I 3 2 I 3 3. So; that means, this term what we get here this is called the inertia matrix the same thing we have written in terms of inertia of dyadic ok. So now, going back here if you look into this we have written as inertia matrix.

So, if you write the same thing in terms of the matrix notation so, the same thing can be reduce to this is equivalent to writing it I times omega tilde ok. So, this is your inertia matrix and this multiplied by omega that gives you the h value 1 term of the h that we have been writing. So, we will continue in the next lecture.

Thank you very much.