Satellite Attitude Dynamics and Control
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Lecture – 47
Gyroscopic Motion

Welcome to the 47th lecture. So, we have been discussing about the Gyroscope, ok. So, in that context we assume that the outer and the inner frame of the gyroscope they are mass less and therefore, their moment of inertia was not accounted and so, the inner wheel motion then it corresponds to the motion of a top.

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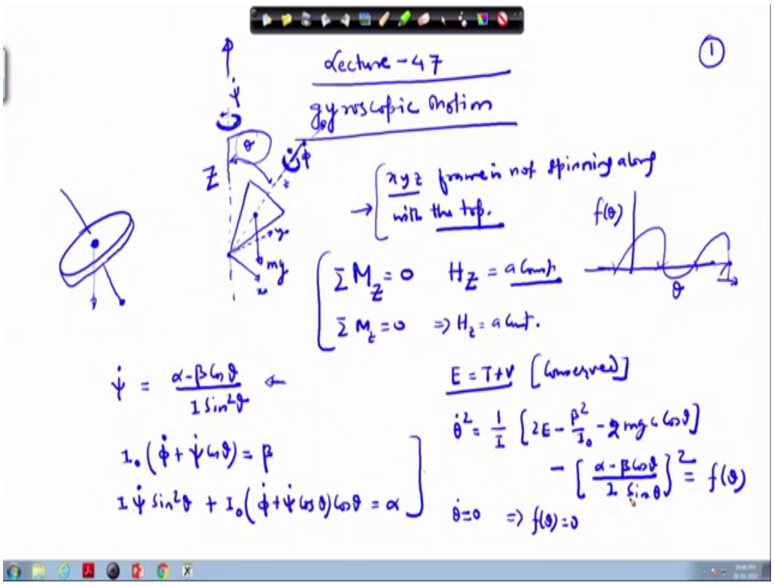

The difference lying that in the case of the gyroscope because the centre of mass it may happen that like you are considering a wheel, ok. So, if it is a pivoted about this point then there will be a torque acting, but if it is pivoted about this point itself about the it is a centre of this disc. So, in that case there would not be any torque, so, that will become a torque free case.

For our case we have taken up that problem that we have a top. It is pivoted about this point by a ball socket joint this is the somewhere the centre of mass, and then mg is acting downward. It is a rotating about this axis at the rate of phi dot, and then there is a motion. So, this angle is theta and there is a motion which is taking place like this which we have called as the precision motion. So, this is psi dot, and then this theta angle can also vary. So, we took this direction to be capital Z direction and this one to be small z direction and then we took x and y like this.

So, if this is x the this is the body axis direction and y is somewhere like this and we assume that xyz frame is not spinning along with the top and this assumption is valid because, here spinning of the top about this axis it does not change the moment of inertia configuration in the xyz frame. And therefore, I have explained it and we choose the body frame such that it is not spinning means the phi dot is not there for the body axis, and then this was the condition. And, also M small z was equal to 0 and then we solve this system and ultimately we through the energy equation and the along this direction your moment of inertia is constant; along this direction also the moment of inertia is constant.

So, this implies it is a small z this will be a constant and here this also implies that H capital Z this will also be a constant ok. So, under this assumption we have worked out solved this problem and in that context we got that psi dot will be given by alpha minus beta cos theta divided by I sin square theta. And, beta I 0 times phi dot plus psi dot cos theta this we have written as beta and alpha we have defined as I times psi dot sin square theta this we worked out in the last class. So, this is just a recalling for the present purpose here, and using these two then we get this equation, ok.

Now, today our objective is fine. So, thereafter one more thing that using the energy equation which is T plus V, this is a total energy is constant means this is conserved ok. Using this we derived one equation about the theta dot which was written as 1 by I times 2E minus 2 mgc cos theta minus alpha minus beta cos theta divided by I sin theta is the whole square, and this we wrote as f theta. So, the question was that what will be the maximum value of theta and the minimum value of theta means the this nutation angle, this particular part. So, that, at the extreme point your theta dot will be equal to 0, ok. So, this implies f theta will be equal to 0 and we need to solve it.

So we have plotted it like this theta and f theta here. So, f theta will be 0 on this axis on the theta axis, ok. So, wherever it cuts theta here so, wherever the this cuts on this axis. So, those are the corresponding solution.

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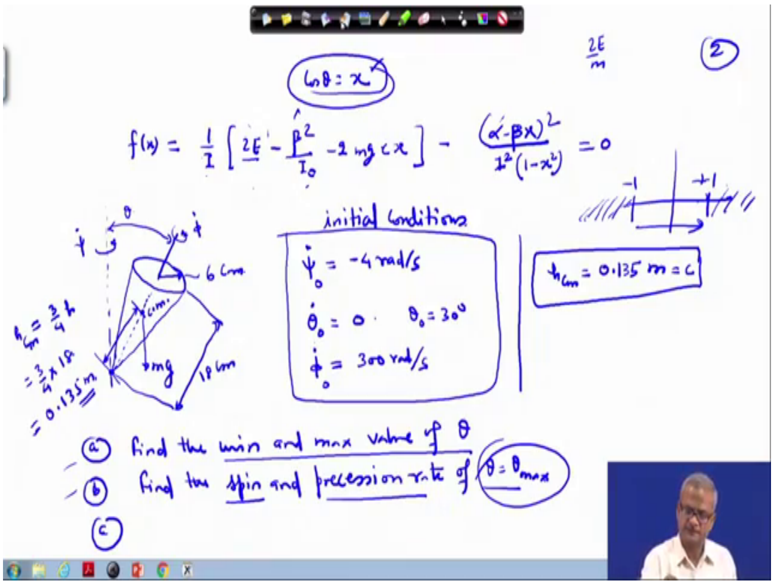

So, this was then converted to by assuming cos theta equal to x, the same equation was rewritten and it was put in this format 1 by I minus 2 mg cos theta becomes x. This is we can put separately as a this as a whole square this is beta times x, ok. I square and in the denominator this sin square theta was there. So, this becomes 1 minus x square ok. Then we need to set it to 0 and we know that it is lies in the range plus 1 and minus 1, ok. So, then we have to look for the corresponding solution where the solution, the solution in this range or either in this range it not acceptable; only within this range minus 1 to plus 1 will be acceptable.

So, we solve this problem one numerical example we will do here. So, this is your cone and it is a here on the ball socket joint with the vertical it is a making angle theta at any time. mg is acting downward and then it is a given that this height of the cone this is 18 centimeter and radius this is 6 centimeter. Mostly there is the psi dot and this is phi dot. So, psi dot is given to be minus 4 radian per second and this is the initial value and theta dot 0 this is 0 ok. So, initial no nutation rate initially and the spin rate is given to be phi dot 0 to be 300 radians per second and theta 0 is given to be 30 degree, and for this we need to work out the problem.

So, what are the things to be determined that, find the minimum and maximum value of theta the nutation angle. So, find the spin and precession rate for theta equal to theta max, ok. So, first we do this two problems, ok. So, here in this case this centre of mass is located here. This is your centre of mass and centre of mass h c m will be equal to 3 by 4 times height of the this cone. So, 3 by 4 times height is 18 centimeter. So, this turns out to be somewhere this is 0.135 meter. So, h c m is 0.135 meter ok. The initial values already we have stated.

So, for solving these problems we need to calculate all this quantities I, E, beta I 0, ok, c already c is the distance this is your distance c. So, h c m these equal to c this we have used in the notation, ok. So, c is known to you from this place then x is cos theta ok. So, for which you are solving mass is here mass of the system, ok, mass of the system it is not given here in this case and let us see how we evolve this problem, anything else? This alpha and beta we need to compute therefore, we go on the next page.

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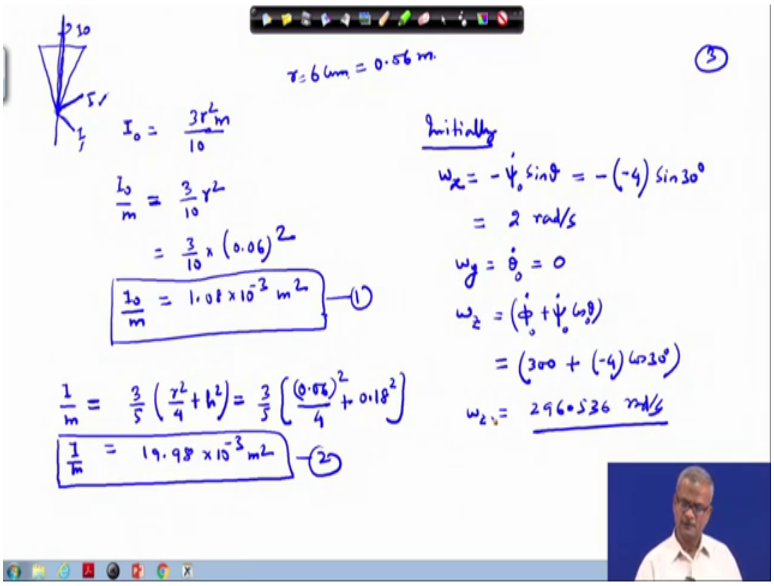

I 0 this will be given by 3r square divided by 10 times m where m is the mass of the cone; that means, you are looking for the moment of inertia along this axis whatever the inertia is that is written as I 0 and this one is written as I and I. So, about this axis this inertia is I 0 equal to 3r square m divided by 10. So, I 0 divided by m becomes 3 by 10 r square ok. This two because of the symmetry this two are also equal and off diagonal terms will not be there because this happens to be the at this point this happens to be the principal axis, ok.

Therefore, from this place we get here 3 by 10 times r is 0.06 meter say 60 mm, r equal to 60 6 centimeter equal to 0.06 meter. Then, I by m quantity also we compute ok. This is given by this relationship. You can refer to any standard book on engineering mechanics and you can find this or either you can using the differential calculus if integral calculus you can work out the whole thing ok. So, this is 3 by 5 and this is 18 centimeter so, 0.18 square. So, I by m this will be equal to ok.

Insert the values initial values, this 0 indicates that is the initial value. Omega y is just theta dot, so, theta dot is initially given to be 0 and omega z this is phi dot plus psi dot cos theta. So, phi dot is given to be 300 radian per second psi dot is initially minus 4 radian per second and cos theta is a cos 30 degree. So, this turns out to be 296.536 radians per second.

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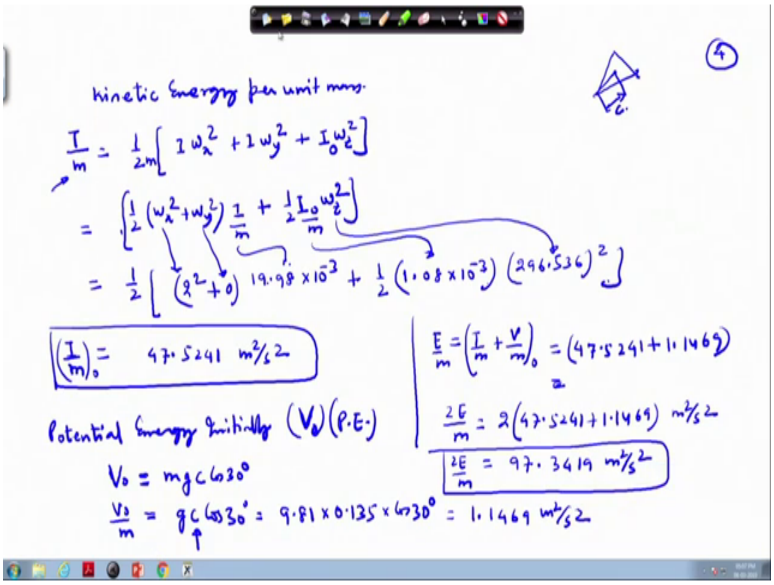

Now, we calculate kinetic energy per unit mass T by m this will be 1 by 2 I times omega x square plus I times omega y square plus I times omega z square, this is the kinetic energy. This is I 0 here; I 0 times omega z square and we need to divide it by m. So, here we need to divide it by m because we have inserted m here in this place. So, that becomes 1 by 2. So, better we can take it inside here, in this place because we have written here. So, we can put here in the I by m, I 0 by m and these quantities are known to us I by m and I 0 by m ok.

So, the initial value of the kinetic energy, that is also known to us. So, 1 by 2 omega x is 19.98, omega x is a 2 radian per second omega y is 0. So, this is 2 radian per second plus 0 times I by m. So, I by m is 19.98 into 10 to the power minus 3. We have converted all the units in terms of meter and I 0 by m we calculated it to be 1.08 into 10 to the power minus 3 and omega z, 296.536. So, this is your I 0 by m and this is I by m omega x, this is omega y and omega z is here.

And, if you compute it, so, this turns out to be T by m this equal to 47.5241. So, this is initial value, and what is the potential energy? So, potential energy that we can write as maybe we are using capital V term for this; V also be used for the velocity, but here let us write this as the potential energy, ok. So, potential energy V 0 then becomes this is your potential energy mgc times cos theta which is 30 degree. So, V 0 by m then will get reduced to g times c cos 30 degree, and g value be chose to be 9.1 meter per second square and see we have already calculated this is 0.135. This is the height of the mass centre of mass of the cone from the vertex.

So, from the vertex this is distance, this is your c and of course, cos 30 degree we have to insert here. So, the total energy then becomes E equal to this is per unit mass T by this is at the time 0. So, adding this up. So, this tells and what is the quantity that we need? We need a quantity if we go back and look here in this equation we require this 2E quantity here. This is the quantity 2E which is entering. Perhaps we have missed out one term. While writing let me check the equation this m is here.

So, if we take this m outside ok, so, whether I have rewritten the equation or not. So, this m is present here in this place. So, if we take this m outside. So, this will get in the form 2E by m, ok, if we take it outside the bracket and this will result in 1 by here m by I which we can write as, so, let me rewrite here. So, if we rewrite this portion. So, this will be 2E by m and this we can write as I 0 by m and from this place then we can remove this m, ok. So, and also here in this place you will have I by m.

So, these are the quantities we are right now working. So, we need 2E by m. So, I will restore it for the time being and write on a first page, ok. So, I will keep it as it is. So, we need a quantity this is the quantity E by m. So, 2E by m will be 2 times 47.5241 plus 1.1469 and this quantity turns out to be 97.3419, ok.

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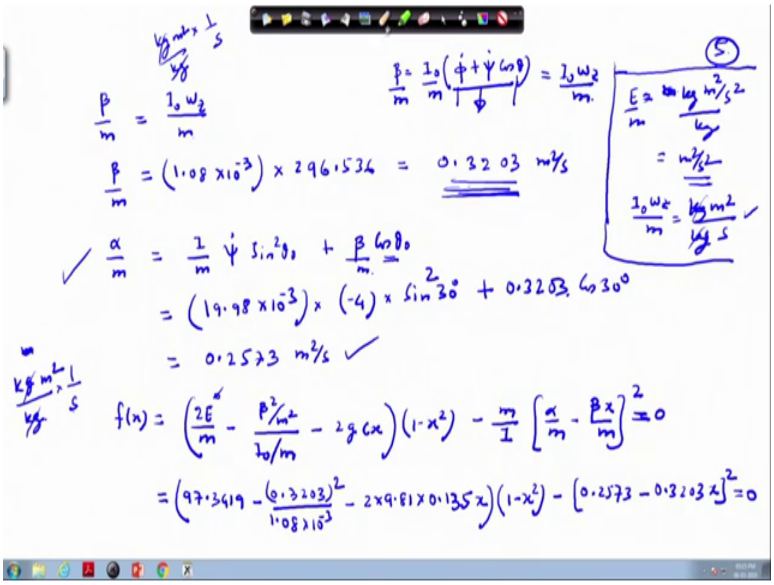

Then, we calculate the quantity beta by m. So, beta is the quantity we have defined as I 0 times phi dot plus psi dot cos theta. So, if we divide it by beta by m. So, we are basically doing like this both side we have to divide. So, beta by m is another quantity that we need. So, this we can write as I 0 omega z divided by m. So, I 0 by m is the quantity we have already calculated. An omega z is the quantity this quantity also we know because the theta phi psi dot a all these things are known right in the beginning. So, this quantity we have calculated 296.536 similarly alpha by m will be the quantity just put in the divided by equation expression for alpha by m.

Now, I by m is known, psi dot is also known and sin 30 degree here beta here beta by m will also be present which is we have calculated here 0.3203 ok. So, if we add it up then cos 30 degree, this part, 257. So, let me check the dimension here I is meter this is I is kg times meter square and m is kg and psi dot this has the dimension of time. So, this we get as kg kg cancels out metre square per second. So, it is here we get as metre square per second. This also be checked, this I is kg meter square omega z is 1 by T. So, 1 by second and there is kg. So, kg-kg cancels out metre square per second. So, this also we have to remove ok.

Going back to the original equation this is the equation we are looking for and we are setting these to 0. So, we divide throughout by m. So, we can write it like this, and we need to insert all the values here. So, your quantity E is a constant E by m this 2E by m we have already computed it. So, this turned out to be a constant here that equation we have to be consistent with the dimensions. So, again I am rechecking that everything is fine kg times meter square divided by kg second this cancel out this (Refer Time: 33:42).

Now this equation we have to insert all the values. So, this is 97.3419 minus beta by m is the quantity which is given here 0.3203 this square because their square term is there and I 0 by m already we have calculated that was 1.08 into 10 to the power minus 3 minus 2 times, g 9.81 and c was three fourth of the height. So, this was 0.135 135 meter and x is your the variable which we want to solve, ok. So, the x remain as it is times 1 minus x square minus m by I. So, I by m the quantity we have already computed.

So, alpha by m is the quantity here this is present this we have computed. So, this is 0.2573 minus beta by m this particular part 3203 and multiplied by x whole square and this quantity is said to 0. Then, we just need to expand it multiplying these terms and simplifying it and then expand it.

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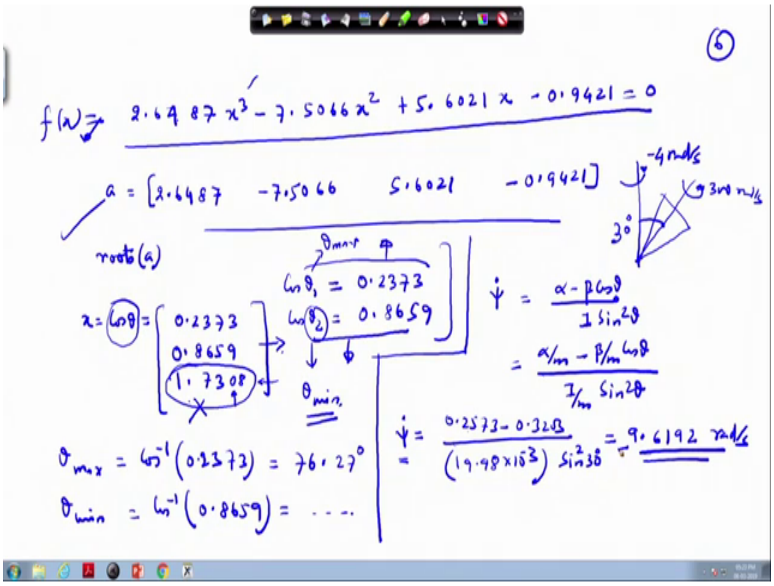

So, once we expand we get the following equation 2.6487 x cube this equal to 0 and in the MATLAB you can defined a vector a tilde like this just give the notation in the MATLAB. a equal to 2.6487 live a gap give a space then 7.5066 then again give a space 5.6021 copy all these coefficients here give a space and then 9.421, and then give command roots a. So, it will list you the roots of this polynomial ok. So, this is your polynomial f x; f x equal to 0 you are solving.

So, this roots they turn out to be x equal to cos theta this equal to 0.2373 and 0.8659 and 1.73 this is 08 or ok. This value I am not sure, this is 68 or 08, but if you give this command. So, you get this result this is 08 only it seems.

Now, you can see that out of this three because it is a third order polynomial. So, you will have three roots this is not acceptable cos theta cannot be greater than 1 or less than 1 minus 1 and therefore, this is rejected and we are left with 0.2373 and 0.8659. So, this is the cos theta one value you are getting say this is cos theta 1 and this is cos theta 2. So, this corresponds to minimum value of theta and this corresponds to maximum value of theta. So, this corresponds to theta max and this one corresponds to theta min, ok.

So, while your top is rotating you are release the top under certain condition that it is a rotating about this axis with minus 4 radian per second means it is a rotating in the opposite direction and here it is a 300 radian per second and this angle you have taken initially to be 30 degree and once you release it so, see if how the theta is varying it is a maximum value it is acquiring which is cos theta value. So, from here theta value can be computed of course.

So, the theta max we can write here theta max equal to cos inverse 0.2373 this will yield 2.72 degree or 276.27 we can write maybe. Similarly, the theta mean can be computed um, but I have not done that part here, ok. So, that will be cos inverse 0.8659 ok. Then the other questions where, find the minimum maximum value of theta find the spin and the precision rate for theta equal to theta max.

So, psi dot this is the precision rate this we need to compute. So, I minus beta cos theta divided by I sin square theta, ok. This is the expression for psi dot divide throughout by m. So, this will come in the format alpha minus m beta divided by m cos theta alpha by m is known to us. So, you see that the mass of the system was not required in our computation.

So, alpha by m this quantity is known to us alpha by m we have already written alpha by m as 0.2573 as it is beta by m 0.3203 and I by m this is a quantity 19.98 8 into 10 to the power minus 3 and sin square 30 degree. So, doing the calculation it is yields result 9.6192 radian per second. So, this is your psi dot, ok. So, you can see that this value is turning out to be here this will turn out with a minus sign minus sign is missing this is 0.2373, 0.3203. So, therefore, here this will turn out with a minus sign, ok.

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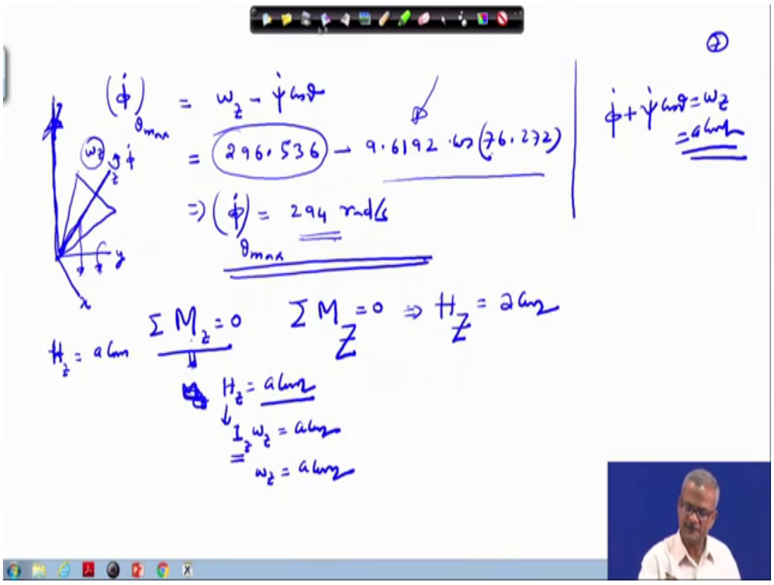

Then we also need to calculate phi dot phi dot corresponding to theta max. So, this quantity will be omega z minus psi dot cos theta and omega z already we have computed. This corresponds to theta max. One thing you should note here that along the z-axis of the cone your omega z is here, ok. This is the z body axis and x body axis is here, y body axis is here. So, omega z we have written phi dot is also along the same line.

Now, because there is no torque acting along this direction and therefore, you have noted earlier that H z this quantity will be a constant because E M a small z this quantity we have set it to 0 and also E M capital Z this quantity we have set it to 0. So, according to both the schemes according to this scheme the angular momentum along this direction because, there is no torque along this direction, ok. Torque is acting only along the y direction because of the gravity along the y body axis. So, along this direction there is no torque and therefore, the corresponding H z this turns out to be a constant and similarly, here m small z this implies that H small z this also be a constant.

And, what is the small H z? This is because this is the principal moments of inertia we are taking x, y and along the z axis. So, this is I z times omega z; this is a constant, ok. Therefore, because I z is a constant so, your omega z this also turns out to be a constant and this we have calculated earlier. If you remember that phi dot plus psi dot cos theta this was written as omega z this was a constant. So, therefore, omega z we have calculated write in the beginning and that quantity was 296.56. So, from there we subtract this corresponding value. Psi dot we just now we have computed on the previous page. So, this gives us the 294 point 294 radian per second. This is the maximum the spin rate here spin spinning of the torque while theta equal to maximum, ok.

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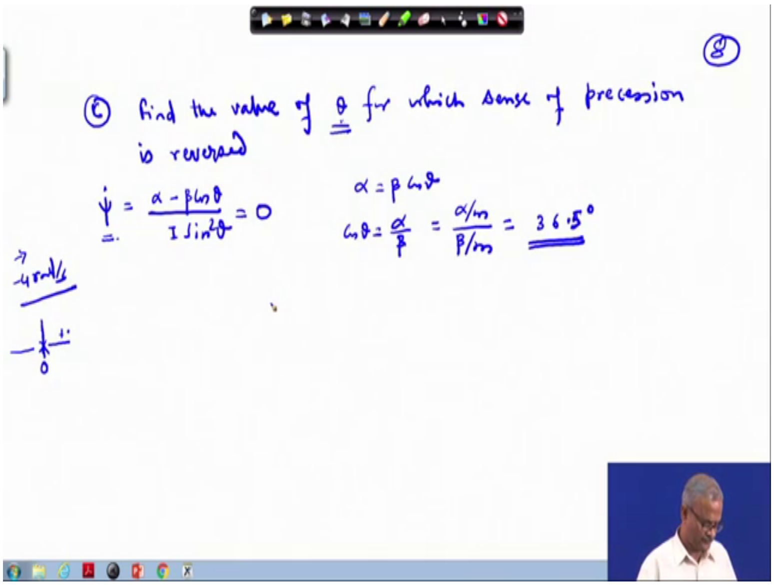

And, one last question is there the last question will state on the next page. So, the question number c find the value of theta for which sense of precision is reversed. So, psi dot equation we have I sin square theta; this is the psi dot equation we have written, ok. When the sense of precision will reverse so, earlier this was negative, minus 4 radian write in the beginning 4 radian per second. So, once it reverses means at the point of reversal it is negative and then it will become positive means somewhere it is a crossing the 0 value ok.

So, your the corresponding theta can value can be found out by setting psi dot equal to 0. So, at this point after this the psi value will reverse and then solve it. So, solving this gives you alpha equal to beta cos theta and this is cos theta equal to alpha by beta which is alpha by m divided by beta by m and these quantities already we have worked out. So, this gives you the result 36.5 degree. So, at this value of theta the sense of precision will change ok.

So, one problem this theory we worked out in the last class, and now we have completed a numerical corresponding to this, ok. So, our objective is finished that we have worked out the theory also and based on that theory then we have solved this problem ok. Then, we take some more problems, but we will do so in the next lecture. Thank you for listening. We will continue in the next lecture with some more examples.

Thank you very much.