Satellite Attitude Dynamics and Control
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Lecture – 20
Rigid Body Dynamics (Contd)

Welcome to the lecture number 20. We have been discussing about the Rigid Body Dynamics so, we will continue with that. And, due course of our analysis we saw that the inertia matrix emerged and we need to know some of the properties inertia matrix, because that we will be helpful in discussing various problems.

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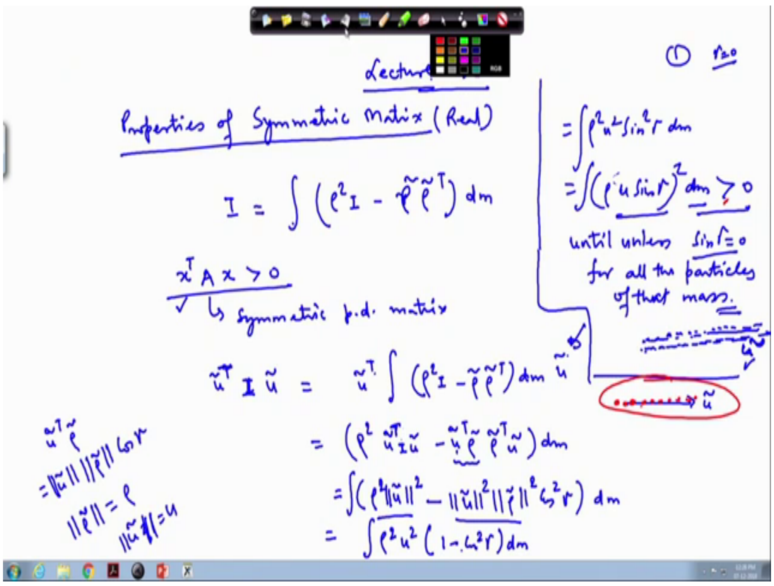

So, let us look into the inertia matrix the equation that we have written, this was written as rho square I minus rho times rho tilde times rho tilde the transpose and times d m ok. So, you know that if I am given a positive definite matrix, then this property x transpose A. If this is a symmetric positive definite matrix, then this must hold this quantity will be greater than 0 ok.

So, let us say that for this v I, we use this type of notation u tilde trans u tilde transpose u tilde well write it like this. So, if you operate from this particular one, take it inside. So, if we take it inside, this will be rho square u tilde transpose rho times rho transpose u tilde, u is an arbitrary matrix which is not depending on m ok.

So, if we look for this quantity, this is vector u and this is vector rho tilde. So, the angle between these 2 u tilde transpose rho, this we can write as u tilde magnitude times rho tilde magnitude and say the angle between these 2 vectors it happens to be gamma. So, we can write it as cos gamma ok. And therefore, this quantity here on the right hand side it can be written as u tilde magnitude square times rho tilde magnitude time square times cos square gamma. And, this is dm on the left hand side and this rho tilde a magnitude this is nothing, but this rho.

So, here we have in the left hand side u tilde transpose times I and here one more part we are missing. So, u tilde is coming from this place. So, we need to put that also. So, we have I matrix here identity matrix times u tilde ok. So, this also constitutes the this is t. This is the transpose which is on here in this place. So, I times u tilde is nothing, but u tilde then therefore, u tilde transpose times u tilde that is equal to u tilde a magnitude square. So, these 2 are the, these 2 terms this term and this term, they are the same term this is integration.

So, what we see here? This is rho square and if we write this as u tilde magnitude, u tilde magnitude equal to u. So, this we can write as 1 minus cos square gamma dm and this quantity is this is equal to rho square u square, sin square gamma d m and therefore, rho square rho times u sin gamma square d m and this is always a positive quantity. So, this is going to be greater than 0 until unless until unless sin gamma equal to 0 for all the particles of that mass. And that is a very extreme case that is the limiting case which will happen only if your u tilde vector, along the u tilde vector your all masses are linearly.

If this is your u tilde vector so, all masses are located along this direction itself then only this vanishes. So, in that case the angle between this 2 will vanish ok; u tilde and rho u tilde and rho tilde vector say here the we have written in terms of sin gamma. So, sin gamma means you are putting this gamma equal to 0 means this vector and this vector the rho vector. So, both are aligned and because this vector is arbitrary, rho tilde is arbitrary your sorry this u tilde is your arbitrary vector you are taken on your own ok.

So, that means, the vector you have taken along that direction itself this rho vector is aligned. So, rho vector is also align in this direction. And this integration is over all the particles; that means, that all the particles are lying over this line only. Means if I have this u tilde vector along this direction, so, all your particles are considering concentrating over this line only ok. So, this is a very extremist distribution and only for that case this will turn out to be equal to 0 otherwise it will never be 0 this will be always positive definite ok. So, we if you note down the following properties of the inertia matrix we shall be helpful due course of time.

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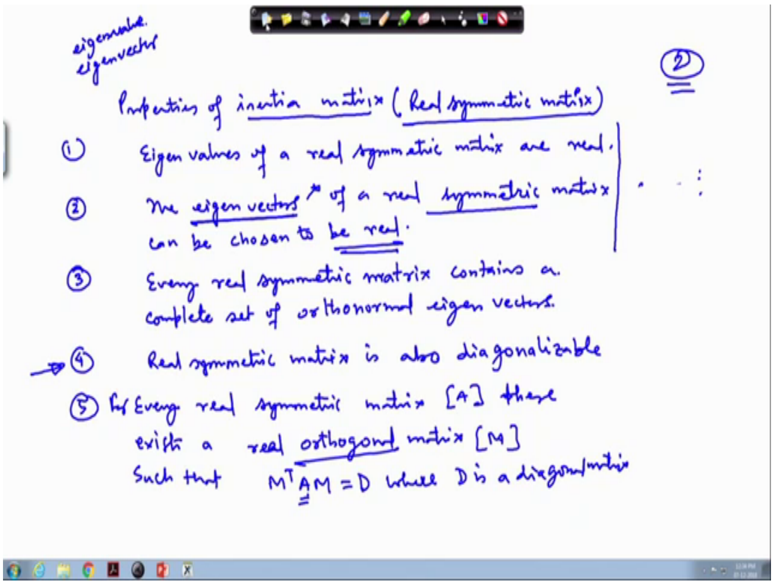

Inertia matrix this is the real symmetric matrix. So, basically these are the what I am going to write it is a property of the real symmetric matrix, and because our inertia matrix also is real symmetric matrix. And therefore, all those properties will apply. Eigen values of a real symmetric matrix are real, because your inertia it cannot be complex quantity it is a physical quantity can be chosen to be real matrix; can be chosen to be.

So, the eigen vectors we can also choose them to be real. Its written like this eigen vectors as one term eigen values as one term ok. And this as a significance that if my eigen vectors are real symmetric of a real symmetric matrix can be chosen to be real means, the eigen vectors they indicate here in this case the directions and here for this particular case they indicate principal x direction as we will see later on.

So, if more over every real symmetric matrix; so, this has got complete set of the orthonormal eigen vectors matrix; ortho real symmetric matrix is also diagonalizable you can diagonalize it, and this is very important because if we can diagonalize. So, we can get a principal moment of inertia. For every real symmetric matrix a there exists a real orthogonal matrix; already we have studied about what is the orthogonal matrix. So, there exists a real orthogonal matrix M such that, M transpose I will not put this bracket where D is a diagonal matrix means if we operate on this a matrix or the inertia matrix I matrix, I can converted it to a diagonal matrix and this is a very important.

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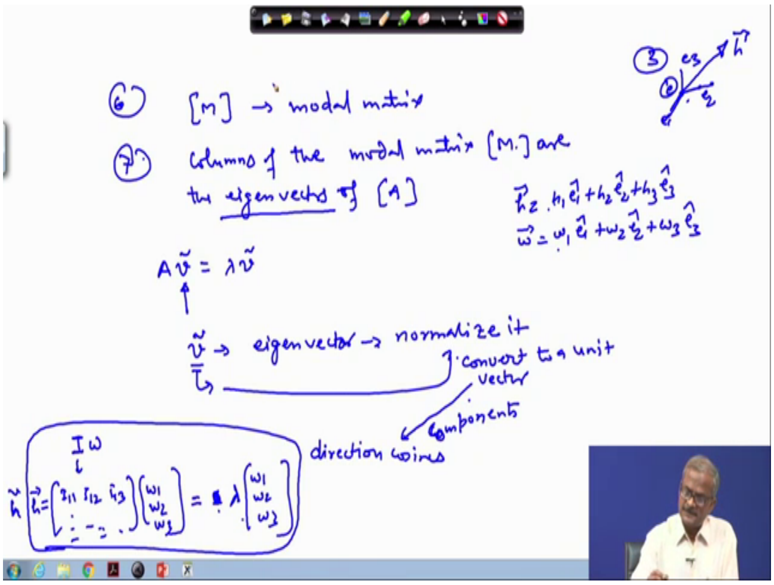

Is called the M is called a modal matrix and if its increase if we normalize, then we get the columns of the of the modal matrix, M are the eigen vectors of A. If it is able to diagnolize A, then this these are the eigen vectors of A. So, in the ortho normal form let us say that, I have this M matrix and this operates on some vector nu and this eigen value problem with a write it like this. So, here this nu tilde this appears as the eigen vector and if we normalize this eigen vector, normalize it means we convert it to a unit vector that is convert to a unit vector then its components they indicate then the components this will indicate the direction cosines.

Let us say that I omega we have written earlier ok. So obviously, I is a matrix where you have the I 11 I 12 I 13 all these terms are there and then you have omega 1 omega 2 omega 3 and what you are looking for that instead of this ok. If it convey reduce to a form where I (Refer Time: 14:52) or instead of writing this I will write in terms of lambda if it can be written like this.

So, can we find a situation or the axis where the rotation can be represented at this is your h vector. So, the h vector or the h tilde, can it be rotated represented it like this? So, this simply indicates that if your h vector if you are writing as h 1 times e 1 cap h 1 times e 2 cap h 3 times e 3 cap. Means in your body frame this is the e frame and this is e 2 direction e 2 direction e 3 direction ok. So, in this frame you are taking the components of the h vector this is your h vector. So, take components along this direction.

So, you can write it like this. Similarly omega you can write at omega 1 e 2 cap omega 2 e 2 cap plus omega 3, e 3 cap ok. So, in general this h and omega 1 they are not parallel we go to the next page.

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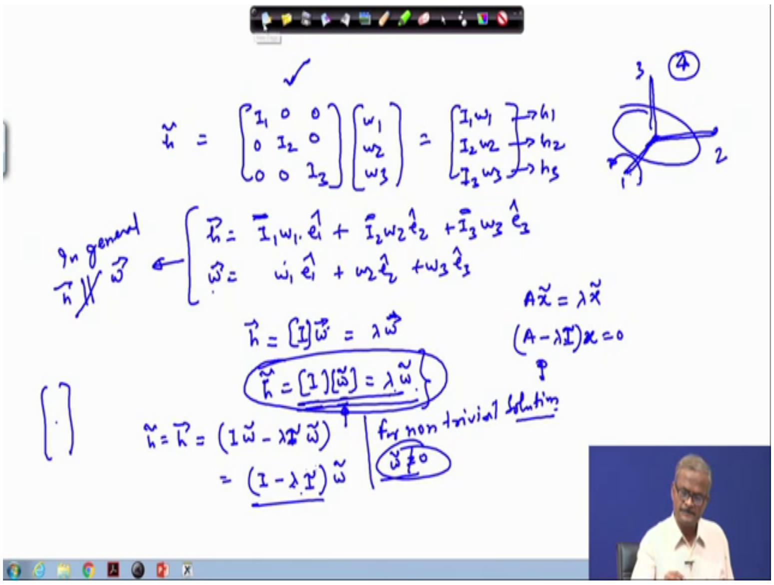

See even if we have this kind of situation I 1 I 2 I 3 and here we have omega 1 omega 2 omega 3 and this is your h which you can write at as I, 1 times omega 1 I 2 times omega 2 I 3 time omega 3. And in vector notion the same thing you can write as this the first term h 1.

So, h 1 equal to I 1 times omega 1 times e 2 cap the second term this is your h 2 this is h 2. So, this is h 1 is h 3; omega 2 e 2 cap plus e 3 cap. While your omega vector we have written as e 2 cap plus omega 3 e 3 cap. What we can see that these 2 vectors cannot be parallel because of the presence of these terms. So, even if we you consider inertia matrix where the off diagonal terms are 0, in that case h and omega in general. So, we can state that in general they are not parallel, in general h is not parallel to omega.

So, what can be that situation where h and omega they are parallel ok. So, if h and omega they are parallel means you should be able to show it like this, this is your I matrix I times omega or writing the same term same thing like; this is your omega vector I matrix omega vector or omega matrix this is a column with matrix basically. So, instead of telling column matrix, I will always pick this as the vector. This is much better than a picking it has a column matrix. So, when this situation will arise that I omega will be equal to lambda times omega. So, if this situation arises so, this simply implies that we are looking for condition where h and omega they are parallel to each other; means if you look here in this side. So, I 1 omega 1, I 2 omega 2.

So, we write this term as I times omega tilde minus lambda I now this is the unit matrix lambda I times omega tilde. So, if you write it this way, we can see that the this is lambda times lambda I tilde; lambda times this identity matrix. So, this is indicating basically your eigen value problem. So, rotation if the rotation happens along a particular principal axis, only then this situation will be satisfied or as earlier we have discussed that or let us say that this is my first axis second axis and the third axis. And, this is the principal axis first principal axis, second principal axis and this is the third principal axis.

So, this kind of notation we can have only in the case where rotation itself is along one o the principal axis otherwise we cannot get this ok. So, if we write it in this way let us say that you have A x equals to lambda x, and then you are writing this as lambda and this is the identity matrix lambda x equal to 0 and then you are solving it ok. So, the same term this the eigen value problem, here also this is the eigen value problem. So, if you try to solve this problem. So, in that case for the nontrivial solution, we look for the nontrivial solution means omega tilde this is not 0 this is not a 0 vector. So, we will particularly explore this expression.

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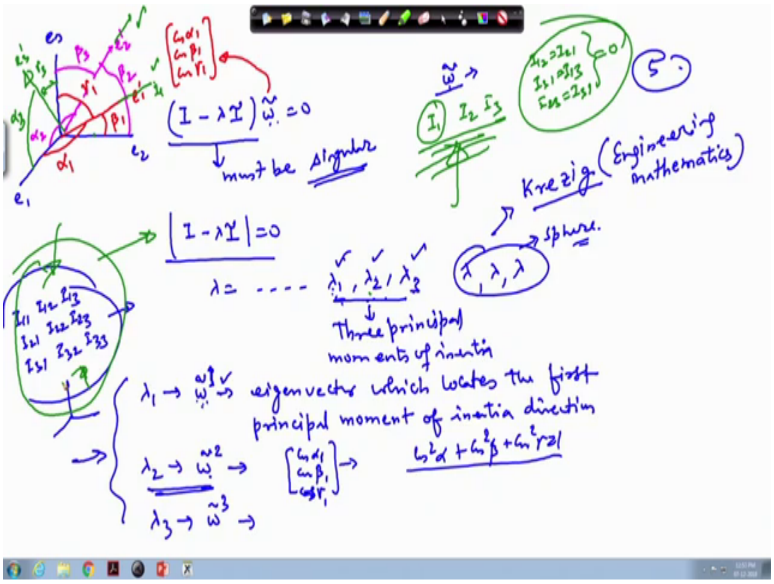

So, what we see that, for our problem to be traceable this must be similar ok. For nontrivial solution determinant of this must vanish this implies that this must be similar only then the determinant will vanish ok. And then this lambda I tilde if you set it to 0 so, you can solve for lambda. So, you get 3 values for lambda say this is lambda 1 lambda 2 lambda 3 and this 3 values they will indicate the 3 principal moments of inertia.

So, the 3 principal moments of inertia; it can happen that all of them like they are lambda lambda lambda all of them are same that happens in the case of the sphere. Where, all the moments of inertia along all the perpendicular direction all the 3 perpendicular direction whichever you choose it happens to be the same ok.

Now, here this omega tilde which is appearing, this is your eigen vector ok. So, for each of the eigen value you can solve for this eigen vector and then this eigen vector this indicates the direction of the principal or the principal moment direction. So, if we have the lambda 1 lambda 2 lambda 3; so, taking this lambda 1 we can solve this. So, one value we will have to assume the rest other 2 we can solve for and if it is repeating. So, you have to apply the process for solving eigen vector getting eigen vectors for the repeated eigen values and you can look for that technique in the book Krezig Engineering Matrix Mathematics by Krezig.

So, this omega then corresponding to this lambda 1 what you get; corresponding to this lambda 1 the omega tilde that you get this keeps you the eigen vector which locates the first principal moment of inertia direction ok. Similarly from lambda 2 you will get omega tilde. So, for this omega tilde now, omega tilde as a whole if you look it may not be a unit vector if it is not unit vector just normalize it ok. So, each component of this omega tilde then, omega tilde the next one let us say for I will write it in the exponent terms not to indicate these are the components of omega, but rather these are the eigen vectors themselves ok.

So, similarly for lambda 3 omega tilde 3; so, the components of this omega tilde, they will be cos alpha cos beta and cos gamma these are the direction cosines. And cos alpha times cos beta square time cos gamma square as we have discussed in our rotation, cos alpha square alpha plus cos square beta plus cos square gamma this will be equal to 1. Once you have normalize it so, the normalized omega tilde which is eigen vector here in this case, it will indicate the direction cosines of the principal direction.

So, let us get into these through a figure, say if I know the mass distribution with respect to this axis e 2 e 2 e 3 where we have all I 1 I 12, I 13 I 21 I 22 I 23 and I 31 I 32 and I 33 all of them are available to us ok. Thereafter while I doing this operation this solving for the eigen values we get to this point that is we solve for eigen values and the eigen vectors. So, if this is my omega 1. So, omega 1 will have components, I can indicate it by say omega alpha 1 beta 1 gamma 1. Similarly for this we can write here as alpha 2 beta 2 gamma 2. So, this eigen vectors. So, now, I have the principal direction, let us this is the first principal direction and I will indicate this as e 1 may be prime ok.

So, if I indicate by e 1 prime. So, the angle from here this will be your alpha 1 this angle will be beta 1 and this angle will be gamma 1 and then the cos alpha 1 cos beta 1 and cos gamma 1 this is your corresponding eigen vector here normalized eigen vector. So, this normalized eigen vector it locates the orientation of the or it is just about the orientation of your principal axis direction similarly you have the second direction, let us say it is along this one. So, this your e 2 prime.

So, the same way the angle from here to here this will be alpha 2 from this place to this place this will be beta 2 and from here to here this will be beta 3. And in the same way than you can take the third direction which is e 3 prime. So, e 3 prime you take the angle from this place to this place this will be your gamma 3 from here to here. This angle will be your alpha alpha 3 and from this axis the angle will be I cannot show it now, this will be beta 3.

So, this way all your principal axis direction principal moment of a inertia axis direction they get located ok. So, if I write here I 1 I 2 and I 3 as the principal moment of inertia, then this indicates that you are taking along this axis I 1, about this axis I 2 about this axis and I 3 about this axis and the mass distribution with respect to this new axis, I 1 e 1 prime e 2 prime and e 3 prime; it is such that all the off diagonal terms I 12 I 21 I 31 I 13 I 23 and I 31 these are all equal to 0 they vanish.

So, the mass distribution then with respect to this axis it becomes such that those terms are vanishing. And this is quite often done that if once say your satellite and you want to get it some moment of inertia. So, first we will you have a big satellite ok. So, there are different components inside. So, you will calculate the there are software available or either you can do it yourself its component you have certain mass. So, you first choose certain point and from that point you can calculate the moment; moment of inertia of the each of the component.

So, this using the parallel axis theorem you can calculate the moment of inertia of which of the components, you can build the moment of inertia for the whole system and once, you have got the inertia for the whole system, which is looking a like this. So, in that case there after you can reduce it to the format by considering the eigen value problem and you can get here lambda 1 lambda 2 lambda 3 and also the principal axis direction and once you get the principal axis direction. So, you can work with that I 1 I 2 I 3 instead of working this full matrix, because this is very easy to handle in controls rather than handling this particular part ok.

So, this is about the of moment of inertia consideration and a lot more can be discussed, but we have limited number of lectures only 30 lectures for this particular course, and already we have covered a around 10 lectures that lecture. Number 20 it is half an hour. So, total 10 hours we have covered. So, rest 20 hours we are having in hand to cover the dynamics part.

So, I will not go in details of this moments of inertia, rather you look into the engineering mechanics books by (Refer Time: 31:31) or either by Beer and Johnston book in both the book itself given in great details. So, after reading this you will be able to solve certain problems. So, in this particular course I am concentrating on the dynamics and controls of the satellite, I will discuss more in more details the dynamics and to some extent the controls because dynamics itself it takes a lot of time. So, we continue in the next lecture.

Thank you very much.