Satellite Attitude Dynamics and Control

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Lecture – 25
Rigid Body Dynamics (Contd)

Welcome, to the lecture – 25. So, we have been discussing about the torque free rigid body dynamics. We will continue with that and there after we will go into the stability of the rigid body dynamics under torque free condition.

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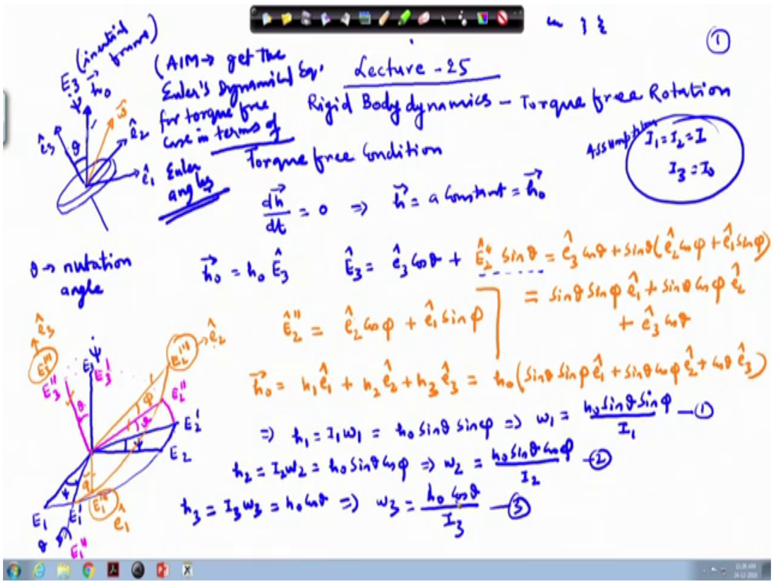

So, what we have done last time that we had the rigid body ah; obviously, for the last week we have been doing the same thing. This may be a disk, this may be a cylinder whatever it is and we assume that this is the direction of the angular momentum. So, because it is a torque free condition torque free condition.

So, this implies dh by dt this will be 0 and this implies h is a constant and that is what we are writing as h 0 and let us assume that it is a acting along this direction which we are fixing as E 3 direction the E 3 which is the inertial frame direction, and then ah; obviously, we will have the E 3 is here then E 1 and E 2 will look like this which we are now drawing here and let us say that we have this is the body axis E 1 and this is body axis E 2 and along this direction we have the body axis E 3. Somewhere here we are showing this is the angular velocity vector, ok. So, this h 0 can be written as h 0 magnitude times unit vector along this direction which we have shown here as E 3. So, this is E 3 cap is the unit vector along this direction, but we need to work it out what will be that value.

So, let us say this angle is theta which is the notation angle and therefore, the E 3 cap this can be described in terms of small E 3 cap cos theta this is here along this direction we will have E 3 cap E 1 cap along this direction E 2 cap along this direction. So, E 3 cap is along this direction. So, it is a component along this axis that will be E 3 cap cos theta and plus now, from this place it is not visible, but we will have to draw another figure we will have another vector here and that time sine theta will appear. So, this vector we need to fill here.

So, let us go back into the situation from where we are start, here along this direction you have the psi dot; psi dot is here, and then we rotate this frame. So, this is your E 1, E 2 and E 3 and once we rotate this, ok. So, this point will go to this point E 2 prime this will come to E 1 prime, this angle is psi and thereafter we are rotating about this axis by theta. So, this will rotate and go to the position here this will rotate by theta and this will also rotate by theta.

So, this is in a circle which I am not showing here. This is your E 3 and E 3 prime is along the same direction and E 3 double prime here E 2 double prime and E 1 double prime and there after we are giving one more rotation which is by phi here. So, this is coming to E 1 triple prime, E 2 triple prime this angle is phi and here E 3 triple prime this is your E 3 along this direction itself and this angle is phi.

So, what we can observe that this line and this line they lie in the same plane, this pink line and this pink line they are in the same plane. So, here the unit vector along this direction will be E 2 cap double prime unit vector along this direction times sine theta and we have to get this quantity E 2 double prime which is the unit vector along this direction. So, this one we have referred to E 2 cap this one we have written as E 3 cap and this one we have written as E 1 cap.

So, taking this so, here it along this direction what will be the unit vector we have to take component of E 2 along this E 2 double prime and component of say this and this they are perpendicular to each other. So, we need to take the component of those vectors only. So, we will have this is E 2 cap cos phi and from this place this will be plus E 1 cap sine phi and once we insert into this. So, this gets reduce to E 3 cap cos theta plus sine theta times E 2 cap cos phi plus E 1 cap sine phi expand it.

So, E 1 we will put it in the front, sine theta and sine phi E 1 cap plus sine theta times cos phi E 2 cap plus E 3 cap cos theta and therefore, if you write h 0 as h 1 times E 1 cap because this is a symmetric symmetrical case h 2 times E 2 cap and h 3 times E 3 cap so, this imply this quantity will be equal to h 0 times cos theta times E 3 cap. So, therefore, from here what we get h 1 equal to I 1 times omega 1, this quantity is h 0 times sine theta sine phi and this implies omega 1 equal to h 0 sine phi divided by I 1 which we have written as here I 1 equal to I 2 equal to I and I 3 we are writing as I 0. So, this is the assumption of the symmetric rigid body we have taken.

Similarly the h 2 equal to I 2 times omega 2 it can be written as from this place h 0 sine theta times cos phi and this implies omega 2 this equal to h 0 sine theta cos phi divided by I 2 and h 3 in the same way this is I 3 times omega 3 cos theta this implies omega 3 equal to h 0 cos theta divided by I 3. So, these are the equation we are having 2 and 3. So, you can see that the omega 1, omega 2, omega 3 we can also express in terms of h 0 and Euler angles, ok. An earlier we have done purely in terms of the Euler angles. So, advantage of doing this you will as we proceed so, you will come to know ok.

Now, what our intention is that here aim is to get the Euler’s dynamical equation for torque free case in terms of of Euler angles, this is our objective, ok. So, we need to put this 1, 2, 3 in the Euler’s dynamical equation under the assumption that the torque is not there and then solve it.

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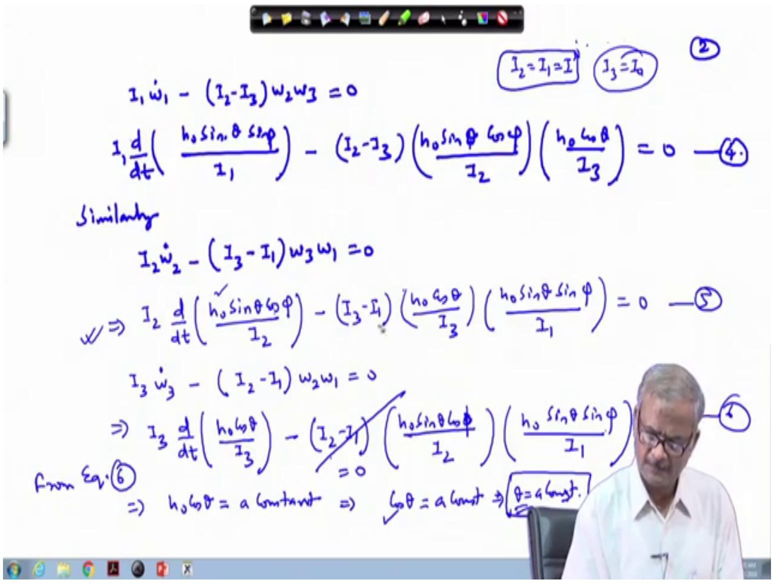

So, we have the first equation I 1 times omega 1 is the dot minus I 2 minus I 3 times omega 2 omega 3 this equal to 0, and here omega 1 if we insert it. So, this is h 0 sine theta times sine phi divided by I 1 minus I 2 minus I 3 times omega 2 omega 3 as we have written h 0 sine theta sine theta times cos phi divided by I 2 and the other one h 0 cos theta divided by I 3, this equal to 0. This cancels out from this place, and here we have I 1 d by dt we need to differentiate this quantity, equation number 4. Similarly, we will have I 2 times omega 2 dot I 1 times omega 3 omega 1 this equal to 0.

Now, we insert the omega 2 dot here. So, omega 2 dot is h 0 omega 2 will be h 0, sine theta times cos phi I 2 minus I 3 minus I 1 and omega 3 is a h 0 cos theta divided by I 3 times, omega 1 is this one h 0 this equal to 0, this is our equation number 5. In the same way I 3 times omega 3 dot this equal to 0 and this implies picking up omega 2 from this place cos phi, this is equation number 6.

Now as we know that I 2 equal to I 1, so, this quantity is 0, ok. So, therefore, this term vanishes and what we get from this place h 0 cos theta this equal to a constant this implies cos theta will be a constant and from here theta is a constant. So, this we directly get from the equation number 5 from equation number 6 because this quantity turns out to be 0, I 2 is equal to I 1. Here I 2 this equal to I 1 equal to I.

Next we use this equation; equation number 5 and solve it.

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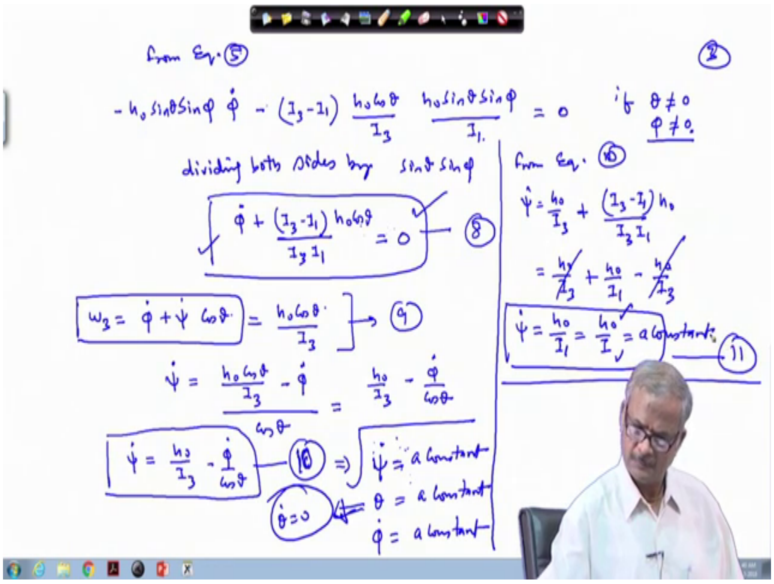

So, the equation number 5, here I 2 times this I 2 – I 2 will cancel out because it is a constant quantity and what we get from here h 0 is a constant because it is a torque free condition h is a constants. From here we have got that this say theta is a constant therefore, h 0 sine theta can be taken outside the bracket I 2 – I 2 cancels out and what we get from this place is a cos phi we have to differentiate this. So, that becomes cos phi differentiation is sine phi with minus sign times phi dot; this is from equation 5, ok.

There after I 3 minus I 1. So, this we can write as I 0 minus I 3 we are writing as I 0 plus this is minus I 3 minus I 1 times h 0 cos theta by I 3 h 0 cos theta divided by I 3 and h 0 sine theta sine phi divided by I 1; h 0 sine theta sine phi divided by I 1 and this quantity is equal to 0. Now, ok, one more thing that we have left out here this h 0 sine theta we have taken it outside the bracket only this part we have differentiated. So, this part we are missing and we should insert here. So, we will put here h 0 sine theta minus sign here, ok.

Now, if theta is not equal to 0 and phi not equal to 0; so, under that condition we can divide both side by dividing both sides by sine theta sine phi. So, what we get from here h 0 also be eliminated. So, we will have here phi dot plus I 3 minus I 1, h 0 gets eliminated and here we get as h 0. So, this is equation number 5, 7 this one can term as 8 and cos theta is missing. So, we put here cos theta also h 0 cos theta in this quantity.

Now, further we go and look for that earlier we have used this equation omega 3 equal to phi dot plus psi dot cos theta. So, if we utilize this equation we get some more thing from this place. So, this quantity can be written as omega 3 if we go back and look here omega 3 somewhere omega 3 equal to h 0 cos theta by I 3 h 0 cos theta divided by I 3. You can utilize this equation here and if you look from this place so, psi dot this will be equal to h 0 cos theta divided by I 3 minus phi dot divided by cos theta. Remember that theta is a constant here, ok. So, this gets reduced to h 0 by I 3 minus phi dot by cos theta. So, your psi dot equal to.

Now, we can work on this further because is a phi dot already we have estimated. Phi dot if we go back and look here in this place phi dot is here phi dot we have derived here in this place. So, we can insert this. So, from this place what we see that. So, from this equation implies that psi dot equal to h 0 divided by I 3 or we can write here on the upper side from equation 10 from phi dot by cos theta from this place will be minus so, that this will come with a minus sign. So, that becomes plus I 3 minus I 1 I 3 I 1 times h 0.

So, h 0 by I 3 plus here I 3 – I 3 will cancels out it will break the bracket h 0 by I 1 minus h 0 I 1 – I 1 cancels out by I 3 and these two get eliminated and the psi dot then becomes h 0 by I 1 and because h 0 is a constant this is a constant and therefore, this turns out to be a constant. So, what we have recovered from this place that psi dot this is a constant, theta this is a constant and phi dot is related to this equation where h 0 is a constant cos theta is a constant and therefore, phi dot also is a constant. So, theta is a constant here and phi dot this is a constant and this implies that theta dot equal to 0.

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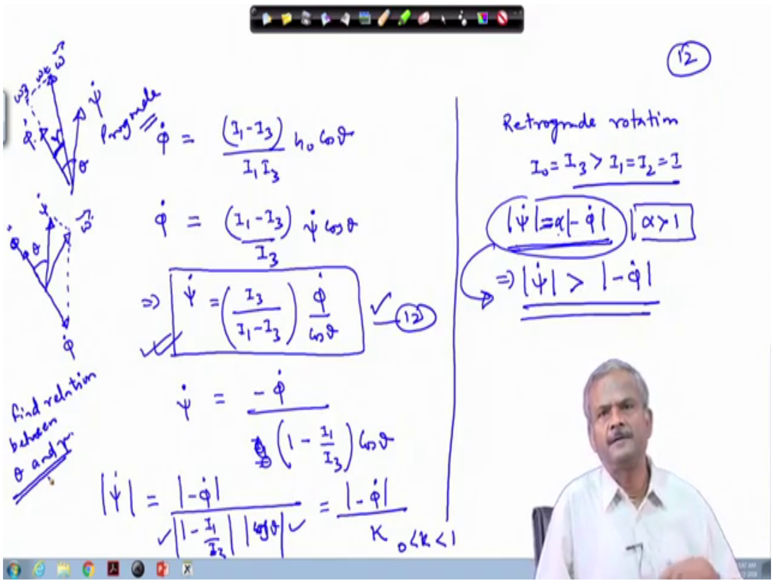

Now, phi dot and psi dot it can be simplified. So, phi dot we have written as phi dot from this place we can pick up from this place phi dot equal to on the right hand side we can write as I 1 minus I 3 divided by I 1 I 3 and then h 0 cos theta cos theta. So, we know that h 0 by I 1 h 0 I 1 here this quantity is psi dot, ok. So, we can write here I 1 minus I 3 divided by I 3 and h 0 by I 1 is psi dot. So, this becomes psi dot cos theta and then psi dot can be expressed in terms of all these quantities. So, from here this implies psi dot this equal to I 3 by I 1 minus I 3 times phi dot by cos theta.

So, see that this equation we derive in some other way also, but using this formulation how smoothly we have been able to work out the whole thing, and you can do it yourself in the way like writing it as phi dot divided by I 1 divided by or minus sign we can take it outside here in this place. So, this will become 1 minus I 1 by I 3 cos theta this is psi dot and psi dot magnitude then you can write as minus phi dot magnitude divided by I 3 magnitude times cos theta magnitude.

And, already we have discussed few things about the retrograde rotation which is the case when I 3 is greater than I 1 which this we have written like this earlier ok. So, if that happens to be the case ok, so, this quantity is going to be less than 1, this quantity is less than 1, both are less than 1. So, you can see that the numerator becomes here denominator becomes less than 1. So, 1 by something like let us say this you write this as minus phi dot divided by K, where K is less than 1 ok, but greater than 0.

So, if we take it upside phi dot and 1 by K then we can write this as say if we write as something like alpha ok. So, obviously, from this place it is a visible that alpha is a quantity where alpha is greater than 1, ok. So, this simply implies that psi dot magnitude will be greater than minus phi dot magnitude from this equation we can get this it is a very simple to say if just check it yourself because alpha is greater than 1, ok.

And, rest other things also you can work out in the same way for the see utilize this figure this phi dot and here in this direction this is your psi dot for prograde rotation this is your omega, this is the situation here. This angle is gamma and this angle is theta, this is for prograde, and for the retrograde case it turns out that phi dot will be not in this direction, but rather in the opposite direction and psi dot will be here in this direction, this is the phi dot.

So, if you are been indicating this as the theta angle ok, here in this case your omega will go along this direction and using the this figure then you can derive many relationships like along this direction omega component you will have omega 3. This is omega 3 along this part will be omega t and obviously, this is omega. So, you can get all sorts of relationship you can derive. So, I need not do all those parts here, ok. Some of the things will appear as the part of the tutorial problems ok.

So, you can take it and find out the relationship as a homework. Find relation between theta and gamma, where remember that gamma is the angle between the omega 3 and the omega vector while theta is the angle between the psi dot and the phi dot vector, ok. So, whatever we have done through the geometry the same thing we have just derived using the basic formulations for the torque free case in terms of the angular momentum.

So, we stop this lecture here and we will continue in the next lecture.

Thank you very much.