Satellite Attitude Dynamics and Control
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Lecture – 44
Control Moment Gyroscope (Contd.)

Welcome to the 44th lecture we have in discussing about the gyroscope and these gyroscope then we are going to apply it on the satellite ok, this gyroscope it is used in a multiple ways it can measure the angles, angular rates and also it can be used as the torquer means if you put it inside the satellite and if you provide torque along one of the axis say along the Y axis, so you can get torque along the another axis which we are going to discuss in the future.

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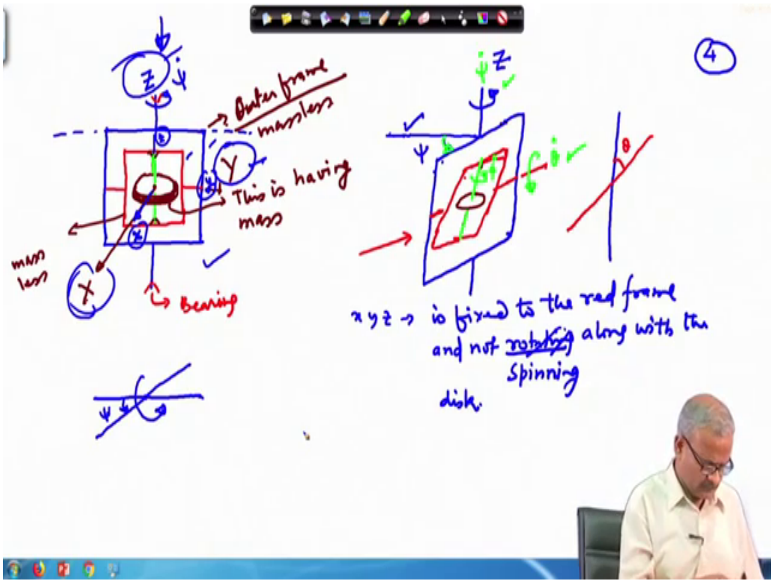

So, we have been discussing about this gyroscope, this was a our problem. So, here it is a X direction, universal X direction, universal Y direction and universal Z direction it is shown.

And with respect to them, the gyroscope it is a rotating like this. So, this with respect to this axis, X axis is here coming out like this and Y axis has been taken like this which is lying in the plane of this frames these are the frames in which plane the your these two frames the shown by the red and the blue they are forming one plane they are right now in the same plane I now rotate the external frame which is the blue one by the psi angle.

So, it will look something like this. So, from this position it has rotated to this position and thereafter if I give rotation about this axis Y axis the capital X, capital Y and capital Z axis they remain fixed ok.

So, here capitals Z axis it is a there and capital X and capital Y they are not of use towards right now ok, but the a small axis which we have chosen as the body axis so what we do that we put a body axis like here say right now in this direction I also choose the x axis and along this direction I choose y axis ok. So, these are the small x axis and small y axis and small z axis will be along here in along this direction, but it is a fixed to the red frame this axis is fixed to the red frame.

So, x, y, z this axis is fixed to the red frame here it is not rotating along with the brown disc; and not rotating what is spinning greater it is a this words we should write here spinning along with the disc. So, this case it will make the whole thing very simple.

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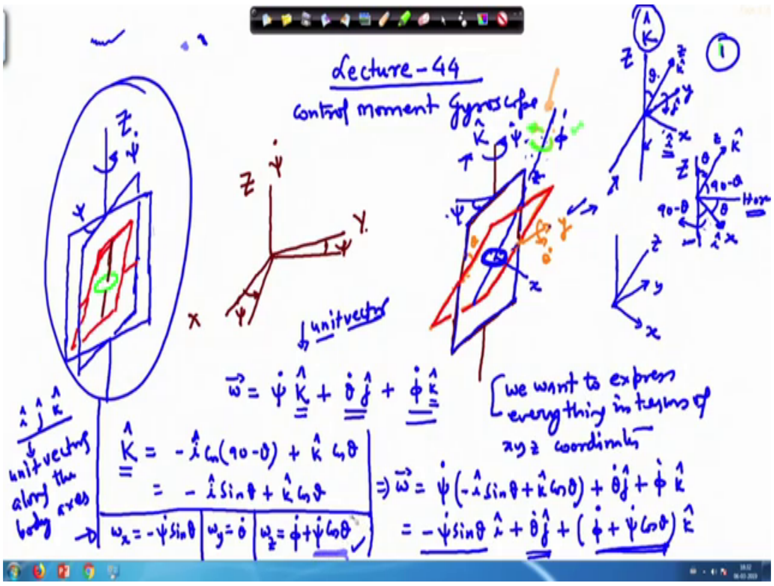

Now, come to the same figure I was trying to reproduce here; so live it that. So, it has rotated by the psi angle, this the then the your red frame and here the blue frame I have show here by so this is your blue flame basically this is the blue frame which is shown here in the rotated position. And then the red frame now it is rotated by theta so its rotation is about the y axis and x axis is here, but this axis x axis is not rotating along with the disc your disc is here this is your disc, but this x axis will not rotate along with the disc rather it is a is a y axis is fixed here so the x axis will be always along this ok.

So, if here this is this is your small z direction ok. So, your x, y, z axis now it is appearing to be tilted. So, if I take this to be the vertical so your z direction is appearing like this a small z and y direction is appearing here and x direction is appearing like this and using this figure now we can describe the motion of the gyroscope. So, omega the angular velocity of this gyroscope already we have discussed how formulate this in details, but still because we are taking the not the torque (Refer Time: 05:07) but the we are going to work with the torque. So, I will take it up again ok.

So, psi dot times in this direction the capital K cap this is the unit vector so this is your unit vector, this is along this direction as shown here plus theta dot, theta dot is along a small y direction so I will write it with a small j cap and then phi; dot phi dot is along the a small Z direction. So, we have write it with a small k. So, you can see the difference this is capital K and this is a small k. So, we will mark with this difference now we have to determine we want to express everything in terms of x, y, z coordinates.

So, this capital K, I need to convert in terms of a small i, small j, small k so these are the unit vectors along the body axis. So, this capital K, then we can write as you can see from this place see if I draw this as the capital K vector and this red line it is inclined to this. So, your x axis is here in this direction this is your x axis and y axis is along this direction and this is your small z axis and this is capital Z axis this is the situation, this is your angle theta ok. So, what will be the this capital K unit vector how we can write in terms of this is small k cap small j cap and a small i cap ok. So, you can see that this i cap unit vector this will have components along this direction and also perpendicular to here if I look from the this place it appears like this say this is your x direction this angle is theta.

This is a little perspective view so I am showing it from the; this side from this side we are seeing it. So, if we see it like this so this is the horizontal and this is your unit k direction and this is x direction, this is small z direction, this is capital Z direction, so this is i cap here. So, what will be the unit vector along the z direction this will be simply we can write here minus i cap because this i cap is along the downward direction so in the negative direction of this capital Z ok. So, we are taking the components along this direction of the unit vector. So, minus i cap cos theta angle we have missed out this angle we are taking as this angle is our theta ok.

So, this is 90 minus theta similarly if you rotate here by theta. So, the x which was horizontal initially this will go down this will become theta. So, this angle will be 90 minus theta and therefore, this will be cos 90 minus theta and then the component of this rotation has taken place about the y axis therefore, y axis is a still remaining horizontal, x has axis has got inclined and z axis have has also got inclined. So, therefore, the unit vector along k can be capital K can be written as k cap cos theta. So, this becomes minus i cap sin theta plus k cap cos theta.

And from here this implies this omega can be written as psi dot times, then copy this part theta dot j cap, then this part phi dot k cap and then i k k terms we can combined together and separate out the terms. So, that gives us minus psi dot sin theta times i cap plus theta dot times j cap and plus phi dot plus psi dot cos theta times k cap. So, you have these are the components of angular velocity along the body axis. So, you can see that here if I right here in this place itself so your omega x this equal to minus psi dot sin theta, similarly omega y that becomes equal to theta dot and omega z it is equal to phi dot plus psi dot cos theta. So, these are the components of the angular velocity along the body axis.

So, see the difference here phi dot is it is already having rotation about this axis if this part I will do it in little better way this is the z axis and about this you have the rotation here there is a gap shown here in this place. So, that gaps in this place you can see a gap. So, that gap is showing that it is a going from the behind means it is a anticlockwise ok, now this gap is much more visible. So, to this phi dot your further this quantity psi dot cos theta this is getting added up. So, we must be careful about this term where psi dot cos theta.

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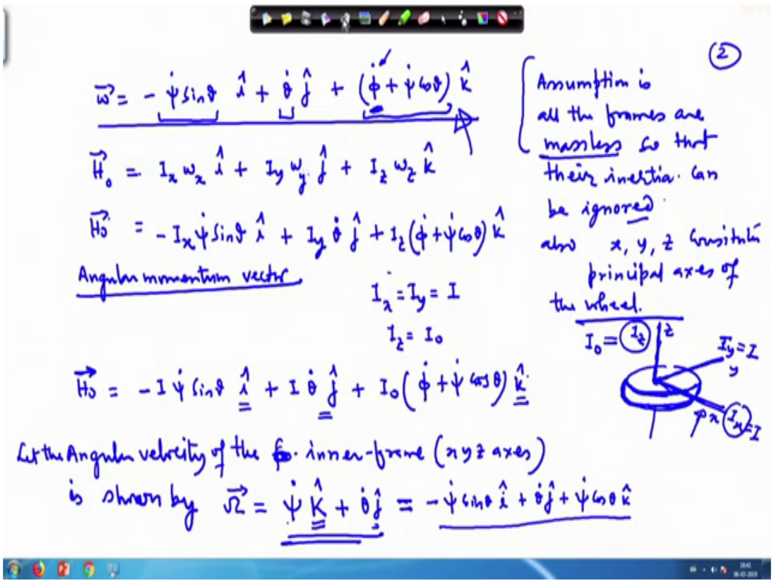

So, what we have got psi dot sin theta times i cap plus theta dot times j cap plus phi dot plus psi dot cos theta k. Now we can calculate the angular momentum of the system. If the system is free from the external torque. So, angular momentum vector will be constant, if it is not free from the external torque then angular momentum vector it cannot be constant and here our assumption is all the frames are massless. So, that their inertias can be moment of inertia basically here inertia implies moment of inertia their moment of inertia can be ignored or is 0, if it is massless.

So, it is a just an ideal case which is never possible in reality and for that we will take care of once the system is having certain mass all the frames are having certain mass so, in that case the system equation will change so there so, that their inertia can be ignored also x, y, z constitutes principal axis of the wheel. Because here we are going to employ one fact that if I put axes like this say this is a x, y and z. So, my wheel can keep rotating, but you can see that i x and i y will not change and therefore, as I have told you earlier that the body frame is not fixed to the wheel, it is located at the centre of mass of the wheel, but it is not rotating along with the wheel.

So, that simplifies the equation of motion otherwise you have to carry a tag for the rotation of the wheel also and why if we are doing so, because this wheel is having symmetry about this axes because of the rotation of this wheel about this axes the moment of inertia does not change and we are using this fact to our advantage. So, H 0 then we can write as I x times angular velocity in this direction omega x times i cap I y times omega z times j cap sorry this is y k cap ok. So, omega the components of omega we know from this place. So, we can insert here in this point. So, this becomes I x with minus sign psi dot sin theta i cap plus I y times theta dot j cap and I z times phi dot plus psi dot cos theta times k cap.

So, this is your H 0 the angular momentum vector ok. Similar type of problem we have done earlier, but what we are doing here we are going to describe further because this case while we are discussing so a similar sort of this kind of device it will it put inside a satellite say here say this is the satellite, inside the satellite this is a cavity in which you have mounted your gyroscope this gyroscope and outside this the satellite is there.

So, I can have a satellite which I can if the cursor is visible on the screen cursor is visible. So, I can assume that like this is my outside the satellite is there, I am showing this by dotted line inside there is a cavity, in that cavity this gyroscope is mounted instead of gyroscope there can be just a wheel which we call as the reaction wheel. So, those things I will take separately because I have to discuss those things in details ok.

So, by actuating about 1 axis you can produce torque about other axes and there is a benefit of using the control moment gyroscope and there are certain disadvantages also. So, we are going to discuss all those things in the future. So, this is your angular momentum vector as we have written here. Now let us write I x equal to I y equal to I. So, here this you are replacing it by I, this you are replacing by I and I z I am replacing by I 0 ok. So, I z we will write as I 0 so in that case your H 0 because the symmetry is there; so we are utilising that symmetry.

So, this becomes I times psi dot minus I times psi dot sin theta i cap plus I times theta dot j cap plus I 0 times phi dot psi dot cos theta times k cap. So, these are all small along the body axes. Now this body axes if you see, the body axes which we have fixed here this is not rotating along with the disc and as a result the angular velocity of the of the inner frame or of the inner frame or x, y, z axes angular velocity of the inner frame let the angular velocity of the inner frame is shown by capital omega ok. So, you can see that while the your x, y, z axes is not spinning with the wheel so this will be constituted of only psi dot capital k cap plus theta dot small j cap.

Only these two will feature out phi dot will not be included because phi dot that gives you the angular velocity of the spinning speed of the wheel, while you have the axes we are taking it should not spinning along the with the wheel. So, the frame angular speed is only this one and already we are we have written here the capital K; so here the only thing this phi dot you can delete and this can be written as psi dot sin theta times i cap plus theta dot j cap because your this frame is not rotating along with the wheel so what will happen in that case only we need to set this phi to 0 and we recover the thing ok.

So, here rest of the things will be psi dot cos theta times a small k cap or either you just put the k here capital K here and this j is at is as it is so as you put you will get this part and there is a directly visible from here this equation also. So, this is your capital omega vector which is the angular velocity of the frame.

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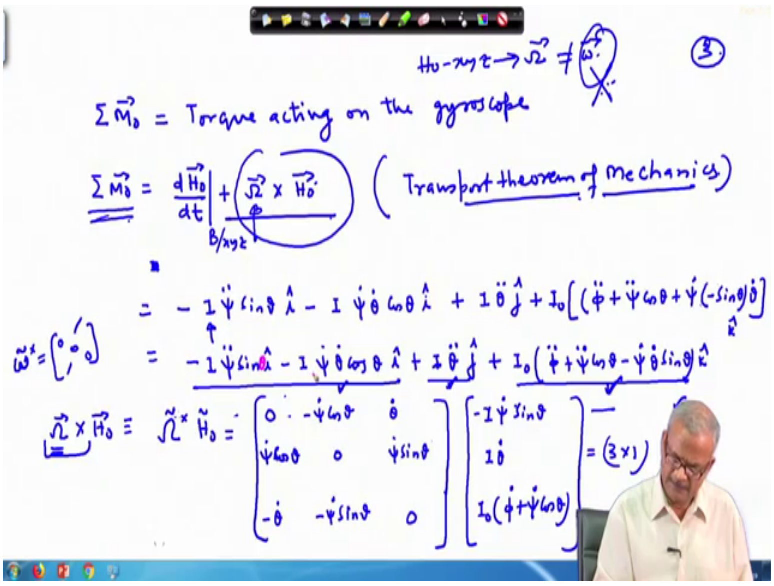

So, if M 0 is the torque acting on the gyroscope. So, M 0 is the torque acting on the gyroscope. So, we can write this as M 0 equal to d H 0 by d t plus capital omega cross H 0, H 0 is your this vector which is in defined in the x, y, z axis you are defining with a axis which is rotating at this rate here i cap, j cap and k cap and therefore, this is rotating at this one not at this one ok.

So, if you apply this, if you write in this equation a small omega here in this place instead of capital omega everything will be wrong because your frame your H 0 you have defined in x, y, z frame and this x, y, z frame is rotating at capital omega not at which is not equal to a small omega ok. So, therefore, including this is will be completely wrong so we are not going to use that part and this part this is with respect to the body axis, so what equation we are using here, we are using transport theorem of mechanics which we have developed while discussing the rigid body dynamics.

So, this is with respect to the body axis means this is body axis is a nothing, but your x, y, z axis so differentiate this. So, H 0 your i, j, k in the body axis they are not changing so we do not need to differentiate only this quantities will be differentiated and we have to insert that there ok. So, once we do that, so that becomes minus I times psi double dot sin theta times i cap minus I times psi dot then sin theta we have to differentiate. So, that becomes cos theta so first we will put theta dot here ok. So, psi dot psi dot theta dot times cos theta times i cap.

So, this is the first term which are differentiated, the second term is I times theta dot only so this is theta double dot I is not getting differentiated remember this is in the body axis means you are in this axis which I am showing it here, in this axis the I is not changing because of the symmetry ok. If there is any other shape so it is possible that depending on the shape basically if the other shape is here so I x and I y they will change then you cannot do like this what we are doing here we are assuming that this frame is this frame is not attached to the wheel so, that will then not be valid in this frame then the moment of inertia will start changing ok. So, this we are using to our advantage this keep in mind because this is very basic this while solving many problems you will find this kind of condition and if you utilise this shortcut so your problem will become very simple.

And then the last term is there which is I 0 and then this particular part this we need to differentiate so phi dot will become phi double dot so we get here phi double dot the other term is psi dot so this will be psi dot and then cos theta is there so cos theta will remain another term is psi dot so this will become psi double dot cos theta then psi dot cos theta will get differentiated so, that becomes then the sin theta with minus sign with minus sign here minus sin theta times theta dot we could have shift here by putting a minus sign here one unnecessary step and then there is a k cap here. So, small k cap thereafter we are left with this term.

So, psi tilde times H 0 cross so this will be given by we can write it as we can write in the form of a determinant rather than expanding multiplying in the form of determinant it will much more easier to work with ok. The components of psi is this capital omega so, components of capital omega we have to pick up and put it here so in the x direction that component is psi dot sin theta we will do one more shortcut rather than doing this, this will also be expensive ok. So, already we are aware of that omega tilde cross this indicates a skew symmetric matrix on this side that will be something this side and the diagonal elements will be 0.

So, here basically this quantity is your you can write it like this omega tilde cross so this is in the here it is written in the form of vector, here you can write the same thing in the form of a matrix means it is a so this becomes your this part basically you are replacing by the skew symmetric matrix and this is H 0 tilde. So, in the skew symmetric matrix this component is 0 and then put the other components of here. So, here minus omega 3 the third component of the capital omega should appear. So, that third component of capital omega we go back here and look into this place here in this place.

So, third component is this one psi dot cos theta. So, here it will come with minus psi dot cos theta and then the second component will appear second component is theta dot so here it goes with theta dot and this part then it will come with plus sign psi dot cos theta diagonal element is 0 and here the first component will appear and the first component is here minus psi dot sin theta so minus psi dot sin theta and that appears with minus sign here. So, minus psi dot sin theta so I will do the simplification here itself and this minus sign, this minus and this minus that makes it plus so I will make it plus here in this place itself.

So, that goes with a plus sign here this is plus so this again this sign I will not put here unnecessary to complicate this and just opposite of this then there will be theta dot you can see here this is minus so this plus so here this is with plus sign then here it will appear with sin dot sin theta and this is 0 and then put the H 0 components here. So, H 0 components is given here in this place this is one part second part and this is third part. So, we need to so this part we have here also we have put the same thing we have written in this format. So, we will take from there because the; I x we have replaced by I so this format we will take ok.

So, this is your first part, this is the second part and this is the third part so we have to put accordingly minus I times psi dot. So, minus I times psi dot this is sin theta so sin theta will appear here the second part was I times theta dot, so that goes here and the third part is I 0 times phi dot plus psi dot cos theta ok, now do the matrix multiplication after the matrix multiplication whatever you get so that becomes your capital omega times H 0. So, here in the first part what we have got here. So, this is one part, this is the second part and this is the third part now think this is the x component, y component and z component.

Similarly, in this multiplication the first line will show because this is a 3 by 3 matrix and this is 3 into 1 matrix ok. So, ultimately you are going to get 3 into 1 matrix ok. So, the first component here will be the x component, y component and z component. So, this x, y, z and the x, y, z here present this can be added and you will get then this capital M 0 so I will write in that format only.

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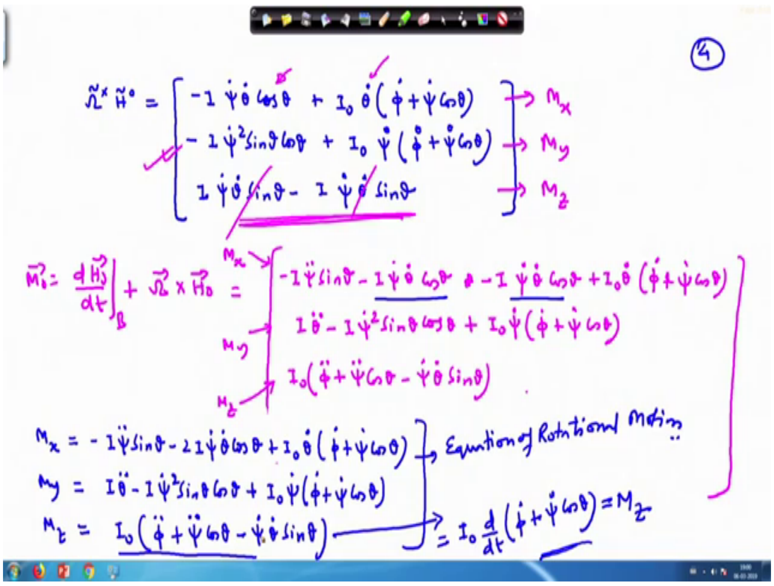

So, if you do the matrix multiplication it appears like this psi dot theta dot cos theta plus this is just our matrix multiplication nothing else it is a straight forward. So, this two will cancel out because they appear with the opposite sign ok.

So, this becomes your M x part, this is the M y part and this is the M z part. So, finally, we can write our equation M 0 which we have written as d H 0 by d t with respect to the body axis plus capital omega cross H 0 ok. So, from the previous page we can carry d H 0 by d t and put it to here in this place ok. So, you can see from this place minus I times psi double dot sin this is sin theta is missing here this sin theta minus I times psi double dot sin theta and then minus I times psi dot theta dot cos theta; minus I times psi dot theta dot cos theta this is what we have got here this particular term I times psi dot theta dot cos theta.

So, this part we have written here and then pick up this part from this place this two terms. So, we if we add here so this term will come with minus sign I times psi dot theta dot cos theta and plus I 0 theta dot phi dot plus psi dot cos theta. So, this is one of the term so this constitutes your M 1 ok. So, this is your M x or M 1.

Similarly the other terms also we can write. So, the second term again going back here second term is just here I times theta double dot. So, this is simple I times theta double dot then pickup this particular term here this one so this is minus I times psi dot square sin theta times cos theta plus I 0 psi dot times phi dot psi dot cos theta and the third term here this is I 0 times this quantity. So, the third term is I 0 times, the first term was differentiated phi double dot thereafter psi dot and then cos theta was there so cos theta and then with minus sign we have psi dot theta dot sin theta appearing here in this place this quantity this cancels out. So, that part is your 0. So, this part is not counting. So, we remove it from this place.

So, therefore, what we have got here this is your M y and this part is your M z. So, we will wind up this M x equal to minus I times psi double dot sin theta and these two will add together that makes it minus two I times psi dot theta dot cos theta and plus I 0 times theta dot plus phi dot psi dot cos theta ok. So, this is the equation of motion angular this is the equation of rotational motion or rotational dynamics as if you can look in this part this particular part especially this can be written as I 0 times this was nothing, but d by d t phi dot plus psi dot cos theta. So, M z is basically this quantity this equal to your M z ok.

So, these are the three equations and using this three equations you can adventurously solve many problems wherever the situation just like the top we were discussing so, in that case there is no external frame involved. So, here we are made the external frame moment of inertia 0. So, this becomes equivalent to a torque and we can work out that problem now ok. So, we will continue in the next lecture.

Thank you very much for listening.