Satellite Attitude Dynamics and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture – 36
Gravity Gradient Satellite (Contd.)

Welcome to the 36th lecture, today we are going to discuss about the Gravity Gradient Satellite what we have been continuing with what we will concentrate on the elliptic orbit. So, if the gravity gradient satellite or satellite which can be gravity gradient has stabilized, if we are working on its motion equation so, how the things we will develop that we are going to look into today.

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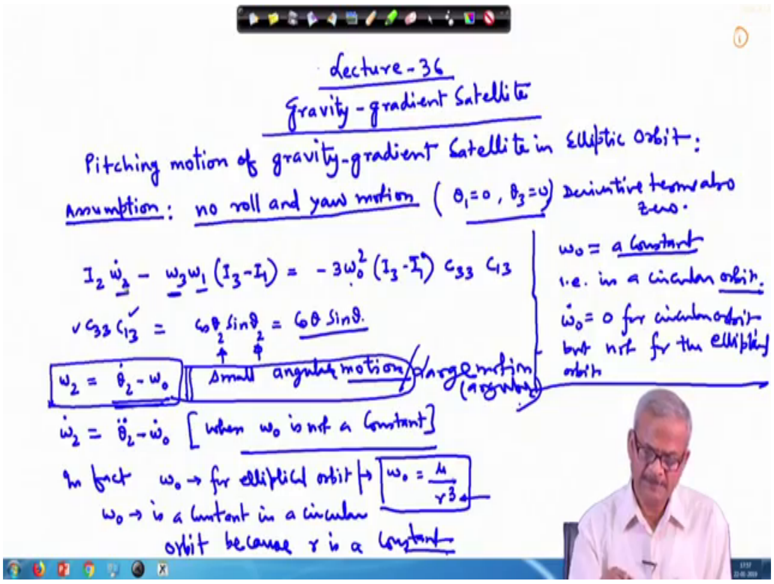

So, we assume here no roll and yaw motion. So, already we have observed that for the circular orbit roll and yaw, this decoupled from the pitching motion. So, let us write that if there is no roll and yaw motion means the theta 1 this will be 0 and theta 3 will be 0. And therefore, the related derivative terms will also be 0 derivative terms also 0.

So, just we have concerned with the pitch motion dynamics and we know that for the motion about the second axis that we can write as I 2 times of omega 2 dot minus omega 3 times omega 1 I 3 minus I 1. And on the right hand side we have 3 and then omega 0 square and we have written the other terms I 3 minus I 1 C 3 C 1, ok. This is the way we have described, but here this we have written for the case where omega 0, this is a constant means it is a moving in a circular orbit.

Now, already we have observed that C 33 into C 13, this quantity is coming from the third column of the direction cosine matrix. So, we need to put or insert this term. So, this term basically we have approximated as cos theta times sin theta. So, I will put here the theta 2 and theta 2 term. Otherwise, we will have to go back into the old set of lectures and from there find out all this quantities which were involved.

And omega 2, we have written as theta 2 dot minus omega 0 and this we have done for small angular motion. So, from this place omega 2 dot, this will be theta 2 double dot minus omega 0 dot. Then omega 0 is not a constant. So, in that case this will not vanish. For the circular orbit omega 0 dot equal to 0 for circular orbit, but not for the elliptical orbit.

In fact, omega 0 this for elliptical orbit, we have to write it properly. See, here in this case omega 0 for the circular orbit. It has got its notation from mu 0 by r cube. In the circular orbit, because or this was simply we have written mu, ok. So, in the circular orbit r is constant and therefore, omega 0 turned out to be a constant. So, omega 0 is a constant in a circular orbit, because r is a constant. However, in the elliptical orbit this r cube is varying and therefore, the things will change, ok. So, we are today going to discuss about all this things.

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So, this theta 2, theta 2 what we are writing here if we replace this in terms of cos theta and sin theta, it will be much convenient to work with, because this 2 will be able to get rid off, ok.

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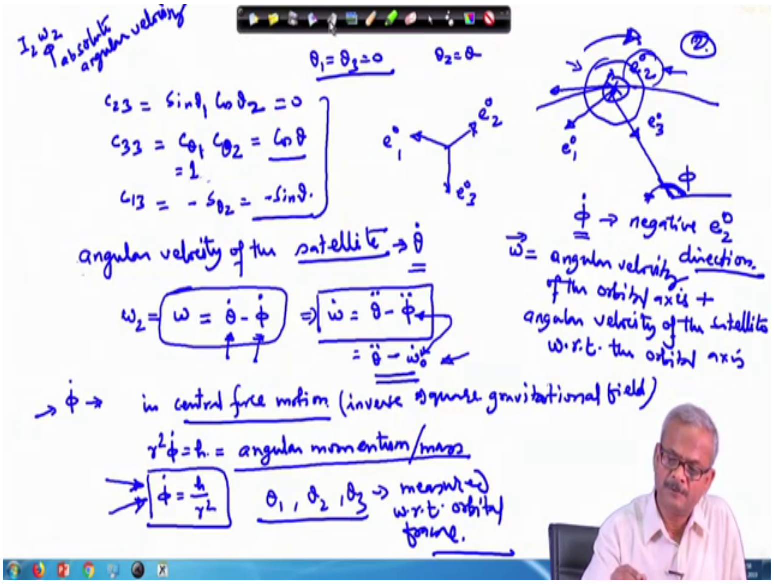

So, to a start with we have C 2 3, we have written as sin theta 1 times cos theta 2, and because the theta 1, here we are choosing to be theta 1 and theta 3 to be 0. So, this quantity vanishes C 3 3 equal to C theta 1 times C theta 2. So, this we are writing as cos theta, because theta 1 this quantity is 0. So, therefore, C theta 1 will become 1. So, better we should write here this quantity to be equal to 1, this is equal to 1 and theta 2, we are writing as theta. So, this we can write as cos theta and C 1 3 equal to minus sin theta 2 minus sin theta.

So, we utilise this information and put it is the our equation of motion. Then therefore, I 2 times pick up this now what will be the value of the; if there is no other motion involved. So, only this motion is involved. So, even for this particular part this is as well valid for the large motion, large angular motion also large motion angular, because if here is the earth is there and this angle let us write this as the phi, ok.

And, then we have taken this direction to be the e three direction orbital and suppose the satellite is going in the elliptical orbit. So, the velocity vector will be along this direction, but in our case e 0 one will be along this and e 0 2 will be directly going inside the paper.

So, you can see the positive direction of the rotation of positive direction of rotation about the e o 2 axis about this particular axis it is in this sense, ok. So, it is opposite to the phi direction, right where this is I am drawing the figure here separately, this is e o 3 e o 1 and this is e o 2 which is going into the paper here in this place, in this place it is a going into the paper.

So, now we will look here in this motion, ok. So, this motion is in the, this particular phi. So, the changes which are taking place for this phi which we will write as the phi dot, ok. So, this is along the negative e o 2 direction, ok. And therefore, because no other motion is involved so, you can check it directly, we can write here the angular velocity of the satellite to be motion with respect to the.

So, omega so, this will be the angular velocity of the orbital axis plus angular velocity of the satellite with respect to the orbital axis. So, angular velocity of the satellite, we are indicating by theta satellite. So, this is pitching motion basically satellite is indicated by theta. So, therefore, you can see that theta and this for angular velocity of the satellite is indicated by theta dot. So, phi dot and theta dot they are opposite to each other, ok.

So, the omega then becomes omega can be written like omega equal to this is only in one direction. And therefore, we can write this as theta dot minus phi dot, ok. And therefore, the omega dot it can be written as theta double dot minus phi double dot which in the on the earlier page we have written this as omega 0 dot,.

Now, this phi dot, you will have to be go back and look into my lecture on space flight mechanics and how this phi is derived. So, basically in a, in central force motion of inverse square field gravitational, we can write here r square phi dot this equal to h. And this is called the angular momentum per unit mass; this is angular momentum per unit mass ok. So, phi dot at any instant it is a related to related by this expression. So, h is your angular momentum per unit mass.

So, the set of equation which is required to indicate this omega 0 dot here, is nothing, but your this quantity omega 0 dot is equal to phi double dot. So, it will directly appear from this place, ok. So, you can look back into that lecture it is perhaps in the beginnings from the 6th, 7th lecture you will find it, it is a part of the central force motion,.

So, what I wanted to tell here that this omega the angular velocity of the satellite, it will be indicated like this. So, here in this case, this is your omega 2 equal to omega, what we are writing here omega 2. Why we are writing omega 2? This is I 2 times omega 2 is the here the absolute angular velocity absolute, angular velocity, this is the absolute angular velocity.

So, this notation should be little bit clear, see here omega 2, we have written omega 3 we have written. So, all these are absolute angular velocity and this then we have converted in terms of theta 2. So, theta 1 theta 2 theta 3 ah, let us write here theta 1 theta 2 theta 3, this we are measured with respect to the orbital frame, ok. And therefore, it appears in this format so, this is measured with the respect to the orbital frame. And this is the motion of the orbital frame itself, ok. Now we are ready to face the problem, ok.

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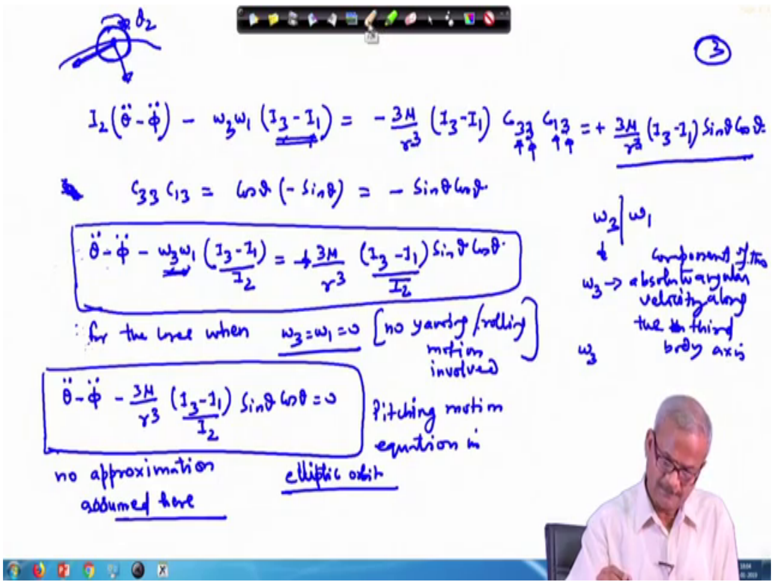

So, we have I 2 times omega 2 here in this case we will pick up from this place theta dot minus phi dot. So, this becomes theta double dot minus phi double dot omega 3 times omega 1 I 3 minus I 1 this equal to, on the right hand side we have minus 3 mu by r cube times I 3 minus I 1, C 3 1 and C 3 3 and C 1 3, this 3 is that for this 3 appears here in this place. For this one this one is appearing here then this is the 3 3 appearing here. For the case we are ah, we are considering the orbital third axis toward the centre of the earth,.

And this thing, so, already we have discussed in detail. So, there is no need of further going into this ok. So, C 3 3 all this things we have to write here C 3 3 times C 1 3 then becomes C 3 3 is here cos theta and C 1 3 is minus sin theta. So, therefore, this is minus sin theta into cos theta. So, here we can write plus 3 mu by r cube times I 3 minus I 1 sin theta into cos theta.

Ah what we can do I 2, we can take it on the right hand side. So, we will simplify the case here itself, theta double dot minus phi double dot is equal to plus 3 mu by r cube other equations have already been eliminated, ok. Now, if we are considering the large motion, ok. So, omega 3 and omega 1, if you look into the omega 3 and omega 1 motion. So, omega 3 is nothing, but your related to the, this is the absolute angular velocity, absolute angular velocity along the or this is the component of the component of the absolute angular velocity along the third axis body axis third along the third body axis.

So, the quantity omega 3, because we are considering that is no motion along the see what we have done that if this is my satellite, it is moving in the orbit. And there is only motion involved which is about this axis, which you are indicating by theta 2, the motion theta 1 which is one direction and three direction along this there is no motion involved. And therefore, this will drop out and the special case where I 3 equal to I 1 also this will drop out.

So, for this particular case so, for the for the case when omega 3 is equal to omega 1, this is equal to 0, ok. So, now, yawing slash rolling motion, this motion is not involved. So, in that case your equation get simplified otherwise if you take it the equation will be very complicated itself, we will see that in the elliptical orbit this turns out to be a nightmare. So, this is plus. So, this minus 3 mu by r cube I 3 minus I 1, here also we need to divided by I 2 divided by I 2 sin theta times cos theta, this equal to 0.

This is the pitching motion equation and elliptic orbit and remember that that we have not done any approximation here. So, no approximation, no approximation assumed here. So, we will explore this particular part, write in a particular way and then develop this equation.

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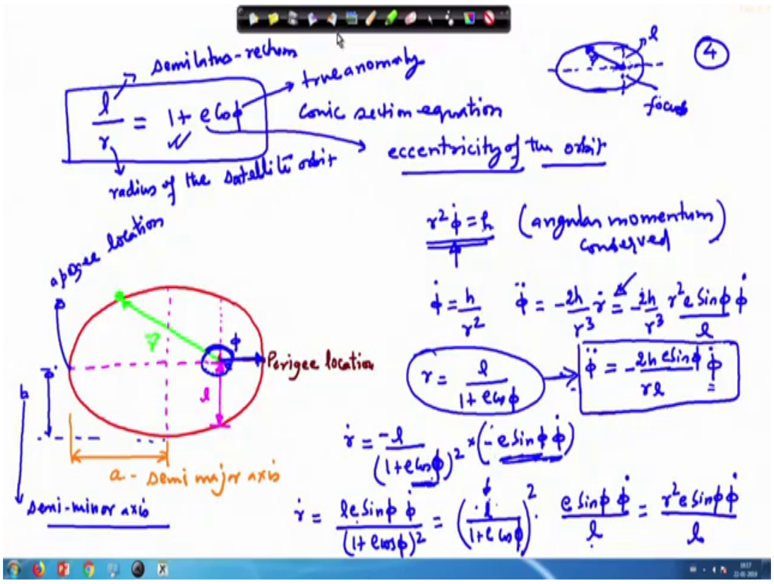

So, let us start with as I told you that go and refer back to refer to our lecture on space flight mechanics. So, this equation, I am writing here, it is a called the conic section, equation conic this is called the semi latus rectum. And r is; obviously, the radius of the satellite orbit and this theta I will change it to phi instead of using theta I would notation, I will be using here phi. So, this called the true anomaly and this e, it is the eccentricity, eccentricity of the orbit.

So, we are dealing with the elliptical orbit, elliptical orbit. How does it look like, something like this? Here this is the focus, this is the focus. So, your earth is situated here and because already we have discussed that this will be considered as a point mass. So, the whole if it is considered as if basically what we are, we have assumed that if earth is assumed to be circular and of uniform density. Then we can assume the earth mass to be concentrated at a point at this focus here in this case. And then your satellite is moving in this orbit.

So, this part is your from these distance to these distance, this is your focus. So, this part is l what we have written here, ok. And then if I draw a vector like this may be little bigger figure I will draw here. So, this is the focus I as discussed earlier and this is called the Periapsis or Perigee location. So, distance from here to here, this is the shortest distance. And then we have the radius vectors so, the satellite is located here in this place this angle we show this as phi. So, this we have written as the true anomaly,. And of course, your l this is the distance vertical to this. So, either up or down this distance is your l from this place to this place, this is l. Then the length from this place to this place is called the semi major axis, ok.

Similarly, the distance from this point, this distance from here to here is indicated by b and this is called semi minor axis. This point is your apo axis or the apogee location, because of this discussion given in the space flight mechanics lecture. So, here this is the conic section equation. So, from there we can get the what will be the rate at which this phi changes. So, this is a simple differentiation and moreover, because the angular momentum is conserved in such kind of orbit. So, this equation is also applicable angular under central force motion so, this is always valid.

Now we are ready to take it take it to the next stage. So, phi dot from here we get as r square and phi double dot. So, the rate at which this angle is changing, it is directly available to you, ok. As r changes along this direction, you can see the this is the smallest r possible from this place to this place, this is the smallest possible r as we move along the orbit. So, we go from this place to this place. So, r is varying, from here to here this r will keep varying.

. So, phi double dot from this place this is minus h by r cube times r dot. Now r dot we have to insert here in this place to get the complete solution r equal to l by 1 plus e cos phi. Differentiate this, if we differentiate this we get here r dot equal to 1 by 1 plus e cos phi whole square, which minus sin here and then differentiating this quantity.

So, this will be e sin phi times phi dot and plus 1 minus sin here from minus sin here. So, e sin phi for this, we have written here e and then differentiation of cos phi with a sin phi and thereafter the phi dot and cos phi differentiation then minus sign also appears simultaneously,

So, therefore, r dot which is appearing here in this place ok. So, this becomes l e sin phi times phi dot divided by 1 plus e cos phi whole square. Then again utilise this particular equation so, we can write this as, utilise this. So, we can write this as l by 1 plus e cos phi whole square, and then rest of the things on this side e sin phi times phi dot divided by l, e sin phi times phi dot we have taken here l, because the l square term is getting introduced here. So, therefore, we have divided here by l. So, this can be written as r square e sin phi times phi dot divided by l. So in this equation then minus h by r cube and r dot is r square e sin phi times phi dot divided by l, ok. So, minus h by r l times e sin phi. So, this is your phi double dot, this is phi dot here. Some more simplification, we can do to this equation. So, I will have to go to the next page.

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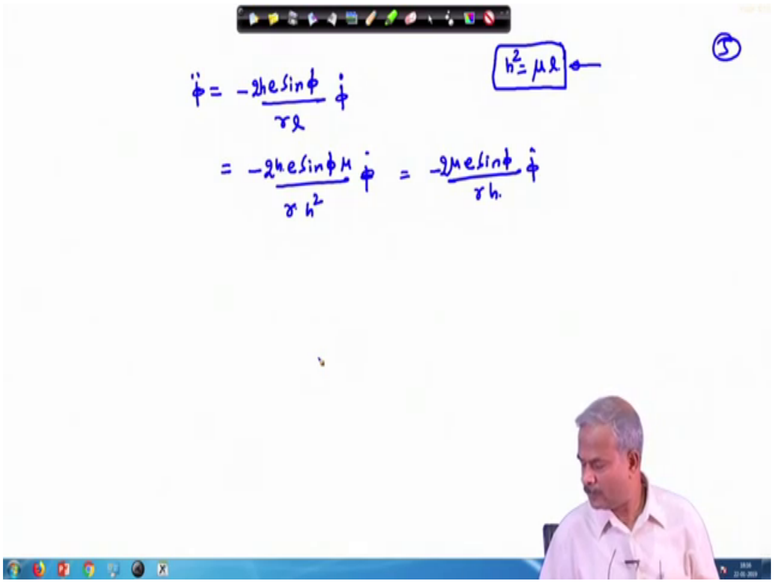

So, we have phi double dot minus h e sin phi divided by r times l and then phi dot. Now, h square this quantity is related to l by this relationship and this is derived in the space flight mechanics lecture. So, we will utilise this relationship directly here and eliminate this l, ok. So, this becomes h e sin phi divided by r and l is equal to h square mu goes upside. So, this is h square divided by mu ah. So, mu will be upside and then we of course, we have phi dot here in this place minus mu times e sin phi divided by r h phi dot.

And here we have missed out a term see if while h by r square we are writing here. So, there are 2 factor will appear here in this place. So, this 2 we have missed out everywhere, ok. So, we have to write here this 2 factor [FL].

Student: [FL].

[FL] repeat [FL], ok.

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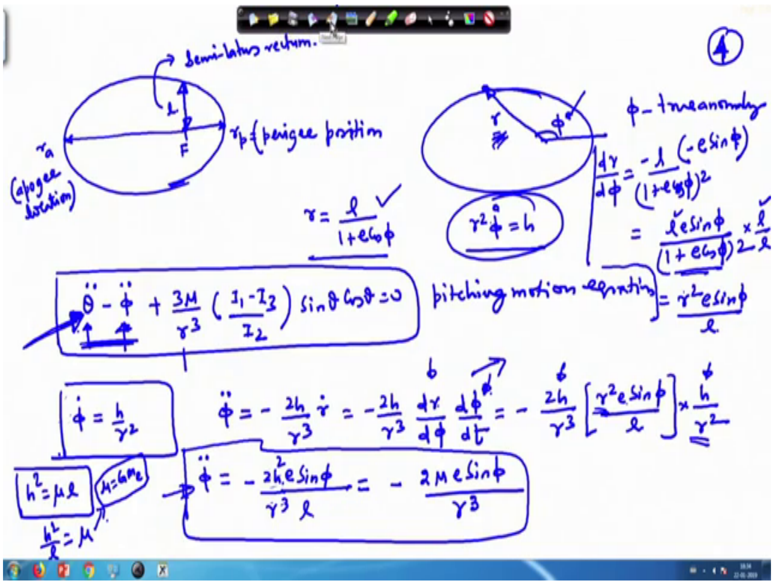

So, what we have been discussing about the quantity or so, this is the focus and this is the shortest distance which we write as r p and we can call as the this is related to the perigee position perigee position. And this part here from the distance from this place to this place, this is your focus F. So, this is r a and this is your apogee location and the length from this place to this place this is the vertical distance from here to here, this is your l which we have written as semi latus rectum, ok.

So, in this figure we are measuring phi angle which is the true anomaly phi is your true anomaly this is r, ok. And for this conic section equation is given by r equal to l by 1 plus e cos phi. And this particle just like we can take the case of the sun earth system. So, in that the angular momentum is also conserved and it is written as r square phi dot, this equal to h. So, what we are interested is we will write the equation what we have written here. So, that we do not have to refer back again and again. So, theta double dot minus phi double dot and then plus three mu by r cube I 1 minus I 3 divided by I 2 times sin theta times cos theta, this is the equation we have got for the pitching moment, pitching motion, pitching motion equation, ok.

So, this part we have to resolve and write in a particular way so, that it is a convenient for analytical analysis, ok. So, we use this relationship l if I r say the r is equal to 1 by e plus, we are interested in finding out this quantity, this quantity is already with respect to the orbital frame, your changes are taking place this theta double dot, phi dot double dot, we are interested in finding, ok. So, phi dot so, this is a quantity h by r square. So, phi double dot this quantity, we can write as minus 2 h by r cube times r dot, ok. And this can be further written as r is changing with respect to phi. So, we write it, it rather in this way.

We could have equally gone another way first finding out r dot and then working out. Right now, if we differentiate, because you can see that this r is varying with respect to this phi, as this phi changing r will also change. And therefore, r is a function of phi and phi is a function of t. So, we are utilizing this information to work out here, ok.

So, therefore, phi double dot this becomes minus 2 h by r cube and r phi we can get from this place. So, d r by d phi we can write as l by 1 plus e cos phi whole square and then on the up side minus e sin phi.

So, this becomes l e sin phi divided by 1 plus e cos phi whole square. And we can also summarize this as by multiplying this by in the numerator and denominator by l and l say we multiply it by l and l in the numerator and denominator. So, this becomes so, l square term this term and this term becomes l square. So, l square by this square term that we can write as r square, ok. And rest other things e sin phi divided by this l.

So, this is the quantity which interfere in this place, this is 2 h by r cube and then r square e sin phi divided by l and this quantity is phi dot this is d phi by dt, ok. This quantity is nothing, but phi dot. So, we will pick up it from this place. So, that becomes h by r square. And therefore, phi double dot, we can write this as minus 2 h by r cube and r square r square it will cancel out, we are left by 2 h e sin phi and here we will write as h square, this h and this h makes it h square. So, 2 h square e sign phi r square this r square and this r square they will cancel out,. So, we are left with r cube and divided by l

And the relationship between h and l is so, h square divided by l this is nothing, but mu, which is the gravitational constant of the earth mu, we have written as g times m earth, or it may be any planet. So, this is minus 2 mu e sin phi divided by r cube this is phi double dot, ok.

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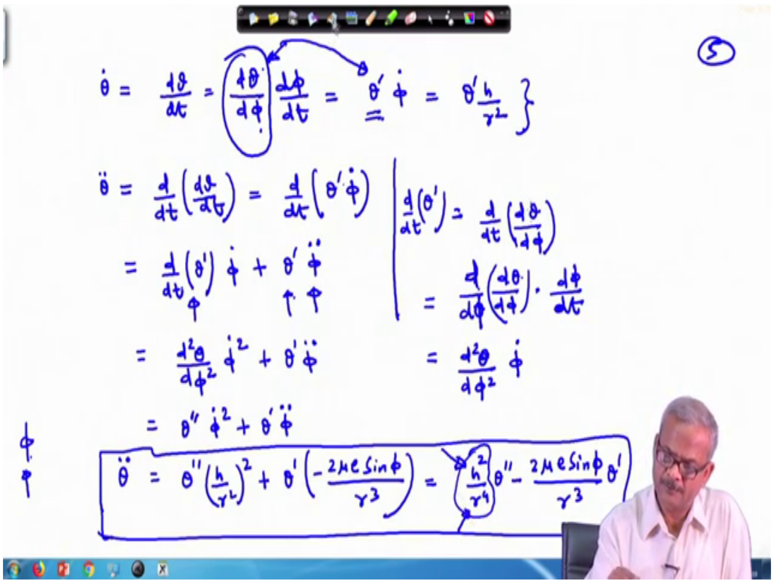

Now, we can proceed and work out this particular part. So, everything we need to express in terms of phi. So, theta dot again this part we can is nothing, but d theta by d t. So, we can write this as d theta by d phi times d phi by d t

And this quantity, we are going to write as theta prime means it is a derivative of theta with respect to phi. So, this is the quantity we are writing here and this part becomes phi dot obliviously this quantity is known to us. So, this is theta prime times h by r square, ok. Similarly, theta double dot is the quantity which we require. So, theta double dot it is present here, and this quantity we require. So, theta double dot, we need to differentiate this again. So, this will be d by d t d theta by d t theta prime times 0.1.

So, if once we differentiate it d by d t plus theta prime phi double dot. So, phi double dot this quantity already we have worked out, theta prime this is just the derivative of theta with respect to phi which is written here. So, this quantity we need to work out. So, theta prime d by d t, we need to work out d theta by, ok. So, we have to write it in the same way d by d phi this theta is there. So, first we differentiate it with respect to d phi and then we write it like this d phi by d t d square theta by this becomes d phi square times phi dot.

So, if we insert here in this equation so, this will become d by d t d prime d square theta divided by d phi square. And then phi dot square plus theta prime plus phi double prime, ok. And this quantity, then we are going to write as theta double prime, phi double dot.

So, now look into this all these quantities are known to us. So, theta double prime and phi dot equal to h by r square. So, this becomes whole square and theta prime and phi double dot phi double dot, we have just now derived this is minus 2 mu e sin phi minus 2 mu e sin phi divided by r cube.

So, we have got the quantity here h square h square by r to the power four theta double prime minus 2 mu e sin phi divided by r cube and theta prime. So, this is our theta double dot. So, by doing so of we can express how your motion will look like as phi keeps varying, instead of t we are expressing the whole thing in terms of phi.

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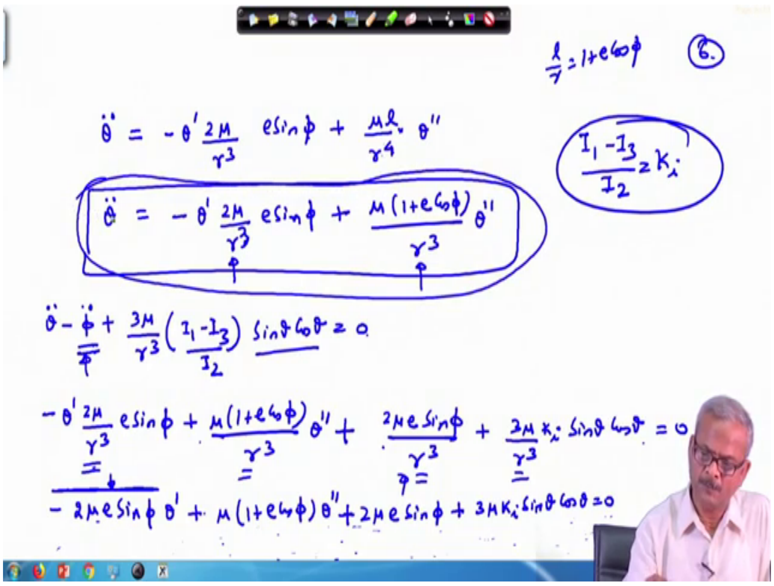

Now, we write the equation therefore, theta double dot equal to minus theta prime 2 mu y the previous equation we have written e sin phi plus mu by r cube and go back and this is the whole thing we are writing. So, this particular part, we have to work out h square by r to the power 4.

So, h square by r to the power 4, this is mu times l. This particular part, this h square by r to the power 4 is equal to mu times l and divided by r to the power 4. So, here let us go one more step by r to the power 4 and then theta double prime, we can eliminate one term here theta prime 2 mu by r cube l by r if this equals to 1 plus e cos phi. So, use this relation so, this becomes 1 plus e cos phi and r cube term is present here.

So, theta double prime, so, theta double dot is expressed like this. And then you can complete the whole equation and it will so, happen that because the r cube is present here, ok. And other terms also they contain r cube so, the this can be eliminated. Now we pick up this equation theta double dot minus phi double dot, and everything we have already worked out. So, theta double dot minus phi double dot plus 3 mu by r cube plus 3 mu by r cube I 1 minus I 3 divided by I 2, I 1 minus I 3 divided by I 2 sin theta times cos theta, sin theta times cos theta, this equal to 0. So, insert all this quantities here. So, this is your minus theta prime 2 mu by r cube e sin phi plus e cos phi divided by r cube times theta double prime minus phi double dot.

So, this quantity phi double dot, we have worked out here in this place. This is minus 2 mu e sin phi by r cube 2 mu e 2 mu e sin phi divided by r cube. So, this is your phi double dot term, ok. And there after the other terms, we need to put here 3 mu by r cube. And let us write this term as usual I 1 minus I 3 by I 2, this we have written as k i. So, we will write this as k i, and then rest of the term sin theta times cos theta, so this equal to 0. So, you can see that in the denominator everywhere this r cube term is appearing and because r is non zero. So, there is no problem, we can take it on the right hand side, eliminate it from this part. Now this gets reduced to 2 mu e sin phi times theta prime plus mu times 1 plus e cos phi and theta double prime minus 2 mu e sin phi plus 3 mu k i sin theta times cos theta this equals to 0.

So, we can see that this is the second derivative term. So, we can write here in the beginning 1 plus e cos phi times theta double dot. And then anywhere the phi prime term, we have to include, ok. So, this is your phi prime term and 2 mu e sin phi this particular term, this one. So, we utilise it here and here you see, 2 mu e sin phi term is appearing.

So, we can write this as this minus 2 mu e sin phi times 1 plus theta prime, and rest other terms 3 mu k i sin theta times cos theta equal to 0. So, this is the equation of motion in elliptical orbit for pitching only, equation of motion in elliptic orbit for pitch only. And this equation can be expanded it can worked out. So, we will continue in the next lecture. Thank you for listening.

Student: Start sir.

So, already we have observed that theta double dot is the quantity which is shown here.

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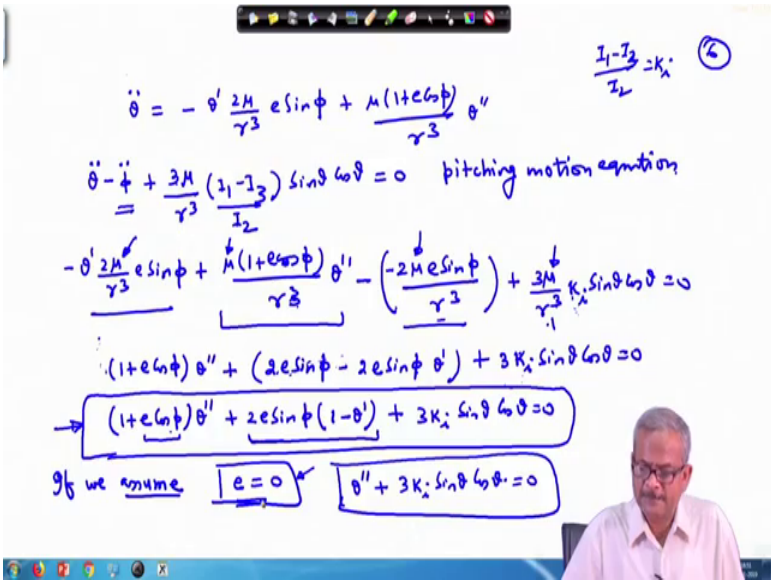

So, theta double dot this is the quantity, minus theta prime 2 mu by r cube minus theta prime 2 mu by r cube and e sin phi plus mu times 1 plus e cos phi divided by r cube and times theta double prime, ok. So, in our equation of motion theta double dot minus phi double dot plus 3 mu by r cube I 1 minus I 3 sin theta times cos theta equal to 0, this is the pitching motion equation. And we need to insert the quantities respective quantities here. So, theta double dot, we can insert from this place 2 mu by r cube e sin phi plus e cos phi divided by r cube.

And then minus theta phi double prime. So, phi double prime, we have derived earlier this is minus 2 mu e sin phi by r cube 2 mu e sin phi divided by r cube which a minus sign. So, this is the quantity phi double dot and then 3 mu by r cube I 1 minus I 3, this is there is the divided by I 2 also. So, I 1 minus I 3 divided by I 2 I 3 divided by I 2, we have written this quantity as k i. So, I will replace this in terms of k i here itself, this is k i and then sin theta times cos theta.

So, if we rearrange this quantity r cube r cube from the denominator, it will get eliminated and we have mu times e cos phi times theta double prime. And moreover, if we see that this mu is also present everywhere. So, this mu also we can eliminate maybe at this stage itself. So, your mu is present here, mu is present here, it is also present here this also present here and this is a non zero quantity. So, therefore, we eliminate here and write it like this.

Then this is plus the quantity which is present here this quantity and this quantity we can combined it together to write 2 mu e sin phi mu again, we do not have to write here plus 2 times e sin phi, ok. And from this place minus 2 times e sin phi times theta prime and plus 3 mu by r cube again this quantity goes here only 3 will remain, ok. So, if mu is eliminated r cube is a eliminated and we get 3 k i sin theta times cos theta this equal to 0 e cos phi times theta double prime plus 2 e sin phi, it can be taken outside the bracket this is 1 minus theta prime plus, as we can observe this is the this is a complicated equation. Now under simplified condition this can be solved.

So, if we assume then this gets simplified to we can see from this place this term will get eliminated and we get here theta double prime, ok. This term we can get rid off and we can write this as 3 k i sin theta times cos theta, this equal to 0. So, this equation is written here only for the e value, we shall continue with this lecture in the; we will continue with this topic in the next lecture and we are stop in the meanwhile.

Thank you very much.